March, 1983 LIDS-P-1287

RELIABLE STABILIZATION USING A MULTI-CONTROLLER CONFIGURATION*

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*This research was supported by the Natural Sciences and Engineering Research Council of Canada under grant no. A-1240, and was carried out while the second author was visiting the University of Waterloo.

1. INTRODUCTION

Suppose P is a given unstable plant. The problem of determining a controller C such that the feedback system of Figure 1 is stable has been studied for several years. Recent results [1-3] provide a characterization of all controllers C that stablize the given plant P. With the availability of this characterization, interest has been created in the problem of reliable stabilization. In [4,5] the object of study is the so-called simultaneous stabilization problem, where one would like to determine whether or not there exists a single controller C that stabilizes each of several given plants $P_0,\ldots,P_n$. The motivation for the problem formulation is that P_0 represents the model of the plant in its normal mode, while $P_1,\ldots,P_n$ represent the same plant under various structural perturbations, such as sensor/actuator failures, changes in the mode of operation etc. Thus, if the simultaneous stabilization problem has a solution, then not only does C stabilize the nominal plant P_0, but this stabilization is reliable against a prespecified set of structural changes in the plant.

The problem studied in this paper is in a sense the dual of the simultaneous stabilization problem. Consider the system shown in Figure 2, where P is a given plant, and C_1, C_2 are controllers to be determined. The objective is to select C_1 and C_2 (if possible) such that the system of Figure 2 is stable as shown, as well as when either C_1 or C_2 is set equal to zero. The structure in Figure 2 is called a multi-controller configuration, and the above requirements on C_1, C_2 mean that C_1 and C_2 together stabilize P, and in addition, both C_1 and C_2 individually stabilize P. The motivation for studying this problem is the following: In the "normal" mode, both controllers C_1 and C_2 are in operation and the system is stable. Should either controller fail (modeled by setting $C_i=0$ for i=1 or 2), the system is still stable (though other properties such as sensitivity might be affected adversely). Thus, if there exist controllers C_1, C_2 satisfying the above requirements, then the stabilization scheme of Figure 2 is reliable against a single controller failure.

It should be emphasized that the reliable stabilization scheme proposed in Figure 2 is quite distinct from the standard technique of having redundancy in key controllers [6]. The redundancy scheme can be represented as in Figure 3. In this scheme, the back-up controller is switched-in once the failure of the main controller is detected. Thus only one controller is connected to P at any one time. In contrast, in the normal mode of operation of the system shown in Figure 2, both controllers are connected to P. There are two reasons for proposing the structure of Figure 2 as an alternative to that in Figure 3: (i) In systems with very fast transients such as aircraft, the system may become unstable during the time it takes to detect the failure of the controller (ii) The structure of Figure 3 is not reliable against the failure of the "switch".

The objective of the paper is to present conditions on P that ensure the existence of controllers C_1 and C_2 that achieve reliable stabilization of P. The problem is of course trivial if a controller C can be found that stabilizes P in such a way that the feedback system has a gain margin greater than two; in such a case, one can simply choose $C_1 = C_2 = C$. If P is a minimum phase plant, the results of [7,8] imply that one can actually find a stabilizing controller with infinite gain margin. However, the case where P is nonminimum phase is still open. The main result of the paper is as complete as it is surprising: It states that, given any plant P and any controller C_1 that stabilizes P, there always exists another controller C_2 such that C_1 and C_2 together reliably stabilize P. Thus, not only does the reliable stabilization problem have a solution for arbitrary plants P, but also one of the two stabilizing controllers can be specified arbitrarily (subject of course to the constraint that it stabilizes P). Further, it is shown that, given any plant P, there exists a stabilizing controller C such that 2C also stabilizes P; hence $C_1 = C_2 = C$ solves the reliable stabilization problem.

The main result of the paper carries over with very little modification to the problem of reliable robust regulation. It is shown that, given any plant P and any controller C_1 that solves the robust tracking problem for P and a given reference input, there exists another controller C_2 such that C_2 and $C_1 + C_2$ also solve the same problem. Moreover, there exists a C such that C and 2C both solve the robust tracking problem. Similar results apply to disturbance rejection.

The present results considerably extend those of [9], in which sufficient conditions of the weak-coupling type are given for a plant P to be reliably stabilizable. In contrast, the present result show that every plant can be reliably stabilized.

