Remarks about the unification type of several non-symmetric non-transitive modal logics — Source link

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Remarks about the unification type of several non-symmetric non-transitive modal logics

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Abstract

The problem of unification in a normal modal logic L can be defined as follows: given a formula φ, determine whether there exists a substitution σ such that $\sigma(\varphi)$ is in L. In this paper, we prove that for several non-symmetric non-transitive modal logics, there exists unifiable formulas that possess no minimal complete set of unifiers.

Keywords: Normal modal logics, non-symmetric modal logics, non-transitive modal logics, unification problem, types of modal logics,

1 Introduction

The problem of unification in a normal modal logic L can be defined as follows: given a formula $\varphi(x_1, \ldots, x_m)$ where $x_1, \ldots, x_m$ are variables, determine whether there exist formulas $\psi_1, \ldots, \psi_m$ such that $\varphi(\psi_1, \ldots, \psi_m)$ is in L. The computability of the problem of unification in transitive normal modal logics like *S4* and *GL* has been solved by Rybakov [22–24] who proved that it is decidable. With respect to its computational complexity, the problem of unification was mostly unexplored before the work of Jer˘ábek [20] who established its membership in *NEXPTIME* in several normal modal logics extending *K4* such as *S4* and *GL*. See also [15, 17, 18, 25, 29] for a study of the problem of unification in different normal modal logics.

Within the context of the problem of unification in a normal modal logic L, an important question is the following: when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is 'yes', how large is this set? This question concerns the determination of the type of L for the problem of unification. Considering the type of unification in a normal modal logic L is justified from the following perspectives: deciding the unifiability of equivalences like $\varphi \leftrightarrow \psi$ in L helps us to understand what is the overlap between the properties φ and ψ correspond to in L [1]; methods for deciding the unifiability of formulas in L can be used to improve the efficiency of automated theorem provers in L [2].

Ghilardi [16] has proved that the unification type of transitive normal modal logics like *S4* and *GL* is finitary. Within the context of tense logics and epistemic logics, Dzik [11–13] has studied the relationships between the unification type of a fusion of modal logics and the unification types of the modal logics composing this fusion. He has also proved that some variants of the normal modal logics studied by Jansana [19] are unitary; these variants being sound and complete with respect to

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classes of frames satisfying conditions generalizing symmetry and transitivity. The unification type of normal modal logics such as common knowledge logics and linear temporal logics has also been studied by Babenyshev and Rybakov [3] and Rybakov [26–28].

Nevertheless, still, very little is known about the problem of unification in some of the most popular normal modal logics. For example, the types of the problem of unification in the normal modal logics *KB*, *KDB* and *KTB* are still unknown [12, Chapter 5]. As well, the types of the problem of unification in the normal modal logics *K5*, *KD5*, *K45*, *K4.3* and *KD4.3* are unknown too. In this paper, following a line of reasoning suggested by Jeřàbek [21] within the context of *K* and furthered by Balbiani and Gencer [4] within the context of *KD*, we prove that for several non-symmetric non-transitive modal logics like *KT*, there exists unifiable formulas that possess no minimal complete set of unifiers. Such modal logics are called nullary.

The section-by-section breakdown of the paper is as follows. Sections 2–4 introduce the basic definitions about normal modal logics. In Section 5, we introduce the basic definitions about unification. In Section 6, we analyse a specific formula from the point of view of some of its unifiers in normal modal logics. In Section 7, we elaborate a sufficient condition for the nullariness of normal modal logics (the adequacy condition) and we give examples of adequate and non-adequate normal modal logics. In Section 8, we prove that if a normal modal logic is adequate then it is nullary.

2 Syntax

Formulas Let *VAR* be a set of *variables* (with typical members denoted x, y, etc) and *CON* be a set of *constants* (with typical members denoted p, q, etc). The set of all *formulas* (with typical members denoted φ, ψ, etc) is inductively defined as follows:

- $\varphi, \psi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$.

We adopt the standard rules for omission of the parentheses. Let $(x_1, x_2, \ldots)$ be an enumeration of *VAR* without repetition and $(p_1, p_2, \ldots)$ be an enumeration of *CON* without repetition. We write $\varphi(x_1, \ldots, x_m, p_1, \ldots, p_n)$ to denote a formula whose variables form a subset of $\{x_1, \ldots, x_m\}$ and whose constants form a subset of $\{p_1, \ldots, p_n\}$. Let $\varphi(x_1, \ldots, x_m, p_1, \ldots, p_n)$ be such a formula. The result of the uniform replacement in each of their occurrences of the variables $x_1, \ldots, x_m$ by the formulas $\psi_1, \ldots, \psi_m$ and of the constants $p_1, \ldots, p_n$ by the formulas $\chi_1, \ldots, \chi_n$ is denoted $\varphi(\psi_1, \ldots, \psi_m, \chi_1, \ldots, \chi_n)$.

Abbreviations The Boolean connectives $\top$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined by the usual abbreviations. Let $\Diamond$ be the modal connective defined as follows:

- $\Diamond\varphi ::= \neg\Box\neg\varphi$.

For all $k \geq 0$, the modal connective $\Box^k$ is inductively defined as follows:

- $\Box^0\varphi ::= \varphi$,
- $\Box^{k+1}\varphi ::= \Box\Box^k\varphi$.

For all $k \geq 0$, the modal connective $\Diamond^k$ is inductively defined as follows:

- $\Diamond^0\varphi ::= \varphi$,
- $\Diamond^{k+1}\varphi ::= \Diamond\Diamond^k\varphi$.

Example: For all formulas φ, $\Box^2\varphi$ is $\Box\Box\varphi$ and $\Diamond^2\varphi$ is $\Diamond\Diamond\varphi$.

For all finite words w over CON, the modal connective $[w]$ is inductively defined as follows:

- $[\epsilon]\varphi ::= \varphi$,
- $[pw]\varphi ::= \Box(p \to [w]\varphi)$.

