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Remainder terms in a higher order Sobolev inequality

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Abstract. For higher order Hilbertian Sobolev spaces, we improve the embedding inequality for the critical L^p-space by adding a remainder term with a suitable weak norm.

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1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be any domain and for an integer m consider the space $\mathcal{D}^{m,2}(\Omega)$, namely the completion of the space of real-valued C^∞-functions with compact support in Ω with respect to the norm

$$\|u\| = \left(\int_\Omega (-\Delta)^m u \cdot u\right)^{1/2} = \begin{cases} |\Delta^{m/2} u|_2 & \text{if } m \text{ is even,} \\ |\nabla \Delta^{(m-1)/2} u|_2 & \text{if } m \text{ is odd,} \end{cases} \tag{1.1}$$

where $|u|_p$ denotes the L^p-norm of a function $u \in L^p(\Omega)$. We assume that $m < \frac{N}{2}$, then the so-called critical Sobolev exponent $2^* = 2N/(N-2m)$ is well-defined and the following inequality holds:

$$S\,|u|_{2^*}^2 \le \|u\|^2 \qquad \text{for all } u \in \mathcal{D}^{m,2}(\Omega). \tag{1.2}$$

It is known [12,14] that the best constant

$$S = \inf_{\substack{u \in \mathcal{D}^{m,2}(\Omega) \\ u \neq 0}} \frac{\|u\|^2}{|u|_{2^*}^2}$$

in inequality (1.2) does not depend on the domain Ω, and that S is attained if $\Omega = \mathbb{R}^N$ and

$$u \in \mathcal{M} := \{cU_{\lambda,y} : c \in \mathbb{R} \setminus \{0\}, y \in \mathbb{R}^N, \lambda > 0\} \tag{1.3}$$

where

$$U_{\lambda,y} \in \mathcal{D}^{m,2}(\mathbb{R}^N), \qquad U_{\lambda,y}(x) := \lambda U(\lambda^{\frac{2}{N-2m}}(x-y)),$$

and $U \in \mathcal{D}^{m,2}$ is given by $U(x) = (1+|x|^2)^{-\frac{N-2m}{2}}$. In the sequel we will also write U_λ in place of $U_{\lambda,0}$. The minimization property of the functions $U_{\lambda,y}$ implies that they satisfy the equation

$$(-\Delta)^m U_{\lambda,y} = \tau_m |U_{\lambda,y}|^{2^*-2} U_{\lambda,y} \qquad \text{with } \tau_m = \frac{\|U_{\lambda,y}\|^2}{|U_{\lambda,y}|_{2^*}^{2^*}} = 2^{2m} \frac{\Gamma\left(\frac{N}{2}+m\right)}{\Gamma\left(\frac{N}{2}-m\right)}. \tag{1.4}$$

In the present paper, we are interested in bounded domains $\Omega \subset \mathbb{R}^N$. In this case, the space $\mathcal{D}^{m,2}(\Omega)$ is usually denoted by $H_0^m(\Omega)$ and we stick to this notation. Since S is not attained when Ω is bounded, it is natural to wonder if some lower bounds exist for the remainder term $\|u\|^2 - S|u|_{2^*}^2$ whenever $u \in H_0^m(\Omega)$. Generalizing a result of Brezis and Lieb [4] for the first order case $m = 1$, Gazzola and Grunau [7] proved that for any bounded domain $\Omega \subset \mathbb{R}^N$ there exists $C = C(\Omega, m) > 0$ such that

$$\|u\|^2 - S|u|_{2^*}^2 \geq C|u|_w^2 \qquad \text{for all } u \in H_0^m(\Omega) \tag{1.5}$$

where $|u|_w$ denotes the weak $L^{2^*/2}$-norm (see [11]) defined by

$$|u|_w = \sup_{\substack{A\subset\Omega \\ |A|>0}} |A|^{-\frac{2m}{N}} \int_A |u|.$$

The space $H_0^m(\Omega)$ is of interest for the study of boundary value problems for the polyharmonic operator $(-\Delta)^m$ complemented with Dirichlet boundary conditions $u = u_\nu = \cdots = \frac{\partial^{m-1}}{\partial\nu^{m-1}}u = 0$ on $\partial\Omega$. If these boundary conditions are replaced by Navier boundary conditions $u = \Delta u = \Delta^2 u = \ldots \Delta^{m-1}u = 0$ on $\partial\Omega$, one is led to consider the space

$$H_\theta^m(\Omega) = \left\{ u \in H^m(\Omega) \ : \ \Delta^j u = 0 \text{ for } 0 \leq j < \frac{m}{2} \right\}$$