2. PROBLEM STATEMENT AND MAIN RESULT

Let R(s) denote the field of rational functions with real coefficients, and let $\mathcal{S}$ denote the subset of R(s) consisting of proper stable rational functions; in other words, $\mathcal{S}$ consists of functions in R(s) that do not have poles in the closed right half-plane nor at infinity. Let $M(R(s))$ (resp. $M(\mathcal{S})$) denote the set of matrices, of whatever order, whose elements all belong to R(s) (resp. $\mathcal{S}$).

Consider now the system of Figure 1 and suppose $P, C \in M(R(s))$. Then it is easy to verify that

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = H(P,C) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1}$$

where

$$H(P,C) = \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix} \tag{2}$$

assuming the indicated inverses exist. We say that the pair (P,C) is <u>stable</u>, and that C <u>stabilizes</u> P, if H(P,C) is well-defined and belongs to $M(\mathcal{S})$. This is equivalent to requiring that e_1, e_2 be bounded whenever u_1, u_2 are bounded.

The problem studied in this paper can now be stated precisely.

<u>Reliable Stabilization Problem (RSP)</u>. Given $P \in M(R(s))$, find $C_1, C_2 \in M(R(s))$ of compatible dimensions such that

(i) (P, C_1) is stable

(ii) (P, C_2) is stable

(iii) $(P, C_1 + C_2)$ is stable.

Let $S(P)$ denote the set of all controllers that stabilize P; i.e.

$$S(P) = \{C \in M(R(s)) : (P,C) \text{ is stable}\} \tag{3}$$

Then the reliable stabilization problem is one of finding C_1, C_2 in $S(P)$ such that $C_1 + C_2$ also belongs to $S(P)$. If such C_1, C_2 can be found, we say that P can be <u>reliably stabilized</u>, and that C_1 and C_2 together <u>reliably stabilize</u> P.

We present at once the main result of the paper.

<u>Theorem 1</u>. Every plant $P \in M(R(s))$ can be reliably stablized. Further given any $P \in M(R(s))$ and any $C_1 \in S(P)$, there exists a $C_2 \in S(P)$ such that $C_1 + C_2 \in S(P)$, i.e. such that C_1 and C_2 together reliably stabilize P.

<u>Theorem 2</u>. Given any $P \in M(R(s))$, there exists a $C \in S(P)$ such that $2C \in S(P)$, i.e. such that $C_1 = C_2 = C$ together reliably stabilize P.

The proof of Theorem 1 requires the following lemma.

<u>Lemma 1</u>[1]. Suppose $A \in S^{m\times n}$, $B \in S^{n\times m}$. Then there exists a matrix $Q \in M(S)$ such that $I - AB + QBAB$ is unimodular in $M(S)$ (i.e. has an inverse in $M(S)$.

<u>Proof</u>. Define the norm on $M(S)$ in the usual way, namely,

$$||F|| = \sup_{\omega} \bar{\sigma}(F(j\omega)), \quad \forall F \in M(S) \tag{3}$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value of a matrix. Then $I + F$ is unimodular whenever $||F|| < 1$. In particular, $I - rAB$ is unimodular whenever $|r| < ||AB||^{-1}$. Let k be an <u>integer</u> larger than $||AB||$. Then $(I - k^{-1}AB)$ is unimodular, and so is $(I - k^{-1}AB)^k$. By the binomial expansion,

$$(I - k^{-1}AB)^k = I - AB + \sum_{i=2}^{k} f_i (AB)^i \tag{4}$$

where the f_i are appropriate real numbers. Now define

$$Q = \sum_{i=o}^{k-2} f_{i+2} (AB)^i A \in M(S) \tag{5}$$

Then clearly

$$I - AB - QBAB = (I - k^{-1}AB)^k \tag{6}$$

is unimodular.

Following [8], we say that a plant P is strongly stabilizable if it can be stabilized by a stable compensator. Thus Lemma 1 shows that every plant of the form $BAB(I-AB)^{-1}$ is strongly stabilizable, irrespective of the matrices A and B.