Example: For all constants p, q and for all formulas φ, $[pq]\varphi$ is $\Box(p \to \Box(q \to \varphi))$.

For all finite words w over CON and for all $k \geq 0$, the modal connective $[w]^k$ is inductively defined as follows:

- $[\in]^0\varphi ::= \varphi$,
- $[pw]^{k+1}\varphi ::= [w][w]^k\varphi$.

Example: For all constants p, q and for all formulas φ, $[pq]^2\varphi$ is $\Box(vp \to \Box(q \to \Box(p \to \Box(q \to \varphi))))$.

For all finite words w over CON and for all $k \geq 0$, the modal connective $[w]^{\leq k}$ is inductively defined as follows:

- $[w]^{\leq 0}\varphi ::= \varphi$,
- $[w]^{\leq k+1}\varphi ::= [w]^{\leq k}\varphi \wedge [w]^{k+1}\varphi$.

Example: For all constants p, q and for all formulas φ, $[pq]^{\leq 2}\varphi$ is $\varphi \wedge \Box(p \to \Box(q \to \varphi)) \wedge \Box(p \to \Box(q \to \Box(p \to \Box(q \to \varphi))))$.

Degrees The *degree* of a formula φ (in symbols $\deg(\varphi)$) is the non-negative integer inductively defined as follows:

- $\deg(x) = 0$,
- $\deg(p) = 0$,
- $\deg(\bot) = 0$,
- $\deg(\neg\varphi) = \deg(\varphi)$,
- $\deg(\varphi \vee \psi) = \max\{\deg(\varphi), \deg(\psi)\}$,
- $\deg(\Box\varphi) = \deg(\varphi) + 1$.

Example: For all constants p, q and for all formulas φ, $\deg([pq]^2\varphi)$ is $\deg(\varphi) + 4$.

Substitutions A *substitution* is a function σ associating to each variable x a formula $\sigma(x)$. Following the standard assumption considered in the literature about unification [1, 12, 16], we will always assume that substitutions move at most finitely many variables. For all formulas $\varphi(x_1, \ldots, x_m, p_1, \ldots, p_n)$, let $\sigma(\varphi(x_1, \ldots, x_m, p_1, \ldots, p_n))$ be $\varphi(\sigma(x_1), \ldots, \sigma(x_m), p_1, \ldots, p_n)$.

Example: If σ is the substitution such that $\sigma(x) = p$, $\sigma(y) = q$ and for all variables z distinct from x and y, $\sigma(z) = z$ and φ is the formula $(x \to p) \wedge (y \to q) \wedge (x \to [q]y) \wedge (y \to [p]x)$ then $\sigma(\varphi)$ is $(p \to p) \wedge (q \to q) \wedge (p \to [q]q) \wedge (q \to [p]p)$.

The *composition* $\sigma \circ \tau$ of the substitutions σ and τ is the substitution associating to each variable x the formula $\tau(\sigma(x))$.

3 Semantics

Frames A *frame* is a couple $F = (W, R)$ where W is a non-empty set of *possible worlds* and R is a binary relation on W. In a frame $F = (W, R)$, for all $n \geq 0$, the binary relation R^n on W is inductively defined as follows:

- $R^0 = Id$, i.e. the identity relation on W,
- $R^{n+1} = R \circ R^n$, i.e. the composition of the binary relations R and R^n on W.

For all $n \geq 1$, we shall say that a frame $F = (W, R)$ is *n-bounded* if for all $s, t \in W$, not $sR^n t$. We shall say that a frame $F = (W, R)$ is *serial* if for all $s \in W$, there exists $t \in W$ such that sRt. We shall say that a frame $F = (W, R)$ is *reflexive* if for all $s \in W$, sRs. For all $n \geq 1$, we shall say that a frame $F = (W, R)$ is *n-ancestral* if for all $s, t \in W$, if $sR^n t$ then there exists $u \in W$ such that tRu. For all $n \geq 1$, we shall say that a frame $F = (W, R)$ is *n-deterministic* if for all $s \in W$ and for all $t_1, \ldots, t_{n+1} \in W$, if sRt_1, …, sRt_{n+1} then there exists distinct $i, j \geq 1$ such that $i, j \leq n + 1$ and $t_i = t_j$. We shall say that a frame $F = (W, R)$ is *symmetric* if for all $s, t \in W$, if sRt then tRs. We shall say that a frame $F = (W, R)$ is *transitive* if for all $s, t, u \in W$, if sRt and tRu then sRu. For all $m, n \geq 1$, if $(m, n) \neq (2, 1)$ then we shall say that a frame $F = (W, R)$ is *(m, n)-compositional* if for all $s, t \in W$, if $sR^m t$ then $sR^n t$. For all $n \geq 1$, let C_n^{bou} be the class of all n-bounded frames. Let C_K, C_{KD} and C_{KT} be, respectively, the class of all frames, the class of all serial frames and the class of all reflexive frames. Remark that $C_{KT} \subseteq C_{KD}$. For all $n \geq 1$, let C_n^{anc} be the class of all n-ancestral frames. Remark that for all $n \geq 1$, $C_{KD} \subseteq C_n^{anc}$. For all $n \geq 1$, let C_n^{det} be the class of all n-deterministic frames. Let C_{KB}, C_{KDB} and C_{KTB} be, respectively, the class of all symmetric frames, the class of all symmetric serial frames and the class of all symmetric reflexive frames. Let C_{K4} be the class of all transitive frames. For all $m, n \geq 1$, if $(m, n) \neq (2, 1)$ then let C_m^n be the class of all (m, n)-compositional frames.

Models A *model* based on a frame $F = (W, R)$ is a triple $M = (W, R, V)$ where V is a function assigning to each variable x a subset $V(x)$ of W and to each constant p a subset $V(p)$ of W. Given a model $M = (W, R, V)$, the *satisfiability* of a modal formula φ at $s \in W$ (in symbols $M, s \models \varphi$) is inductively defined as follows:

- $M, s \models x$ if $s \in V(x)$,
- $M, s \models p$ if $s \in V(p)$,
- $M, s \not\models \bot$,
- $M, s \models \neg\varphi$ if $M, s \not\models \varphi$,
- $M, s \models \varphi \vee \psi$ if $M, s \models \varphi$ or $M, s \models \psi$,
- $M, s \models \Box\varphi$ if for all $t \in W$, if sRt then $M, t \models \varphi$.

