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REMARK ON SIEGEL DOMAINS OF TYPE III[1]

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ABSTRACT. Bounded symmetric domains have standard realizations as "Siegel domains of type III." Pjateckiĭ-Šapiro has introduced a more restrictive notion of "Siegel domain of type III." Here we give a direct proof that those standard realizations satisfy the additional conditions of the new definition.

1. **Introduction.** In his book on the geometry and function theory of the classical domains [1], I. I. Pjateckiĭ-Šapiro introduced the very useful concepts of Siegel domains of types I, II and III for application to the theory of automorphic functions. Given an irreducible classical bounded symmetric domain D and an equivalence class $\{B\}$ of boundary components of D, he worked out an *ad hoc* realization of D as a Siegel domain of type III with base B. Later, A. Korányi and I [4] worked out the type III Siegel domain realizations of all bounded symmetric domains in an intrinsic and classification free manner. At about the same time, Pjateckiĭ-Šapiro [2] revised his concept of type III Siegel domain for convenience of application. In this note I extract a few small pieces of [4] to show directly that the type III Siegel domains (old sense), that Korányi and I constructed in [4], all satisfy the revised conditions of Pjateckiĭ-Šapiro for Siegel domains of type III (new sense).

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I wish to thank Professor I. I. Pjateckiĭ-Šapiro for his comment on the manuscript. It turns out that the result had been obtained by Pjateckiĭ-Šapiro in somewhat more generality [3], using the detailed theory of bounded homogeneous domains. The present proof applies only to bounded symmetric domains, but it is somewhat more direct and elementary for that important case.

2. **Definitions.** Let U_R be a real vector space and $\Omega \subset U_R$ a nonempty convex open cone that does not contain a straight line. This data defines a *Siegel domain of type I* (=*tube domain*) in the complex vector space $U = U_R + iU_R = U_R \otimes C$, which is

$$(\mathrm{S_I}) \qquad \{u \in U : \mathrm{Im}\, u \in \Omega\}.$$

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Let V be a second complex vector space and $F: V\times V\to U$ a map that is hermitian relative to complex conjugation of U over U_R. Suppose that F is positive definite in the sense that $0\neq v\in V$ implies $0\neq F(v, v)\in\bar{\Omega}$ (=closure of Ω). This data defines a *Siegel domain of type II* in $U\oplus V$, which is

$$(\mathrm{S_{II}}) \qquad \{(u, v) \in U \oplus V: \mathrm{Im}\, u - F(v, v) \in \Omega\}.$$

Siegel domains of type I are the special case $V=0$.

There is no variation in these original definitions of Pjateckiĭ-Šapiro for Siegel domains of types I and II.

Let W be a third complex vector space, $B\subset W$ a bounded domain containing the origin 0, and $w\mapsto F_w$ a smooth map of B into $\mathrm{Hom}_R(V\otimes_R V, U)$ such that

(1) $F_w=F_w^h+F_w^b$ where F_w^h is hermitian and F_w^b is C-bilinear,

(2) $F_0=F$, so $F_0=F_0^h$ and $F_0^b=0$, and

(3) if $v\in V$ and either $F_w(v, V)=0$ or $F_w(V, v)=0$ then $v=0$.

This data defines a *Siegel domain of type III* (old sense) in $U\oplus V\oplus W$, which is

$$(\mathrm{S_{III}\ old}) \qquad \{(u, v, w) \in U \oplus V \oplus W: w \in B \text{ and } \mathrm{Im}\, u - \mathrm{Re}\, F_w(v, v) \in \Omega\}.$$

B is its *base*. Siegel domains of type II are the case $W=0$, i.e. $B=(0)$.

Start again with the data (U_R, Ω, V, F) for a Siegel domain of type II. That defines a complex vector space

$$(4) \qquad W_{\mathrm{univ}} = \{p: V \to V \text{ conjugate-linear}: F(pv, v') = F(pv', v) \text{ for all } v, v' \in V\}.$$

That vector space contains a bounded domain

$$(5) \qquad B_{\mathrm{univ}} = \{p \in W_{\mathrm{univ}}: \text{if } 0 \neq v \in V \text{ then } 0 \neq F(v, v) - F(pv, pv) \in \bar{\Omega}\}.$$