Proof of Theorem 1. Suppose $P \in M(R(s))$ and $C_1 \in S(P)$ are specified. Let (N,D), $(\tilde{D},\tilde{N})$ be any right-coprime factorization and left-coprime factorization, respectively, of P over $M(S)$. The fact that C_1 stabilizes P implies [2,3] that $C_1 = Y^{-1}X = \tilde{X}\tilde{Y}^{-1}$, where $X, \tilde{X}, Y, \tilde{Y} \in M(S)$ satisfy

$$XN + YD = I, \; \tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I \tag{7}$$

Moreover, $Y^{-1}X = \tilde{X}\tilde{Y}^{-1}$ implies that

$$Y\tilde{X} = X\tilde{Y} \tag{8}$$

Using Lemma 1, select a matrix $Q \in M(S)$ such that $I - XN + QNXN$ is unimodular. From the results of [2,3], the controller $\bar{C}$ defined by

$$\bar{C} = (Y - Q\tilde{Y}\tilde{N})^{-1}(X + Q\tilde{Y}\tilde{D}) \tag{9}$$

is in S(P). Let $\bar{C} = C_1 + C_2$. We now show that $C_2 = \bar{C} - C_1$ is also in S(P), which shows that C_1 and C_2 together reliably stabilize P. Now

$$C_2 = \bar{C} - C_1 = (Y - Q\tilde{Y}\tilde{N})^{-1}(X + Q\tilde{Y}\tilde{D}) - \tilde{X}\tilde{Y}^{-1}$$

$$= (Y - Q\tilde{Y}\tilde{N})^{-1}[(X + Q\tilde{Y}\tilde{D})\tilde{Y} - (Y - Q\tilde{Y}\tilde{N})\tilde{X}]\tilde{Y}^{-1}$$

$$= (Y - Q\tilde{Y}\tilde{N})^{-1}[X\tilde{Y} - Y\tilde{X} + Q\tilde{Y}(\tilde{D}\tilde{Y} + \tilde{N}\tilde{X})]\tilde{Y}^{-1}$$

$$= (Y - Q\tilde{Y}\tilde{N})^{-1}\, Q\tilde{Y} \quad \tilde{Y}^{-1} \quad \text{by (7) and (8)}$$

$$= (Y - Q\tilde{Y}\tilde{N})^{-1}Q$$

$$= \tilde{D}_{C_2}^{-1} \quad \tilde{N}_{C_2} \tag{10}$$

where $\tilde{D}_{C_2} = Y - Q\tilde{Y}\tilde{N}$, $\tilde{N}_{C_2} = Q$. At this stage, it has not been shown that $\tilde{D}_{C_2}$, $\tilde{N}_{C_2}$ are left-coprime. But let us anyway compute the "return difference" matrix $\tilde{D}_{C_2} D + \tilde{N}_{C_2}$ as in [2,3]. This gives

$$\tilde{D}_{C_2} D + \tilde{N}_{C_2} N = (Y - Q\tilde{Y}\tilde{N})D + QN$$

$$= YD - Q\tilde{Y}\tilde{N}D + DN$$

$$= YD - Q\tilde{Y}\tilde{D}N + QN, \quad \text{since } \tilde{N}D = \tilde{D}N \tag{11}$$

Now (7) and (8), together with $\tilde{N}D = \tilde{D}N$, can be written as

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = I \tag{12}$$

Thus the two matrices in (12) are the inverses of each other. Hence interchanging the order of multiplication does not affect the result; i.e.

$$\begin{bmatrix} D & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} \begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{D} \end{bmatrix} = I \tag{13}$$

In particular, $N X + \tilde{Y}\tilde{D} = I$, so that $\tilde{Y}\tilde{D} = I - NX$. Similarly, from (7) we get $YD = I - XN$. Substituting these in (11) gives

$$\begin{aligned} D_{c_2} D + N_{c_2} N &= I - XN - QN + QNXN + QN \\ &= I - XN + QNXN \end{aligned} \tag{14}$$

which is unimodular by construction. Hence $C_2 \in S(P)$. This also shows, a fortiriori, the left-coprimeness of $(\tilde{D}_{c_2}, \tilde{N}_{c_2}) = (Y - Q\tilde{Y}\tilde{N}, Q)$.

The proof of Theorem 2 depends on the following lemma.

Lemma 2: Given a plant $P \in M(R(s))$, let (N,D), $(\tilde{D},\tilde{N})$ be any r.c.f. and l.c.f. of P, and let (X,Y) be any solution of the equation $XN+YD = I$. Then there exists an $R \in M(S)$ such that $I + XN + R\tilde{D}N$ is unimodular.