We shall say that a formula φ is *true* in a model $M = (W, R, V)$ (in symbols $M \models \varphi$) if φ is satisfied at all $s \in W$.

Validity We shall say that a formula φ is *valid* in a frame F (in symbols $F \models \varphi$) if φ is true in all models based on F. We shall say that a formula φ is *valid* in a class C of frames (in symbols $C \models \varphi$) if φ is valid in all frames in C.

4 Normal modal logics

A *normal modal logic* is a set L of formulas such that

- L contains all tautologies,
- L contains all formulas of the form $\square(\varphi \rightarrow \psi) \rightarrow (\square\varphi \rightarrow \square\psi)$,
- L is closed under the rule of modus ponens $\frac{\varphi\ \varphi\rightarrow\psi}{\psi}$,
- L is closed under the rule of generalization $\frac{\varphi}{\square\varphi}$,
- L is closed under the rule of uniform substitution $\frac{\varphi(x_1,\ldots,x_m,p_1,\ldots,p_n)}{\varphi(\psi_1,\ldots,\psi_m,\chi_1,\ldots,\chi_n)}$.

It is evident that for all classes C of frames, the set of all C-valid formulas is a normal modal logic.

LEMMA 4.1
Let L be a normal modal logic. For all $k \geq 0$, for all formulas φ and for all constants p, q,

- $[qp]^k[q]\bot \rightarrow [qp]^{k+1}[q]\bot \in L$,
- $[qp]^{\leq k}\varphi \wedge [qp]^k[q]\bot \rightarrow [q][pq]^{\leq k}[p]\varphi \in L$,
- $[qp]^{\leq k}[q]\varphi \rightarrow [q][pq]^{\leq k}\varphi \in L$,
- $[qp]^k[q]\bot \rightarrow [q][pq]^k[p]\bot \in L$.

PROOF. Left to the reader. □

LEMMA 4.2
Let L be a normal modal logic. For all $k \geq 0$, for all formulas φ, ψ and for all constants p, q, if $\varphi \rightarrow [q]\psi \in L$ and $\psi \rightarrow [p]\varphi \in L$ then $\varphi \rightarrow [qp]^{\leq k}(\varphi \wedge [q]\psi) \in L$.

PROOF. Left to the reader. □

It is evident that the set of all normal modal logics is closed under arbitrary intersections. For all $n \geq 1$, let K_n^{bou} be the least normal modal logic containing the formula $\square^n\bot$. Let K, KD and KT be, respectively, the least normal modal logic, the least normal modal logic containing all formulas of the form $\square\chi \rightarrow \lozenge\chi$ and the least normal modal logic containing all formulas of the form $\square\chi \rightarrow \chi$. Remark that $KD \subseteq KT$. For all $n \geq 1$, let K_n^{anc} be the least normal modal logic containing the formula $\square^{2n-1}\lozenge\top$. Remark that for all $n \geq 1$, $K_n^{anc} \subseteq KD$. For all $n \geq 1$, let Alt_n be the least normal modal logic containing all formulas of the form $\bigvee\{\square(\bigwedge\{\chi_j : j \geq 1$ is such that $j \leq i\} \rightarrow \chi_{i+1}) : i \geq 0$ is such that $i \leq n\}$. Let KB, KDB and KTB be, respectively, the least normal modal logic containing all formulas of the form $\chi \rightarrow \square\lozenge\chi$, the least normal modal logic containing all formulas of the form $\chi \rightarrow \square\lozenge\chi$ and $\square\chi \rightarrow \lozenge\chi$ and the least normal modal logic containing all formulas of the form $\chi \rightarrow \square\lozenge\chi$ and $\square\chi \rightarrow \chi$. Let $K4$ be the least normal modal logic containing all formulas of the form $\square\chi \rightarrow \square\square\chi$. For all $m, n \geq 1$, if $(m, n) \neq (2, 1)$ then let K_m^n be the least normal modal logic containing all formulas of the form $\lozenge^m\chi \rightarrow \lozenge^n\chi$.

PROPOSITION 4.3
For all $n \geq 1$, K_n^{bou} is sound and complete with respect to C_n^{bou}.

PROOF. Let $n \geq 1$. It is evident that the canonical frame for K_n^{bou} is in C_n^{bou}. Hence, the result follows from the standard canonical model construction in normal modal logics as developed in [8, Chapter 4]. □

PROPOSITION 4.4
K, KD and KT are, respectively, sound and complete with respect to C_K, C_{KD} and C_{KT}.

PROOF. See [8, Chapter 4] for a proof. □

PROPOSITION 4.5
For all $n \geq 1$, K_n^{anc} is sound and complete with respect to C_n^{anc}.

PROOF. Let $n \geq 1$. It is evident that the canonical frame for K_n^{anc} is in C_n^{anc}. Hence, the result follows from the standard canonical model construction in normal modal logics as developed in [8, Chapter 4]. □

PROPOSITION 4.6
For all $n \geq 1$, Alt_n is sound and complete with respect to C_n^{det}.

PROOF. See [9, Chapters 3 and 4] for a proof. □

PROPOSITION 4.7
KB, *KDB* and *KTB* are, respectively, sound and complete with respect to C_{KB}, C_{KDB} and C_{KTB}.

PROOF. See [8, Chapter 4] for a proof. □

PROPOSITION 4.8
K4 is sound and complete with respect to C_{K4}.

PROOF. See [8, Chapter 4] for a proof. □

PROPOSITION 4.9
For all $m, n \geq 1$, if $(m, n) \neq (2, 1)$ then K_m^n is sound and complete with respect to C_m^n.