If $p\in B_{\mathrm{univ}}$ then $I+p$ is invertible, for $(I+p)v=0$ implies $F(v, v) = F(pv, pv)$. Thus we have maps

$$(6) \qquad L_p: V \times V \to U \quad \text{defined by} \quad L_p(v, v') = F(v, (I+p)^{-1}v') \quad \text{for } p \in B_{\mathrm{univ}}.$$

Now the ***universal Siegel domain of type III*** associated to (U_R, Ω, V, F) is the domain in $U\oplus V\oplus W_{\text{univ}}$ which is

$$(\mathrm{S_{III}\ univ})\quad \Big\{(u, v, p) \in U \oplus V \oplus W_{\text{univ}}: \\ p \in B_{\text{univ}} \text{ and } \operatorname{Im} u - \operatorname{Re} L_p(v, v) \in \Omega\Big\}.$$

LEMMA 1. *If $p\in B_{\text{univ}}$ then $I-p^2$ is invertible and the map L_p satisfies* (1), (2) *and* (3) *with L_p^h and L_p^b given by*

$$(7)\quad \begin{aligned} L_p^h(v, v') &= F(v, (I - p^2)^{-1}v') \quad \textit{and} \\ L_p^b(v, v') &= -F(v, (I - p^2)^{-1}pv'). \end{aligned}$$

In particular, universal Siegel domains of type III are Siegel domains of type III in the old sense.

PROOF. If $(I-p^2)v=0$ then $F(v, v)=F(p^2v, v)=F(pv, pv)$ by (4), so $v=0$ by (5); that proves $I-p^2$ invertible. Now $(I-p^2)^{-1}(I-p)=(I+p)^{-1}$ by power series expansion in a neighborhood of 0 and then analytic continuation to B_{univ}. Given L_p^h and L_p^b as in (7) it follows that $L_p=L_p^h+L_p^b$. Now (1) and (2) are immediate, and (3) follows. □

Again let (U_R, Ω, V, F) be the data for a Siegel domain of type II. Let W be a third complex vector space, $B\subset W$ a bounded domain and $\phi:B\to B_{\text{univ}}$ a holomorphic map with $0\in\phi(B)$. This data defines a *Siegel domain of type III* (new sense) in $U\oplus V\oplus W$ with *base* B, which is

$$(\mathrm{S_{III}\ new})\quad \Big\{(u, v, w) \in U \oplus V \oplus W: \\ w \in B \text{ and } \operatorname{Im} u - \operatorname{Re} L_{\phi(w)}(v, v) \in \Omega\Big\}.$$

Lemma 1 says

LEMMA 2. *A Siegel domain of type III in the new sense is a Siegel domain of type III in the old sense for which $F_w(v, v')=F(v, (I+\phi(w))^{-1}v')$.*

3. **Siegel domain realizations.** We use the notation and conventions of [4] for our verification, even though they may be too complicated for other purposes. Now D is a bounded symmetric domain embedded in its antihomomorphic tangent space $\mathfrak{p}^-$ by the method of Harish-Chandra. Δ is the corresponding maximal set of strongly orthogonal noncompact positive roots, Γ is a subset of Δ, D_Γ is the subdomain of D whose maximal set of strongly orthogonal noncompact roots is Γ, and $c_{\Delta-\Gamma}$ is the partial Cayley transform involving the elements of $\Delta-\Gamma$. The space $\mathfrak{p}^-=U\oplus V\oplus W$ where

$U = \mathfrak{p}^-_{\Delta-\Gamma,1}$ is the $(+1)$-eigenspace of $\mathrm{ad}(c_{\Delta-\Gamma})^4$ on the subspace of $\mathfrak{p}^-$ for $\Delta-\Gamma$,

$V = \mathfrak{p}_2^{\Gamma-}$ is the (-1)-eigenspace of $\mathrm{ad}(c_{\Delta-\Gamma})^4$ on $\mathfrak{p}^-$, and

$W = \mathfrak{p}^-_\Gamma$ is the subspace of $\mathfrak{p}^-$ for Γ, and is the ambient space of the domain D_Γ.

We also have

$U_R = \mathfrak{n}_1^{\Gamma-}$ real form of $U = \mathfrak{p}^-_{\Delta-\Gamma,1}$ defined in [4, 6.1.3], and

$\Omega = \mathfrak{c}^\Gamma$ self dual cone in $U_R = \mathfrak{n}_1^{\Gamma-}$ defined in [4, §7.1].

