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REMARK ON INTERPOLATION BY L-ALMOST PERIODIC FUNCTIONS

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In [1] the notion of I_0-set is discussed. We say that a set Λ in an LCA-group G is I_0 if any complex-valued bounded function on Λ can be extended to an almost periodic function on G. The aim of this remark is to look at these sets Λ for which every function on Λ can be extended to an L-almost periodic (L-ap) function in the sense of Levitan [4]. There are several equivalent definitions of L-ap functions. We need two of them. Before quoting them let us first recall that, for a compact $K \subset G$, a (K, ε)-*almost period* of f is a $\tau \in G$ such that

$$(*) \qquad \forall_{t \in K} |f(t+\tau)-f(t)| < \varepsilon.$$

Secondly, we fix the meaning of a relative density as follows: A set $E \subset G$ is called *relatively dense* (in G) if there is a compact C such that $G = C+E$.

The first definition we adopt is that given by Reich [6] who proved its equivalence (for $G = \boldsymbol{R}$) with the original concept of Levitan, which was both formally stronger and more complicated.

Definition 1. A complex-valued continuous function f on an LCA-group G is called an *L-ap function* if, for any $\varepsilon > 0$ and any compact $K \subset G$, there is a relatively dense set $E \ni 0$ such that every $\tau \in E-E$ is a (K, ε)-almost period of f.

For the sake of exactness it should be added that in Reich's definition the compact sets K and C are supposed finite whereas we do not admit this restriction. We shall return to this matter in a moment but previously we write down the second definition.

Definition 2. A complex-valued function on G is called *L-ap* if it is continuous in the weak topology of the group G, i.e. in that induced by its Bohr compactification.

This characterization of *L-ap* functions is stated for $G = \boldsymbol{R}$ implicitely already in [3] and explicitely in [5]. For arbitrary LCA-groups it is proved in [6], however, under assumption that the compact set C satisfying $E+C=G$ is finite (*strong* relative density of E). This is no real objection. In fact, given ε and K, let U be a symmetric compact neighborhood of zero element such that

$$|f(t+h)-f(t)|<\varepsilon/2 \quad \text{for } t\in K \text{ and } h\in U.$$

So, if τ is a $(K+U, \varepsilon/2)$-almost period, then $\tau+h$ is a (K, ε)-almost period for every $h\in U$. Hence, if in Definition 1 we let E correspond to $K+U$ and $\varepsilon/2$, and choose a symmetric neighborhood W of zero such that $W+W\subset U$, then $(E+W)-(E+W)$ consists of (K, ε)-almost periods; on the other hand, it is evident that $E+W$ is strongly relatively dense. So we are free to assume the strong relative density in Definition 1 as well. It may be added that the weak continuity implies Definition 1 in a simple manner and even in this strengthened sense, first for K finite and then for K compact, in view of the fact that a weakly continuous function has a uniform modulus of weak continuity on compact sets. The converse implication is far from being obvious.

We now assume that the group G is metric separable. Then we have

LEMMA 1. *The set LAP of L-ap functions in the space $C(G)$ of continuous functions on G endowed with compact-open topology is Borel.*

Proof. We use (*) to express Definition 1 by means of quantifiers:

$$(**) \qquad \forall_{\varepsilon>0}\forall_{K}\exists_{C}\forall_{x}\exists_{\tau_x}(\tau_x\in x-C)$$
$$\forall_{x_1,x_2}\Big(\forall_{t\in K}|f(t+\tau_{x_1}-\tau_{x_2})-f(t)|<\varepsilon,\ |f(t\pm\tau_{x_1})-f(t)|<\varepsilon\Big).$$

All quantifiers in (**) can be made countable. For the first one this is obvious. In view of σ-compactness of G,

$$G=\bigcup_{n=1}^{\infty}K_n \quad (K_n\subset K_{n+1}),$$

and we can choose K and C among K_n's. Since G is separable, we can restrict x, τ_x and t to a fixed dense countable set. Clearly, the set under the last quantifier is open in $C(G)$. Thus (**) defines a Borel set. This achieves the proof.

LEMMA 2. *The set $C_b(G)$ of bounded functions in $C(G)$ is F_σ.*

The proof is obvious.

THEOREM 1. *If every function on Λ can be extended to an L-ap function, then every bounded function on Λ can be extended to a bounded L-ap function. If every bounded function on Λ is extendable to an L-ap function, then every function on Λ is extendable to an L-ap function. The necessary and sufficient condition for any of these properties is that Λ has no accumulation points in the weak topology of G.*

Proof. The sufficiency follows immediately from Definition 2. Before proving the necessity observe that if every bounded function on Λ can be extended to an L-ap function, then Λ has no (strong) accumulation point, and so, since G is separable, Λ is countable, $\Lambda = \{t_n\}$. If Λ had a weak accumulation point in G, the function defined by $\underset{n}{\forall}\varphi(t_n) = n$ would not be extendable to a weakly continuous function.