PROOF. See [10, Chapter 3] for a proof. □

5 Unification

Let L be a normal modal logic. In this section, we shall introduce the basic definitions about unification in L.

Unifiers We shall say that a formula φ is *L-unifiable* if there exists a substitution σ such that $\sigma(\varphi) \in L$. In that case, σ is an *L-unifier* of φ.

Example: If σ is the substitution such that $\sigma(x) = p$, $\sigma(y) = q$ and for all variables z distinct from x and y, $\sigma(z) = z$ and φ is the formula $(x \to p) \wedge (y \to q) \wedge (x \to [q]y) \wedge (y \to [p]x)$ then σ is a *K*-unifier of φ.

We shall say that a substitution σ is *L-equivalent* to a substitution τ (in symbols $\sigma \simeq_L \tau$) if for all variables x, $\sigma(x) \leftrightarrow \tau(x) \in L$.

Example: If σ and τ are the substitutions such that $\sigma(x) = \Box p$, $\tau(x) = \Box p \wedge p$, $\sigma(y) = \Diamond q \vee q$, $\tau(y) = \Diamond q$ and for all variables z distinct from x and y, $\sigma(z) = z$ and $\tau(z) = z$ then $\sigma \simeq_{KT} \tau$.

LEMMA 5.1
The binary relation $\simeq_L$ is reflexive, symmetric and transitive on the set of all substitutions.

PROOF. Left to the reader. See [1, 12] for details about the binary relation $\simeq_L$. □

We shall say that a substitution σ is more *L-general* than a substitution τ (in symbols $\sigma \preceq_L \tau$) if there exists a substitution υ such that $\sigma \circ \upsilon \simeq_L \tau$.

LEMMA 5.2
The binary relation $\preceq_L$ is reflexive and transitive on the set of all substitutions. Moreover, it contains $\simeq_L$.

PROOF. Left to the reader. See [1, 12] for details about the binary relation $\preceq_L$. □

We shall say that a set Σ of substitutions is *L-minimal* if for all $\sigma, \tau \in \Sigma$, if $\sigma \preceq_L \tau$ then $\sigma \simeq_L \tau$. We shall say that a set Σ of L-unifiers of an L-unifiable formula φ is *L-complete* if for all L-unifiers σ of φ, there exists $\tau \in \Sigma$ such that $\tau \preceq_L \sigma$.

Types An important question is the following: when a formula is L-unifiable, has it an L-minimal L-complete set of L-unifiers? When the answer is 'yes', how large is this set? We shall say that an L-unifiable formula

- φ is *L-nullary* if there exists no L-minimal L-complete set of L-unifiers of φ,
- φ is *L-infinitary* if there exists an L-minimal L-complete set of L-unifiers of φ but there exists no finite one,
- φ is *L-finitary* if there exists a finite L-minimal L-complete set of L-unifiers of φ but there exists no with cardinality 1,
- φ is *L-unitary* if there exists an L-minimal L-complete set of L-unifiers of φ with cardinality 1.

We shall say that

- L is of unification type *nullary* if there exists an L-nullary formula,
- L is of unification type *infinitary* if every L-unifiable formula is L-infinitary or L-finitary or L-unitary and there exists an L-infinitary formula,
- L is of unification type *finitary* if every L-unifiable formula is L-finitary or L-unitary and there exists an L-finitary formula,
- L is of unification type *unitary* if every L-unifiable formula is L-unitary.

See [1] for a proof that *S5* is unitary, [4] for a proof that *KD* is nullary, [6] for a proof that Alt_1 is nullary, [16] for a proof that *K4* and *S4* are finitary and [21] for a proof that *K* is nullary. By the way, the proof given in [1] that *S5* is unitary can be easily adapted to a proof that K_1^{bou} is unitary (the unification type of K_n^{bou} is not known when $n \geq 2$). In other respect, the proof given in [6] that Alt_1 is nullary can be easily adapted for all $n \geq 2$ to a proof that Alt_n is nullary. We shall say that L is *filtering* if for all L-unifiable formulas φ and for all L-unifiers σ, τ of φ, there exists an L-unifier μ of φ such that $\mu \preceq_L \sigma$ and $\mu \preceq_L \tau$. When L is filtering, given two L-unifiers of an L-unifiable formula, there is always an L-unifier that is more L-general than both of them. Hence, in this case, it is known that L is unitary or L is nullary. See [17] for a proof that if $K4 \subseteq L$ then L is filtering iff $K4.2^+ \subseteq L$. See also [5] for a proof that *K45* and *KD45* are filtering. The purpose of Sections 6–8 is to elaborate a sufficient condition for the nullariness of L.

Remarks Note that the proof of the nullariness of *K* given by Jeřàbek [21] only assumed that the language contains at least one variable. As well, note that the proof of the nullariness of *KD* given by Balbiani and Gencer [4] only assumed that the language contains at least one variable and one constant. As for the nullariness of Alt_1 given by Balbiani and Tinchev [6], it only assumed that the language contains at least one variable. This means that when the language contains no constant, *K* and Alt_1 are still nullary whereas the unification type of *KD* is still unknown. In the case where the language contains infinitely many constants, one is talking about *unification with constants* and in the case where the language contains no constant, one is talking about *elementary unification.* In Sections 6–8, we will always assume that the language of modal logic contains at least two distinct variables and two distinct constants. Hence, our results in the forthcoming sections only concern unification with constants.

6 Analysis of a specific formula

Let L be a normal modal logic. In this section, we shall analyse a specific formula from the point of view of some of its L-unifiers, namely the formula

- $\varphi = (x \to p) \wedge (y \to q) \wedge (x \to [q]y) \wedge (y \to [p]x)$

in which x, y are distinct variables and p, q are distinct constants, i.e. φ is the conjunction of the 4 following formulas:

- $x \to p$,
- $y \to q$,
- $x \to [q]y$,
- $y \to [p]x$.