which may also be endowed with the norm (1.1). Clearly, whenever $m \geq 2$, the space $H_\theta^m(\Omega)$ is strictly larger than $H_0^m(\Omega)$. Nevertheless, it has been shown in [8] (see also previous work in [9,15]) that the Sobolev inequality (1.2) holds with the same optimal constant S also for functions in $H_\theta^m(\Omega)$. Whenever $m \geq 2$, this fact does not follow by a trivial extension argument, as is most easily seen in the special case $m = 2$. Indeed, in this case any extension of a function in $H_\theta^2(\Omega)$ with nontrivial outer normal derivative u_ν on $\partial\Omega$ to a function in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ increases the norm $\|\cdot\|$ if $\mathbb{R}^N \setminus \overline{\Omega} \neq \varnothing$. We also point out that the optimal constant changes for subcritical embeddings, namely embeddings in L^p with $p < 2^*$, see [5]. In this paper we prove a remainder term estimate of type (1.5) for functions $u \in H_\theta^m(\Omega)$. We note that the proof of (1.5) in [7] does not carry over to functions in this larger space since one cannot trivially extend functions in $H_\theta^m(\Omega)$ to functions in $H_\theta^m(B)$ where B is a ball containing Ω; moreover, a further nontrivial radial extension outside this larger ball B

was needed in [7] and this extension seems not to be possible in $H^m_\theta(\Omega)$ even if Ω is itself a ball. The following is the main result of the present paper.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $\partial\Omega$ of class C^m. Then there exists a constant $C = C(\Omega, m) > 0$ such that*

$$\|u\|^2 - S|u|^2_{2^*} \geq C|u|_w \qquad \textit{for all } u \in H^m_\theta(\Omega).$$

The exponent of the weak norm is sharp. Indeed, using functions of the form ψU_λ as test functions with $0 \in \Omega$, large λ and a cut off function ψ, it is easily seen that an estimate of this type cannot hold for $q > 2^*/2$. For expansions of different norms of ψU_λ as $\lambda \to \infty$, see [6,9,10]. On the other hand, Theorem 1.1 implies that for all $q \in [1, 2^*/2)$ there exists a constant $C_q = C_q(n, \Omega) > 0$ such that

$$\|u\|^2 \geq S|u|^2_{2^*} + C_q|u|^2_q \qquad \text{for all } u \in H^m_\theta(\Omega).$$

Our proof of Theorem 1.1 is based on the following tools. First, we use Talenti's comparison principle [13] to reduce the problem to radial positive functions in a ball. Second, we apply the extension map constructed in the recent paper [8] in order to pass to radial functions in $\mathcal{D}^{m,2}(\mathbb{R}^N)$. Finally, we use a remainder term estimate proved in [2]. In Section 2 below we collect and discuss these tools, and in Section 3 we complete the proof of Theorem 1.1.

2. Preliminaries. In the following, for the sake of clarity we will sometimes specify the domain of integration in the norms we use, that is, we write $|\cdot|_{p,\Omega}$, $\|\cdot\|_\Omega$ and $|\cdot|_{w,\Omega}$. We denote by B the unit ball in $\mathbb{R}^N$, by $e_N = |B|$ its measure and by $f^* \in L^2(B)$ the spherical rearrangement of $f \in L^2(\Omega)$ when $|\Omega| = |B|$. Here we use the definition of f^* given in [13, p. 701], so the superlevel sets $\{x \in B : f^*(x) > t\}$ are concentric balls centered at zero with the same measure as $\{x \in \Omega : |f(x)| > t\}$. With this definition, $f^* = |f|^*$ is always a nonnegative and radially decreasing function - even if f is sign changing.

The first crucial tool for the proof of Theorem 1.1 is the following comparison principle due to Talenti [13, Theorem 1].

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a C^m-smooth bounded domain such that $|\Omega| = |B| = e_N$. Let $m = 2k$ be an even number. Let $g \in L^2(\Omega)$ and let $u \in H^m_\theta(\Omega)$ be the unique strong solution to*

$$\begin{cases} (-\Delta)^k u = g & \textit{in } \Omega, \\ \Delta^j u = 0 & \textit{on } \partial\Omega, \qquad j = 0, \ldots, k-1. \end{cases}$$

Let $g^* \in L^2(B)$ and $u^* \in H^1_0(B)$ denote respectively the spherical rearrangements of g and u, and let $v \in H^m_\theta(B)$ be the unique strong solution to

$$\begin{cases} (-\Delta)^k v = g^* & \textit{in } B, \\ \Delta^j v = 0 & \textit{on } \partial B, \qquad j = 0, \ldots, k-1. \end{cases} \tag{2.1}$$

Then, $v \geq u^*$ a.e. in B.