Proof. It is first shown that the matrices $I + XN$, $\tilde{D}N$ are right-coprime. From [11,12], one can select $\tilde{X}$, $\tilde{Y}$ such that

$$\begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{D} \end{bmatrix} \begin{bmatrix} D & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} = \begin{bmatrix} D & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} \begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{D} \end{bmatrix} = I \tag{15}$$

Suppose M is a right divisor of both $I+XN$ and $\tilde{D}N$, denoted by $M|(I+XN)$, $M|\tilde{D}N$. This implies, successively, that

$M|\tilde{Y}\tilde{D}N$, $M|(I-NX)N$ since $NX + \tilde{Y}\tilde{D} = I$ (16a)

$M|N(I-XN)$, $M|(I+NX)N$ (16b)

$M|N$ since $N = [(I-NX)N + (I+NX)N]/2$ (16c)

$N|XN$ (16d)

$M|I$ since $N|(I+XN)$, $M|XN$ (16c)

This last step shows that M is unimodular.

Now let C_{+e} denote the extended right half-plane, i.e. $\{s: \text{Re } s \geq 0\}$ $\{\infty\}$. The next step is to show that $|I+X(s)N(s)| > 0$ whenever $s \in C_{+e}$ is real and $\tilde{D}(s)\ N(s) = 0$. It would then follow from [5,10] that $I + XN + R\tilde{D}N$ is unimodular for some $R \in M(S)$. Suppose $(\tilde{D}N)(s) = 0$. Then

$$(\tilde{Y}\tilde{D}N)(s) = 0 \Rightarrow [(I-NX)N](s) = 0$$

$$\Rightarrow N(s) = (NXN)(s)$$

$$\Rightarrow (XN)(s) = (XN\ XN)(s) = [(XN)(s)]^2 \qquad (17)$$

Let $\alpha = \sqrt{2} - 1 \cong 0.414$. Then it is easy to verify that $1-2\alpha = \alpha^2$. Thus

$$I + (XN)(s) = I + 2\alpha(XN)(s) + \alpha^2(XN)(s)$$

$$= I + 2\alpha(XN)(s) + \alpha^2[XN(s)]^2 \quad \text{by (17)}$$

$$= [I + \alpha(XN)(s)]^2 \qquad (18)$$

$$|I + (XN)(s)| = |I + \alpha(XN)(s)|^2 \geq 0 \qquad (19)$$

However, since I+XN and $\tilde{D}N$ are right-coprime, the smallest invariant factor of $\tilde{D}N$ and $|I + XN|$ are coprime. Hence $|I + (XN)(s)| \neq 0$, which implies, in conjunction with (19) that $|I + (XN)(s)| > 0$.

Proof of Theorem 2. Let $C = (Y-R\tilde{N})^{-1}(X+R\tilde{D})$. Then $2C = (Y-R\tilde{N})^{-1} \cdot 2(X+R\tilde{D})$. Clearly C stabilizes P, from [2,3]. The return difference matrix corresponding to P and 2C is

$$(Y-RN)D + 2\cdot(X+R\tilde{D})N = I + XN + R\tilde{D}N \qquad (20)$$

which is unimodular by construction. Thus $2C \in S(P)$.

The preceding results extend readily to the problem of reliably stabilizing a plant while at the same time tracking a given reference input, or rejecting a disturbance. In order to present this extension, a few facts are recalled from [13].

Given a plant $P \in M(R(s))$, a basic neighborhood of P is a set $N(P)$ $M(R(s))$ of the form

$$N(P) = \{N_1D_1^{-1}: ||N_1-N|| < \varepsilon,\ ||D_1-D|| < \varepsilon,\ (N,D) \text{ an r.c.f. of } P\} \qquad (21)$$

A property (such as stability, tracking or disturbance rejection) is said to be robust against perturbations in P if there is a basic neighborhood $N(P)$ such that the property continues to hold for all plants in $N(P)$.

Consider first the problem of robust tracking, as depicted in Figure 4. The reference signal r is the output of an unstable system $\tilde{D}_r^{-1}\tilde{N}_r$, where $\tilde{D}_r$, $\tilde{N}_r$ are left-coprime. The controller C solves the robust tracking problem if

(i) C stabilizes P

(ii) $(I+PC)^{-1}D_r^{-1}N_r \in M(S)$

(iii) Both (i) and (ii) are robust against perturbations in P.

The following result is proved in [13].