In [4] following Lemma 7.2 one finds the following definitions, which we rewrite according to the "dictionary" just above. If $w \in D_\Gamma$ then $\mu(w): V \to V$ is the conjugate-linear map given by $\mu(w)v = \mathrm{ad}(w)\mathrm{ad}(c_{\Delta-\Gamma})^2 \cdot \nu(v)$ where ν is a certain complex conjugation, and $\Lambda_w: V \times V \to U$ is map given by

$$\begin{aligned}\Lambda_w(v, v') &= -\frac{i}{2}\,[v, \mathrm{ad}(c_{\Delta-\Gamma})^2\nu(I + \mu(w))^{-1}v'] \\ &= -\frac{i}{2}\,\mathrm{ad}(c_{\Delta-\Gamma})^2\nu[\mathrm{ad}(c_{\Delta-\Gamma})^2\nu v, (I + \mu(w))^{-1}v'].\end{aligned}$$

In [4, §7] it is shown that

$$c_{\Delta-\Gamma}(D) = \{(u, v, w) \in U \oplus V \oplus W: \qquad w \in D_\Gamma \text{ and } \mathrm{Im}\, u - \mathrm{Re}\, \Lambda_w(v, v) \in \Omega\} \tag{8}$$

and that the Λ_w, $w \in D_\Gamma$, satisfy (1), (2) and (3), so that

(9) $c_{\Delta-\Gamma}(D)$ is a Siegel domain of type III (old sense) in $\mathfrak{p}^-$.

THEOREM. *μ is a holomorphic map from D_Γ to the domain B_{univ} for the type II Siegel domain data $(U_R, \Omega, V, \Lambda_0)$. Thus $c_{\Delta-\Gamma}(D)$ is a Siegel domain of type III (new sense) in $\mathfrak{p}^-$.*

PROOF. Let W_{univ} and B_{univ} be defined as in (4) and (5) for the type II Siegel domain data $(U_R, \Omega, V, \Lambda_0)$. If $w \in D_\Gamma$ then $\mu(w)$ acts on V by: the complex conjugation ν, then $\mathrm{ad}(c_{\Delta-\Gamma})^2$, then $\mathrm{ad}(w)$. Thus μ is a holomorphic map of D_Γ into the space of conjugate-linear transformations of V. If $v, v' \in V$ and $w \in D_\Gamma$ then [4, Lemma 7.3 (iii)] says that $\Lambda_0(v, \mu(w)v') = \Lambda_0(v', \mu(w)v)$; after complex conjugation of U over U_R this says that $\mu(w) \in W_{\mathrm{univ}}$. Now μ is a holomorphic map of D into W_{univ}.

Let $w \in D_\Gamma$ and $0 \neq v \in V$. Define $v' = (I - \mu(w)^2)v$ so $0 \neq v' \in V$. We compute

$$
\begin{aligned}
\Lambda_0(v, v) - \Lambda_0(\mu(w)v, \mu(w)v) \\
&= \Lambda_0(v, v) - \Lambda_0(\mu(w)^2v, v) \quad \text{because } \mu(w) \in W_{\text{univ}}, \\
&= \Lambda_0((I - \mu(w)^2)v, v) \\
&= \Lambda_0(v', (I - \mu(w)^2)^{-1}v') \quad \text{by definition of } v', \\
&= \Lambda_0\left(v', \sum_{n=0}^{\infty} \mu(w)^{2n}v'\right) \quad \text{by power series expansion}, \\
&= \sum_{n=0}^{\infty} \Lambda_0(\mu(w)^n v', \mu(w)^n v') \quad \text{because } \mu(w) \in W_{\text{univ}}.
\end{aligned}
$$

If $\mu(w)^n v' \neq 0$ then $0 \neq \Lambda_0(\mu(w)^n v', \mu(w)^n v') \in \bar{\Omega}$. As $\bar{\Omega}$ is convex, and is strictly convex at 0, now $\Lambda_0(v, v) - \Lambda_0(\mu(w)v, \mu(w)v) \in \bar{\Omega}$ and $\Lambda_0(v, v) - \Lambda_0(\mu(w)v, \mu(w)v) = 0$ if and only if $\mu(w)^n v' = 0$ for all $n \geqq 0$. But $v' \neq 0$, i.e. $\mu(w)^0 v' \neq 0$, so now

$$0 \neq \Lambda_0(v, v) - \Lambda_0(\mu(w)v, \mu(w)v) \in \bar{\Omega}.$$

We have proved $\mu(w) \in B_{\text{univ}}$, completing the proof of the theorem. □

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