As we shall see, Proposition 2.1 enables us to reduce the proof of Theorem 1.1 to the case where $\Omega = B$ and to the subspace of H^m_θ of radially symmetric and decreasing functions, which we denote by $R^m_\theta(B)$.

The second tool needed in the proof of Theorem 1.1 is an extension argument taken from [8] which we now explain in some detail. Consider first the case where m is even, namely $m = 2k$ for some $k \geq 1$. For any $g : [0, \infty) \to \mathbb{R}$ with appropriate integrability conditions, we define

$$(\mathcal{G}g)(r) := \int_r^\infty \int_0^\rho \left(\frac{s}{\rho}\right)^{N-1} g(s) ds d\rho.$$

If g goes to 0 fast enough for $r \to \infty$ (e.g. like $r^{-\gamma}$ with $\gamma > 2$), then an integration by parts gives

$$(\mathcal{G}g)(r) = \frac{1}{N-2} r^{2-N} \int_0^r s^{N-1} g(s) ds + \frac{1}{N-2} \int_r^\infty s g(s) ds, \tag{2.2}$$

and

$$-\Delta (\mathcal{G}g)(|x|) = g(|x|) \text{ for } x \in \mathbb{R}^N.$$

Moreover, we denote by $\mathcal{G}^k$ the kth iteration of the operator $\mathcal{G}$. With these notations we recall a result by Gazzola, Grunau, and Sweers [8]:

Proposition 2.2. *Let $m = 2k$ and let $u \in R_\theta^m(B) \setminus \{0\}$. Let $w(r) = \left(\mathcal{G}^k f\right)(r)$ for*

$$f(r) = \begin{cases} (-\Delta)^k u(r) & \text{if } r \leq 1, \\ 0 & \text{if } r > 1, \end{cases}$$

then $w \in \mathcal{D}^{m,2}(\mathbb{R}^N)$, $\|w\|_{\mathbb{R}^N} = \|u\|_B$, and $|w|_{2^*,\mathbb{R}^N} > |u|_{2^*,B}$.

In particular, if $m = 2$ the extension of a radial function $u = u(r)$ in $R_\theta^2(B)$ is given by

$$w(r) = \begin{cases} u(r) + \dfrac{1}{N-2}|u'(1)| & \text{if } r \in (0,1), \\ \dfrac{r^{N-2}}{N-2}|u'(1)| & \text{if } r \in [1, \infty). \end{cases}$$

Proposition 2.2 also enables us to treat the case of odd m, namely $m = 2k+1$ for some $k \geq 1$. Since $H_\theta^{2k+1}(B) \subset H_\theta^{2k}(B)$, by Proposition 2.2 we know that any $u \in R_\theta^{2k+1}(B) \setminus \{0\}$ allows to define an entire function w such that

$$w > u \text{ in } B, \quad \Delta^k(w - u) = 0 \text{ in } B, \quad \Delta^k w = 0 \text{ in } \mathbb{R}^N \setminus B.$$

In particular, this implies that also

$$\nabla(\Delta^k(w - u)) = 0 \text{ in } B, \quad \nabla(\Delta^k w) = 0 \text{ in } \mathbb{R}^N \setminus B. \tag{2.3}$$

The construction for the $2k$-case also enables us to conclude that $w \in C^{2k-1}(\mathbb{R}^N)$, a regularity which is not enough to obtain $w \in \mathcal{D}^{2k+1,2}(\mathbb{R}^n)$, here we need one more degree of regularity. This is obtained by recalling the extra boundary condition that appears by going from $H_\theta^{2k}(B)$ to $H_\theta^{2k+1}(B)$, namely $\Delta^k u = 0$ on ∂B, and that $\Delta^k w = 0$ in $\mathbb{R}^N \setminus B$.

Next, we recall a result by Bartsch, Weth, and Willem [2]:

Proposition 2.3. *There exists a constant $\alpha > 0$ such that*

$$\|u\|^2 - S|u|_{2^*}^2 \geq \alpha \mathrm{dist}(u, \mathcal{M})^2 \qquad \text{for all } u \in \mathcal{D}^{m,2}(\mathbb{R}^N).$$

Here $\mathrm{dist}(u, \mathcal{M}) = \inf\{\|u - v\| : v \in \mathcal{M}\}$ *is the distance of u from $\mathcal{M}$ in $\mathcal{D}^{m,2}(\mathbb{R}^N)$.*

For $m = 1$ this result is due to Bianchi and Egnell [3], solving a problem posed by Brezis and Lieb [4].