Lemma 3. Let $(\tilde{D},\tilde{N})$ be any l.c.f. of P, and let α_r denote the largest invariant factor of $\tilde{D}_r$. Then the robust tracking problem has a solution if and only if $\tilde{N}$ and $\alpha_r I$ are right-coprime. Suppose $C \in S(P)$ and let (N_C, D_C) be any r.c.f. of C. Then C solves the robust tracking problem if and only if α_r divides every element of D_C.

A ready consequence of Lemma 3 is the following:

Lemma 4 with all symbols as in Lemma 3, suppose $\alpha_r I$, and $\tilde{N}$ are right-coprime. Then C solves the robust tracking problem if and only if $\alpha_r C$ stabilizes P/α_r. Thus the set of solutions to the robust tracking problem is given by $\alpha_r^{-1} S(P/\alpha_r)$.

Proof. The coprimeness of $\alpha_r I$ and $\tilde{N}$ implies that $(\alpha_r \tilde{D}, \tilde{N})$ is a l.c.f. of P/α_r.

"if" suppose $\alpha_r C$ stabilizes P/α_r, and let $C_1 = \alpha_r C$. Then, from [2,3] it follows that C_1 has an r.c.f. (B,A) such that

$$\alpha_r \tilde{D} A + \tilde{N} B = I \tag{21}$$

or equivalently

$$\tilde{D}\, \alpha_r A + \tilde{N} B = I \tag{22}$$

Now (22) implies that $C = B(A\alpha_r)^{-1} = \alpha_r^{-1} \cdot BA^{-1} = C_1/\alpha_r$ stabilizes P. Moreover, since $\alpha_r A$ and B are clearly coprime, it follows from Lemma 3 that C solves the robust tracking problem.

"only if" Suppose C solves the robust tracking problem, and let (N_C, D_C) be an r.c.f. of C such that

$$\tilde{D}D_C + \tilde{N}N_C = I \tag{23}$$

By Lemma 3, $\alpha_r I$ divides D_C. Accordingly, suppose $D_C = \alpha_r M$. Then (23) implies that

$$\alpha_r \tilde{D}M + \tilde{N}N_C = I \tag{24}$$

Hence $N_C N^{-1} \triangleq C_1$ stabilizes $(\alpha_r \tilde{D})^{-1}\tilde{N} = P/\alpha_r$. Clearly $C_1 = N_C(D_C/\alpha_r)^{-1} = \alpha_r N_C D_C^{-1} = \alpha_r C$.

Combining Lemma 4 with Theorems 1 and 2 now gives the following result.

Theorem 3. Suppose a plant P and a reference input generator $\tilde{D}_r^{-1}\tilde{N}_r$ are specified, together with a controller C_1 that solves the robust tracking problem. Then there exists a C_2 such that both C_2 and $C_1 + C_2$ solve the robust tracking problem. In particular, there exists a C such that both C and 2C solve the robust tracking problem.

3. CONCLUSIONS

In this paper, a complete solution has been given to the problem of designing a pair of controllers C_1 and C_2 for a given plant P such that C_1, C_2, $C_1 + C_2$ all stabilize P. This problem was previously studied in [9] and can be thought of as a dual to the simultaneous stabilization problem considered in [4,5].

A much more interesting problem which is as yet unsolved is the following: Given a plant P and a controller C that stabilizes P, when can it be decomposed as a sum of two controllers C_1 and C_2, each of which stabilizes P? This problem is more natural than the one studied here in the following sense. During the normal (i.e., unfailed) node, C is the controller that is applied, and can be chosen to have desirable properties such as optimality, low sensitivity, etc. In contrast, in the design algorithm described in this paper, the normal mode controller $C_1 + C_2$ is obtained as a by-product of the algorithm, and is only guaranteed to stabilize P, or to regulate P. Still, it is hoped that the techniques presented in this paper will eventually lead to a resolution of the above problem as well.

FOOTNOTES

[1] The values of the integers n,m are unimportant, what is important is that both AB and BA are well-defined and square.