Remark that Jeřàbek [21] has used the formula $x \to \Box x$ to prove that K is nullary and Balbiani and Gencer [4] have used the formula $(x \to p) \wedge (x \to [p]x)$ to prove that KD is nullary.

> Remark that in order to present our line of reasoning, we have to assume that the language of modal logic contains at least two distinct variables and two distinct constants.

Let $\sigma_\perp$ be the substitution defined as follows:

- $\sigma_\perp(x) = \perp$,
- $\sigma_\perp(y) = \perp$,
- for all variables z distinct from x and y, $\sigma_\perp(z) = z$.

LEMMA 6.1
$\sigma_\perp$ is an L-unifier of φ.

PROOF. Remark that $\sigma_\perp(\varphi)$ is the conjunction of the 4 following formulas:

- $\perp \to p$,
- $\perp \to q$,
- $\perp \to [q]\perp$,
- $\perp \to [p]\perp$.

Hence, $\sigma_\perp(\varphi) \in L$. Thus, $\sigma_\perp$ is an L-unifier of φ. □

Let $\sigma_\top$ be the substitution defined as follows:

- $\sigma_\top(x) = p$,
- $\sigma_\top(y) = q$,
- for all variables z distinct from x and y, $\sigma_\top(z) = z$.

LEMMA 6.2
$\sigma_\top$ is an L-unifier of φ.

PROOF. Remark that $\sigma_\top(\varphi)$ is the conjunction of the 4 following formulas:

- $p \to p$,
- $q \to q$,
- $p \to [q]q$,
- $q \to [p]p$.

Hence, $\sigma_\top(\varphi) \in L$. Thus, $\sigma_\top$ is an L-unifier of φ. □

LEMMA 6.3
Let σ be a substitution. The following conditions are equivalent:

1. $\sigma_\top \circ \sigma \simeq_L \sigma$,
2. $\sigma_\top \preceq_L \sigma$,
3. $\sigma(x) \leftrightarrow p \in L$ and $\sigma(y) \leftrightarrow q \in L$.

PROOF. (1) $\Rightarrow$ (2). By definition of $\preceq_L$.
(2) $\Rightarrow$ (3). Suppose $\sigma_\top \preceq_L \sigma$. Let τ be a substitution such that $\sigma_\top \circ \tau \simeq_L \sigma$. Hence, $\tau(\sigma_\top(x)) \leftrightarrow \sigma(x) \in L$ and $\tau(\sigma_\top(y)) \leftrightarrow \sigma(y) \in L$. Since $\tau(\sigma_\top(x)) = p$ and $\tau(\sigma_\top(y)) = q$, therefore $\sigma(x) \leftrightarrow p \in L$ and $\sigma(y) \leftrightarrow q \in L$.
(3) $\Rightarrow$ (1). Suppose $\sigma(x) \leftrightarrow p \in L$ and $\sigma(y) \leftrightarrow q \in L$. Since $\sigma(\sigma_\top(x)) = p$ and $\sigma(\sigma_\top(y)) = q$, therefore $\sigma(\sigma_\top(x)) \leftrightarrow \sigma(x) \in L$ and $\sigma(\sigma_\top(y)) \leftrightarrow \sigma(y) \in L$. Moreover, since for all variables z distinct from x and y, $\sigma(\sigma_\top(z)) = \sigma(z)$, therefore for all variables z distinct from x and y, $\sigma(\sigma_\top(z)) \leftrightarrow \sigma(z) \in L$. Hence, $\sigma_\top \circ \sigma \simeq_L \sigma$ □

For all $k \geq 0$, let σ_k be the substitution defined as follows:

- $\sigma_k(x) = p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\bot$,
- $\sigma_k(y) = q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\bot$,
- for all variables z distinct from x and y, $\sigma_k(z) = z$.

LEMMA 6.4
For all $k \geq 0$, σ_k is an L-unifier of φ.

PROOF. Let $k \geq 0$. Remark that $\sigma_k(\varphi)$ is the conjunction of the 4 following formulas:

- $p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\bot \to p$,
- $q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\bot \to q$,
- $p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\bot \to [q](q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\bot)$,
- $q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\bot \to [p](p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\bot)$.

Hence, by Lemma 4.1, $\sigma_k(\varphi) \in L$. Thus, σ_k is an L-unifier of φ. □

LEMMA 6.5
Let σ be a substitution. If σ is an L-unifier of φ then for all $k \geq 0$, the following conditions are equivalent:

1. $\sigma_k \circ \sigma \simeq_L \sigma$,
2. $\sigma_k \preceq_L \sigma$,
3. $\sigma(x) \to [qp]^k[q]\bot \in L$ and $\sigma(y) \to [pq]^k[p]\bot \in L$.

PROOF. Suppose σ is an L-unifier of φ. Let $k \geq 0$.

(1) $\Rightarrow$ (2). By definition of $\preceq_L$.

(2) $\Rightarrow$ (3). Suppose $\sigma_k \preceq_L \sigma$. Let τ be a substitution such that $\sigma_k \circ \tau \simeq_L \sigma$. Hence, $\tau(\sigma_k(x)) \leftrightarrow \sigma(x) \in L$ and $\tau(\sigma_k(y)) \leftrightarrow \sigma(y) \in L$. Since $\tau(\sigma_k(x)) = p \wedge [qp]^{\leq k}(\tau(x) \wedge [q]\tau(y)) \wedge [qp]^k[q]\bot$ and $\tau(\sigma_k(y)) = q \wedge [pq]^{\leq k}(\tau(y) \wedge [p]\tau(x)) \wedge [pq]^k[p]\bot$, therefore $p \wedge [qp]^{\leq k}(\tau(x) \wedge [q]\tau(y)) \wedge [qp]^k[q]\bot \leftrightarrow \sigma(x) \in L$ and $q \wedge [pq]^{\leq k}(\tau(y) \wedge [p]\tau(x)) \wedge [pq]^k[p]\bot \leftrightarrow \sigma(y) \in L$. Thus, $\sigma(x) \rightarrow [qp]^k[q]\bot \in L$ and $\sigma(y) \rightarrow [pq]^k[p]\bot \in L$.