We finally note that if $u \in \mathcal{D}^{m,2}(\mathbb{R}^N)$ is a function with $\mathrm{dist}(u, \mathcal{M}) < \|u\|$, then there exists $v \in \mathcal{M}$ with $\mathrm{dist}(u, \mathcal{M}) = \|u-v\|$ since $\mathcal{M}$ is relatively closed in $\mathcal{D}^{m,2}(\mathbb{R}^N) \setminus \{0\}$. If, in addition, u is a radial positive function, then the distance minimizing $v \in M$ can be chosen as a positive and radial function, i.e. $v = cU_\lambda$ with $c, \lambda > 0$. To see this, we note that every positive function $v \in M$ is a translation of a radially decreasing function. Therefore $v \in M$ implies $v^* \in M$, whereas by (1.4) and [1, Theorem 2.2] we have

$$\int\limits_{\mathbb{R}^N} (-\Delta^m v)u = \tau_m \int\limits_{\mathbb{R}^N} v^{2^*-1} u \leq \tau_m \int\limits_{\mathbb{R}^N} (v^*)^{2^*-1} u = \int\limits_{\mathbb{R}^N} (-\Delta^m v^*)u$$

and therefore

$$\begin{aligned}\|u - v\|^2 &= \|u\|^2 + S^2|v|_{2^*}^2 - 2\int\limits_{\mathbb{R}^N} (-\Delta^m v)u \\ &\geq \|u\|^2 + S^2|v^*|_{2^*}^2 - 2\int\limits_{\mathbb{R}^N} (-\Delta^m v^*)u = \|u - v^*\|^2.\end{aligned}$$

3. Proof of Theorem 1.1. With no loss of generality we may assume that $|\Omega| = |B| = e_N$.

Assume first that m is even, $m = 2k$ for some $k \geq 1$. Take any function $u \in H_\theta^m(\Omega)$, put $g := (-\Delta)^k u$, and let $v \in H_\theta^m(B)$ the unique solution to (2.1). Then by the properties of symmetrization, see [1], we obtain both that

$$\|v\|_B^2 = |\Delta^k v|_{2,B}^2 = |g^*|_{2,B}^2 = |g|_{2,\Omega}^2 = |\Delta^k u|_{2,\Omega}^2 = \|u\|_\Omega^2 \tag{3.1}$$

and

$$\|u\|_{2^*,\Omega}^2 = |u^*|_{2^*,B}^2 \leq |v|_{2^*,B}^2 \tag{3.2}$$

where, for the last inequality, we used Proposition 2.1. Moreover, for any $A \subset \Omega$ such that $|A| > 0$ we have

$$|A|^{-\frac{2m}{N}} \int\limits_A |u| = |A|^{-\frac{2m}{N}} \int\limits_\Omega \chi_A |u|$$

where χ_A denotes the characteristic function of A. Since by [1, Theorem 2.2] we know that

$$\int\limits_\Omega \chi_A |u| \leq \int\limits_B \chi_A^* u^*,$$

for any such A we have

$$|A|^{-\frac{2m}{N}} \int\limits_A |u| \le |A^*|^{-\frac{2m}{N}} \int\limits_{A^*} u^*$$

and therefore, by taking the supremum over all such A, we deduce that $|u|_{w,\Omega} \le |u^*|_{w,B}$. In turn, by Proposition 2.1, we infer that

$$|u|_{w,\Omega} \le |v|_{w,B}. \tag{3.3}$$

Putting together (3.1), (3.2), and (3.3) shows that if we can prove Theorem 1.1 in the symmetric framework where $\Omega = B$ and $u \in R^m_\theta(B)$, then we are done.

A similar conclusion is reached if m is odd, $m = 2k+1$ for some $k \ge 0$. In this case, invoking again [1], (3.1) becomes an inequality:

$$\|v\|^2_B = |\nabla\Delta^k v|^2_{2,B} = |\nabla g^*|^2_{2,B} \le |\nabla g|^2_{2,\Omega} = |\nabla\Delta^k u|^2_{2,\Omega} = \|u\|^2_\Omega,$$

which also allows to consider just the case where $\Omega = B$ and $u \in R^m_\theta(B)$.