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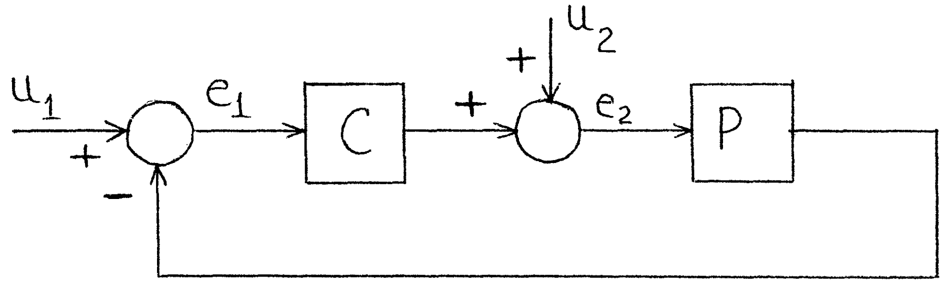

Figure 1

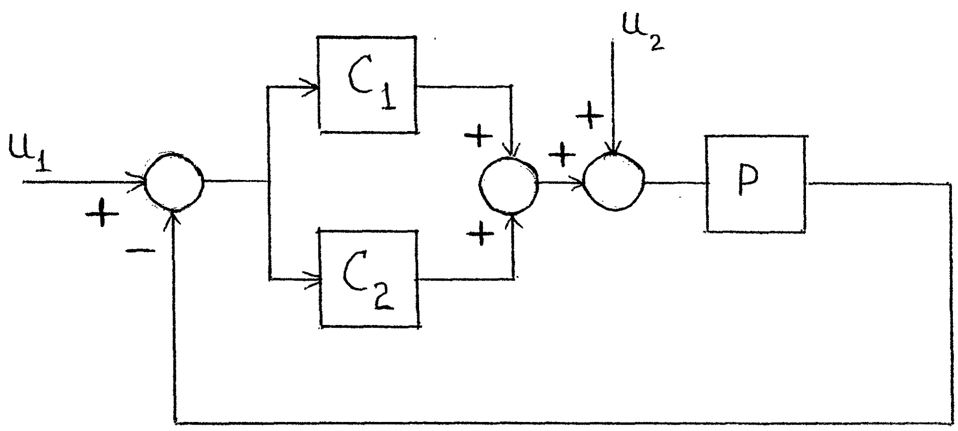

Figure 2

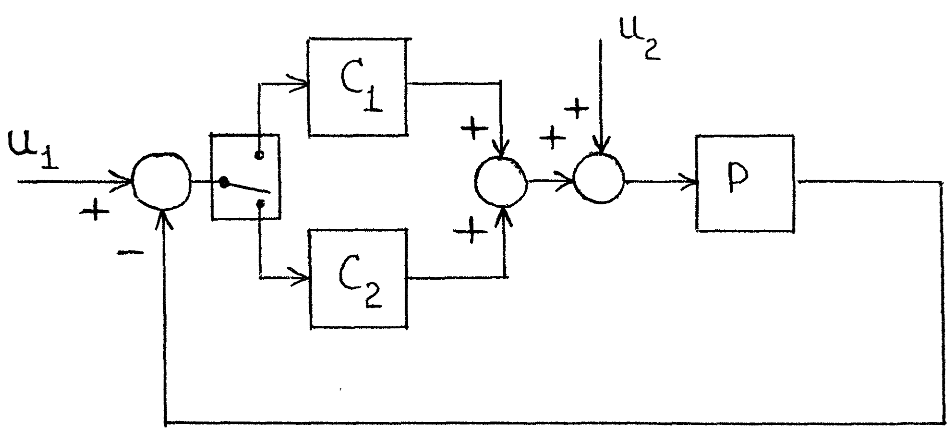

Figure 3

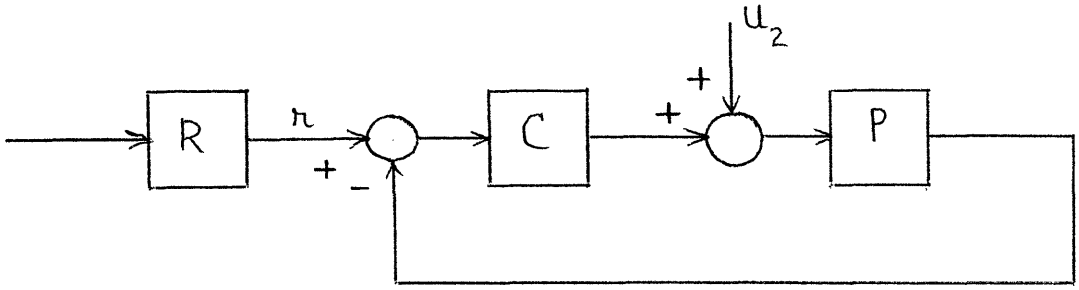

Figure 4