Not so obvious is the proof that if any *bounded* function is extendable to an L-ap function, then Λ still cannot have weak accumulation points. It is clear that such a point could not belong to Λ. Thus we must prove that Λ is weakly closed.

Lemmas 1 and 2 enable us to follow closely the method used by Ryll-Nardzewski [7] for proving that $\boldsymbol{I}_0$-sets are weakly closed. Suppose that Λ is not weakly closed and let $t_0 \notin \Lambda$ be a weak accumulation point of Λ. Then, for every function φ which is bounded on Λ, the value $\bar{\varphi}(t_0)$ is the same for all L-ap extensions $\bar{\varphi}$ of φ and is thus completely determined by φ. Denote this value by $\omega(\varphi)$. Obviously, ω is a homomorphism of the ring $b(\Lambda)$ of bounded sequences with coordinate-wise operations and coordinate-wise convergence into $\boldsymbol{C}$, and so, according to a theorem of Sierpiński [8], it is not Borel measurable unless it is trivial, i.e. of the form $\varphi \to \varphi(\lambda_0)$, where λ_0 is a fixed point in Λ. Since this is impossible, ω cannot be Borel measurable. However, the graph $\{(\varphi, y): y = \omega(\varphi)\}$ of ω in the product space $b(\Lambda) \times \boldsymbol{C}$ is a projection of the set

$$\{(\varphi, y, \bar{\varphi}): \bar{\varphi} \in LAP \cap C_b(G); \bar{\varphi}/\Lambda = \varphi, \bar{\varphi}(t_0) = y\}$$

which, in view of Lemmas 1 and 2, is Borel in the (separable complete metric) space $C(G) \times b(\Lambda) \times \boldsymbol{C}$. Thus the graph of ω is an analytic (Suslin) set and this is equivalent to Borel measurability of ω ([2], p. 398). We arrived at a contradiction.

From now we will refer to sets admitting an L-ap interpolation in the sense of Theorem 1 as to $\boldsymbol{LI}_0$-sets. An immediate consequence of Theorem 1 is

COROLLARY. *The union of two $\boldsymbol{LI}_0$-sets is an $\boldsymbol{LI}_0$-set.*

It can be easily shown that this is not true for $\boldsymbol{I}_0$-sets.

THEOREM 2. *If K is a compact set in G and Λ an $\boldsymbol{LI}_0$-set, then $\Lambda + K$ is an $\boldsymbol{LI}$-set, i.e. every function which is continuous on $\Lambda + K$ has an L-ap extension.*

Proof. We must prove that $\Lambda + K$ is weakly closed and that every (strongly) continuous function on it is weakly continuous. The first statement is obvious since Λ is weakly closed and K a compact set. To get the second it is enough to see that every weak accumulation point of $\Lambda + K$ is a strong one (i.e. a limit point) of $\Lambda + K$. This can be shown as follows: Let t_0 be an arbitrary point of G. Since Λ consists of isolated points, the set $\Lambda_0 = \Lambda \cap (t_0 - K)$ is finite. Since Λ has no weak accumulation points, there is a weak neighborhood W of zero element such that $t_0 - K + W$ is disjoint with $\Lambda \setminus \Lambda_0$. It means that $t_0 + W$ is disjoint with $(\Lambda \setminus \Lambda_0) + K$, and so t_0 cannot be a weak accumulation point of $\Lambda + K$ unless it is a weak accumulation point of the compact set $\Lambda_0 + K$, but then it must be one of its limit points, since the weak and the strong topology coincide on compact sets. The more, t_0 would be a limit point of $\Lambda + K$.

$\boldsymbol{LI}_0$ cannot be too "big", e.g. they cannot be relatively dense, i.e. no compact set K exists such that $K + \Lambda = G$. In view of Theorem 1, this is equivalent to the theorem saying that sets without weak accumulation points in a separable LCA-group are never relatively dense ([1], Part II).

On the other hand, an $\boldsymbol{LI}_0$-set needs not be very rare, it has not even to be regular, i.e. the distance between each two of its points can approach 0. This follows at once from Theorem 2. But even $\boldsymbol{LI}_0$-sets in $\boldsymbol{Z}$ can be of fairly small lacunarity, for example, the set E of prime numbers of the form $5k+2$ or $5k+3$ has no weak accumulation points in $\boldsymbol{Z}$, since every $n \in \boldsymbol{Z}$ can be separated from $E \setminus \{n\}$ by a weak neighborhood determined by a single exponential $e^{2\pi i/s}$ with a suitable integer s. Thus an $\boldsymbol{LI}_0$-set needs by no means be a Sidon set as must be every $\boldsymbol{I}_0$-set.

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