We now proceed by contradiction. If the assertion of Theorem 1.1 is false, then there exists a sequence of functions $u_n \in R^m_\theta(B)$ ($n \in \mathbb{N}$) such that $\|u_n\|_B = 1$ for all n and

$$\frac{1 - S|u_n|^2_{2^*,B}}{|u_n|_{w,B}} \to 0 \qquad \text{as } n \to \infty. \tag{3.4}$$

We denote by w_n the extension of u_n as given by Proposition 2.2 (if m is odd, also the remarks following Proposition 2.2 are needed). Then we know that

$$\|w_n\|_{\mathbb{R}^N} = \|u_n\|_B = 1\ , \qquad |w_n|_{2^*,\mathbb{R}^N} > |u_n|_{2^*,B}.$$

Moreover, recalling that $w_n > u_n$ in B, we also have

$$|w_n|_{w,\mathbb{R}^N} > |u_n|_{w,B}.$$

Consequently,

$$0 \le 1 - S|w_n|^2_{2^*,\mathbb{R}^N} \le 1 - S|u_n|^2_{2^*,B} \to 0$$

and therefore, by Proposition 2.3,

$$\operatorname{dist}(w_n, \mathcal{M}) \to 0 \quad \text{as } n \to \infty.$$

Since $\|w_n\|_{\mathbb{R}^N} = 1$ for all $n \in \mathbb{N}$, it follows by the remarks below Proposition 2.3 that there exists $c_n, \lambda_n > 0$ with $\|w_n - c_n U_{\lambda_n}\|_{\mathbb{R}^N} = \operatorname{dist}(w_n, \mathcal{M})$, and that

$$0 < \inf_{n\in\mathbb{N}} c_n \le \sup_{n\in\mathbb{N}} c_n < \infty.$$

In case that $m = 2k$ is even, we have

$$\begin{aligned}\operatorname{dist}(w_n, \mathcal{M})^2 &= \|w_n - c_n U_{\lambda_n}\|^2_{\mathbb{R}^N} = |\Delta^k(w_n - c_n U_{\lambda_n})|^2_{2,\mathbb{R}^N} \\ &\ge |\Delta^k(w_n - c_n U_{\lambda_n})|^2_{2,\mathbb{R}^N\setminus B} = c_n|\Delta^k U_{\lambda_n}|^2_{2,\mathbb{R}^N\setminus B} \\ &\ge S\, c_n |U_{\lambda_n}|^2_{2^*,\mathbb{R}^N\setminus B}\end{aligned}$$

since $\Delta^k w_n = 0$ a.e. in $\mathbb{R}^N \setminus B$ for $n \in \mathbb{N}$. In case $m = 2k+1$ is odd, we get the same conclusion using (2.3). In both cases necessarily $\lambda_n \to \infty$ and therefore $\lambda_n \geq 1$ for all n after passing to a subsequence. This yields that

$$\frac{|U_{\lambda_n}|^{2^*}_{2^*,\mathbb{R}^N\setminus B}}{Ne_N} = \lambda_n^{2^*} \int_1^\infty \frac{r^{N-1}}{\left[1+\left(\lambda_n^{\frac{2}{N-2m}} r\right)^2\right]^N}\,dr = \int_{\lambda_n^{\frac{2}{N-2m}}}^\infty \frac{r^{N-1}}{(1+r^2)^N}\,dr$$

$$\geq 2^{-N} \int_{\lambda_n^{\frac{2}{N-2m}}}^\infty \frac{dr}{r^{N+1}} = \frac{1}{N\,2^N\,\lambda_n^{2^*}}.$$

We conclude that

$$\operatorname{dist}(w_n, \mathcal{M}) \geq \frac{C_1}{\lambda_n} \qquad \text{with } C_1 > 0 \text{ independent of } n \in \mathbb{N}.$$

On the other hand, a scaling argument shows that $|c_n U_{\lambda_n}|_{w,B} \leq \frac{c_n}{\lambda_n}|U|_{w,\mathbb{R}^N}$; we point out that scaling gives this nice estimate precisely because we deal with the weak $L^{2^*/2}$-norm. Therefore, we have

$$\begin{aligned}|u_n|_{w,B} &\leq |w_n|_{w,B} \leq |c_n U_{\lambda_n}|_{w,B} + |w_n - c_n U_{\lambda_n}|_{w,B}\\ &\leq \frac{c_n}{\lambda_n}|U|_{w,\mathbb{R}^N} + C_2\|w_n - c_n U_{\lambda_n}\|_{\mathbb{R}^N} \leq C_3 \operatorname{dist}(w_n, \mathcal{M}).\end{aligned}$$

with constants $C_2, C_3 > 0$ independent of n. Hence, (3.4) implies that

$$\frac{\|w_n\|^2_{\mathbb{R}^N} - S|w_n|^2_{2^*,\mathbb{R}^N}}{\operatorname{dist}(w_n, \mathcal{M})^2} \to 0 \qquad \text{as } n \to \infty.$$

contrary to Proposition 2.3. This contradiction shows the claim.

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