(3) $\Rightarrow$ (1). Suppose $\sigma(x) \rightarrow [qp]^k[q]\bot \in L$ and $\sigma(y) \rightarrow [pq]^k[p]\bot \in L$. Since σ is an L-unifier of φ, therefore $\sigma(\varphi) \in L$. Since $\sigma(\varphi) = (\sigma(x) \rightarrow p) \wedge (\sigma(y) \rightarrow q) \wedge (\sigma(x) \rightarrow [q]\sigma(y)) \wedge (\sigma(y) \rightarrow [p]\sigma(x))$, therefore $(\sigma(x) \rightarrow p) \wedge (\sigma(y) \rightarrow q) \wedge (\sigma(x) \rightarrow [q]\sigma(y)) \wedge (\sigma(y) \rightarrow [p]\sigma(x)) \in L$. Hence, the 4 following formulas are in L:

- $\sigma(x) \rightarrow p$,
- $\sigma(y) \rightarrow q$,
- $\sigma(x) \rightarrow [q]\sigma(y)$,
- $\sigma(y) \rightarrow [p]\sigma(x)$.

Since $\sigma(x) \rightarrow [qp]^k[q]\bot \in L$ and $\sigma(y) \rightarrow [pq]^k[p]\bot \in L$, therefore by Lemma 4.2, $\sigma(x) \rightarrow p \wedge [qp]^{\leq k}(\sigma(x) \wedge [q]\sigma(y)) \wedge [qp]^k[q]\bot \in L$ and $\sigma(y) \rightarrow q \wedge [pq]^{\leq k}(\sigma(y) \wedge [p]\sigma(x)) \wedge [pq]^k[p]\bot \in L$. In other respect, $p \wedge [qp]^{\leq k}(\sigma(x) \wedge [q]\sigma(y)) \wedge [qp]^k[q]\bot \rightarrow \sigma(x) \in L$ and $q \wedge [pq]^{\leq k}(\sigma(y) \wedge [p]\sigma(x)) \wedge [pq]^k[p]\bot \rightarrow \sigma(y) \in L$. Since $\sigma(\sigma_k(x)) = p \wedge [qp]^{\leq k}(\sigma(x) \wedge [q]\sigma(y)) \wedge [qp]^k[q]\bot$ and $\sigma(\sigma_k(y)) = q \wedge [pq]^{\leq k}(\sigma(y) \wedge [p]\sigma(x)) \wedge [pq]^k[p]\bot$, therefore $\sigma(\sigma_k(x)) \leftrightarrow \sigma(x) \in L$ and $\sigma(\sigma_k(y)) \leftrightarrow \sigma(y) \in L$. Moreover, since for all variables z distinct from x and y, $\sigma(\sigma_k(z)) = \sigma(z)$, therefore for all variables z distinct from x and y, $\sigma(\sigma_k(z)) \leftrightarrow \sigma(z) \in L$. Thus, $\sigma_k \circ \sigma \simeq_L \sigma$ □

LEMMA 6.6
For all $k, l \geq 0$, $\sigma_l \preceq_L \sigma_k$ iff $p \wedge [qp]^k[q]\bot \rightarrow [qp]^l[q]\bot \in L$ and $q \wedge [pq]^k[p]\bot \rightarrow [pq]^l[p]\bot \in L$.

PROOF. Let $k, l \geq 0$.

($\Rightarrow$). Suppose $\sigma_l \preceq_L \sigma_k$. Hence, by Lemma 6.5, $\sigma_k(x) \rightarrow [qp]^l[q]\bot \in L$ and $\sigma_k(y) \rightarrow [pq]^l[p]\bot \in L$. Since $\sigma_k(x) = p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\bot$ and $\sigma_k(y) = q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\bot$, therefore $p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\bot \rightarrow [qp]^l[q]\bot \in L$ and $q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\bot \rightarrow [pq]^l[p]\bot \in L$. Thus, uniformly replacing in each of their occurrences the variables x and y by, respectively, the formulas $\top$ and $\top$, $p \wedge [qp]^k[q]\bot \rightarrow [qp]^l[q]\bot \in L$ and $q \wedge [pq]^k[p]\bot \rightarrow [pq]^l[p]\bot \in L$.

($\Leftarrow$). Suppose $p \wedge [qp]^k[q]\bot \rightarrow [qp]^l[q]\bot \in L$ and $q \wedge [pq]^k[p]\bot \rightarrow [pq]^l[p]\bot \in L$. Hence, $p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\bot \rightarrow [qp]^l[q]\bot \in L$ and $q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\bot \rightarrow [pq]^l[p]\bot \in L$. Since $\sigma_k(x) = p \wedge [qp]^{\leq k}(x \wedge [q]y) \wedge [qp]^k[q]\bot$ and $\sigma_k(y) = q \wedge [pq]^{\leq k}(y \wedge [p]x) \wedge [pq]^k[p]\bot$, therefore $\sigma_k(x) \rightarrow [qp]^l[q]\bot \in L$ and $\sigma_k(y) \rightarrow [pq]^l[p]\bot \in L$. Thus, by Lemma 6.5, $\sigma_l \preceq_L \sigma_k$. □

7 Adequate modal logics

Let L be a normal modal logic. In this section, we shall elaborate a sufficient condition for the nullariness of L: the adequacy condition. Before elaborating it, we shall elaborate a weaker condition: the coherence condition. We shall say that L is *coherent* if for all distinct constants p, q and for all $k \geq 0$,

- $p \wedge [qp]^{k+1}[q]\bot \rightarrow [qp]^k[q]\bot \notin L$.

LEMMA 7.1
Let $n \geq 1$. For all distinct constants p, q, $p \wedge [qp]^{n+1}[q]\bot \rightarrow [qp]^n[q]\bot \in K_n^{bou}$.

PROOF. Left to the reader. □

LEMMA 7.2
For all distinct constants p, q and for all $k \geq 0$, $p \wedge [qp]^{k+1}[q]\bot \rightarrow [qp]^k[q]\bot \notin KT$.

PROOF. Let $k \geq 0$. Let $F = (W, R)$ be the frame such that $W = \mathbb{N}$ and for all $s, t \in W$, sRt iff $t = s$ or $t = s + 1$. Obviously, F is in C_{KT}. Let $M = (W, R, V)$ be the model based on F such that $V(p) = \{2i : i \geq 0 \text{ is such that } i \leq k+1\}$ and $V(q) = \{2i+1 : i \geq 0 \text{ is such that } i \leq k\}$. Obviously, $M, 0 \models p \wedge [qp]^{k+1}[q]\bot$ and $M, 0 \not\models [qp]^k[q]\bot$. Since KT is sound with respect to C_{KT} and F is in C_{KT}, therefore $p \wedge [qp]^{k+1}[q]\bot \rightarrow [qp]^k[q]\bot \notin KT$. □

LEMMA 7.3
Let $n \geq 1$. For all distinct constants p, q and for all $k \geq 0$, $p \wedge [qp]^{k+1}[q]\bot \rightarrow [qp]^k[q]\bot \notin Alt_n$.

PROOF. Let $k \geq 0$. Let $F = (W, R)$ be the frame such that $W = \mathbb{N}$ and for all $s, t \in W$, sRt iff $t = s + 1$. Obviously, F is in C_n^{det}. Let $M = (W, R, V)$ be the model based on F such that $V(p) = \{2i : i \geq 0 \text{ is such that } i \leq k+1\}$ and $V(q) = \{2i+1 : i \geq 0 \text{ is such that } i \leq k\}$. Obviously, $M, 0 \models p \wedge [qp]^{k+1}[q]\bot$ and $M, 0 \not\models [qp]^k[q]\bot$. Since Alt_n is sound with respect to C_n^{det} and F is in C_n^{det}, therefore $p \wedge [qp]^{k+1}[q]\bot \rightarrow [qp]^k[q]\bot \notin Alt_n$. □

LEMMA 7.4
For all distinct constants p, q and for all $k \geq 0$, $p \wedge [qp]^{k+1}[q]\bot \rightarrow [qp]^k[q]\bot \in KB$.

PROOF. Left to the reader. □

LEMMA 7.5
For all distinct constants p, q and for all $k \geq 0$, $p \wedge [qp]^{k+1}[q]\bot \rightarrow [qp]^k[q]\bot \notin K4$.

PROOF. Let $k \geq 0$. Let $F = (W, R)$ be the frame such that $W = \mathbb{Q}$ and for all $s, t \in W$, sRt iff $s < t$. Obviously, F is in C_{K4}. Let $M = (W, R, V)$ be the model based on F such that $V(p) = \{2i : i \geq 0 \text{ is such that } i \leq k+1\}$ and $V(q) = \{2i+1 : i \geq 0 \text{ is such that } i \leq k\}$. Obviously, $M, 0 \models p \wedge [qp]^{k+1}[q]\bot$ and $M, 0 \not\models [qp]^k[q]\bot$. Since $K4$ is sound with respect to C_{K4} and F is in C_{K4}, therefore $p \wedge [qp]^{k+1}[q]\bot \rightarrow [qp]^k[q]\bot \notin K4$. □

LEMMA 7.6
Let $m, n \geq 1$. If $(m, n) \neq (2, 1)$ then for all distinct constants p, q and for all $k \geq 0$, $p \wedge [qp]^{k+1}[q]\bot \rightarrow [qp]^k[q]\bot \notin K_m^n$.

PROOF. Similar to the proof of Lemma 7.5. □

PROPOSITION 7.7
Let $n \geq 1$. If $K_n^{bou} \subseteq L$ then L is not coherent.

PROOF. By Lemma 7.1. □

PROPOSITION 7.8
If $L \subseteq KT$ then L is coherent.

PROOF. By Lemma 7.2. □

PROPOSITION 7.9
Let $n \geq 1$. If $L \subseteq K_n^{anc}$ then L is coherent.

PROOF. By Lemma 7.2. □

PROPOSITION 7.10
Let $n \geq 1$. If $L \subseteq Alt_n$ then L is coherent.

PROOF. By Lemma 7.3. □

PROPOSITION 7.11
If $KB \subseteq L$ then L is not coherent.

PROOF. By Lemma 7.4. □

PROPOSITION 7.12
If $L \subseteq K4$ then L is coherent.

PROOF. By Lemma 7.5. □

PROPOSITION 7.13
Let $m, n \geq 1$. If $(m, n) \neq (2, 1)$ and $L \subseteq K_m^n$ then L is coherent.

PROOF. By Lemma 7.6. □

Now, we are ready to elaborate the adequacy condition. We shall say that L is *adequate* if L is coherent and for all formulas φ, ψ, for all distinct constants p, q and for all $k \geq 0$, if $\deg(\varphi) \leq 2k$ then

- if $\varphi \to [qp]^k[q][p]\varphi \in L$ then $\varphi \to [qp]^k[q]\bot \in L$ or $p \to \varphi \in L$,
- if $\varphi \to [qp]^k[q]\psi \in L$ then $\varphi \to [qp]^k\bot \in L$ or $q \to \psi \in L$.

PROPOSITION 7.14
Let $n \geq 1$. If $K_n^{bou} \subseteq L$ then L is not adequate.

PROOF. By Proposition 7.7. □

PROPOSITION 7.15
The normal modal logic K is adequate.

PROOF. By Proposition 7.8, it suffices to prove that for all formulas φ, ψ, for all distinct constants p, q and for all $k \geq 0$, if $\deg(\varphi) \leq 2k$ then

- if $\varphi \to [qp]^k[q][p]\varphi \in K$ then $\varphi \to [qp]^k[q]\bot \in K$ or $p \to \varphi \in K$,
- if $\varphi \to [qp]^k[q]\psi \in K$ then $\varphi \to [qp]^k\bot \in K$ or $q \to \psi \in K$.

Let φ, ψ be formulas and $k \geq 0$ be such that $\deg(\varphi) \leq 2k$.

Suppose $\varphi \to [qp]^k[q]\bot \notin K$ and $p \to \varphi \notin K$. Since K is complete with respect to C_K, therefore let $F = (W, R)$ be a frame in C_K such that $F \not\models \varphi \to [qp]^k[q]\bot$ and $F' = (W', R')$ be a frame in C_K such that $F' \not\models p \to \varphi$. Let $M = (W, R, V)$ be a model based on F such that $M \not\models \varphi \to [qp]^k[q]\bot$ and $M' = (W', R', V')$ be a model based on F' such that and $M' \not\models p \to \varphi$. Let $s \in W$ be such that $M, s \not\models \varphi \to [qp]^k[q]\bot$ and $s' \in W'$ be such that $M', s' \not\models p \to \varphi$. Hence, $M, s \models \varphi$, $M, s \not\models [qp]^k[q]\bot$, $M', s' \models p$ and $M', s' \not\models \varphi$. Let $t_1, \ldots, t_k, t_{k+1} \in V(q)$ and $u_1, \ldots, u_k \in V(p)$ be such that $sRt_1Ru_1 \ldots Rt_kRu_kRt_{k+1}$. Let $M_s = (W_s, R_s, V_s)$ be the unravelling of M around s. In particular, (s), (s, t_1), (s, t_1, u_1), …, $(s, t_1, u_1, \ldots, t_k)$, $(s, t_1, u_1, \ldots, t_k, u_k)$ and $(s, t_1, u_1, \ldots, t_k, u_k, t_{k+1})$ are in W_s. Moreover, $(s)R_s(s, t_1)R_s(s, t_1, u_1) \ldots R_s(s, t_1, u_1, \ldots, t_k)R_s(s, t_1, u_1, \ldots, t_k, u_k)R_s(s, t_1, u_1, \ldots, t_k, u_k, t_{k+1})$. The reader may easily verify that for all $(s_0, \ldots, s_n) \in W_s$ and for all formulas χ, if

$\deg(\chi)+n\leq 2k$ then $M,s_n\models\chi$ iff $M_s,(s_0,\ldots,s_n)\models\chi$. Since $M,s\models\varphi$, therefore $M_s,(s)\models\varphi$. Let M'' be (W'',R'',V'') where

- $W''=W_s\cup W'$,
- $R''=R_s\cup R'\cup\{((s,t_1,u_1,\ldots,t_k,u_k,t_{k+1}),s')\}$,
- $V''=V_s\cup V'$.

The reader may easily verify that for all $(s_0,\ldots,s_n)\in W_s$ and for all formulas χ, if $\deg(\chi)+n\leq 2k$ then $M_s,(s_0,\ldots,s_n)\models\chi$ iff $M'',(s_0,\ldots,s_n)\models\chi$. Since $M_s,(s)\models\varphi$, therefore $M'',(s)\models\varphi$. The reader may also easily verify that for all $t'\in W'$ and for all formulas χ, $M',t'\models\chi$ iff $M'',t'\models\chi$. Since $M',s'\models p$ and $M',s'\not\models\varphi$, therefore $M'',s'\models p$ and $M'',s'\not\models\varphi$. Since $t_1,\ldots,t_k,t_{k+1}\in V(q)$, $u_1,\ldots,u_k\in V(p)$, $(s)R_s(s,t_1)R_s(s,t_1,u_1)\ldots R_s(s,t_1,u_1,\ldots,t_k)R_s(s,t_1,u_1,\ldots,t_k,u_k)R_s(s,t_1,u_1,\ldots,t_k,u_k,t_{k+1})$ and $M'',(s)\models\varphi$, therefore $M'',(s)\not\models\varphi\rightarrow[qp]^k[q][p]\varphi$. Thus, $\varphi\rightarrow[qp]^k[q][p]\varphi\notin K$.

Suppose $\varphi\rightarrow[qp]^k\bot\notin K$ and $q\rightarrow\psi\notin K$. Since K is complete with respect to C_K, therefore let $F=(W,R)$ be a frame in C_K such that $F\not\models\varphi\rightarrow[qp]^k\bot$ and $F'=(W',R')$ be a frame in C_K such that $F'\not\models q\rightarrow\psi$. Let $M=(W,R,V)$ be a model based on F such that $M\not\models\varphi\rightarrow[qp]^k\bot$ and $M'=(W',R',V')$ be a model based on F' such that and $M'\not\models q\rightarrow\psi$. Let $s\in W$ be such that $M,s\not\models\varphi\rightarrow[qp]^k\bot$ and $s'\in W'$ be such that $M',s'\not\models q\rightarrow\psi$. Hence, $M,s\models\varphi$, $M,s\not\models[qp]^k\bot$, $M',s'\models q$ and $M',s'\not\models\psi$. Let $t_1,\ldots,t_k\in V(q)$ and $u_1,\ldots,u_k\in V(p)$ be such that $sRt_1Ru_1\ldots Rt_kRu_k$. Let $M_s=(W_s,R_s,V_s)$ be the unravelling of M around s. In particular, (s), (s,t_1), (s,t_1,u_1), …, $(s,t_1,u_1,\ldots,t_k)$ and $(s,t_1,u_1,\ldots,t_k,u_k)$ are in W_s. Moreover, $(s)R_s(s,t_1)R_s(s,t_1,u_1)\ldots R_s(s,t_1,u_1,\ldots,t_k)R_s(s,t_1,u_1,\ldots,t_k,u_k)$. The reader may easily verify that for all $(s_0,\ldots,s_n)\in W_s$ and for all formulas χ, if $\deg(\chi)+n\leq 2k$ then $M,s_n\models\chi$ iff $M_s,(s_0,\ldots,s_n)\models\chi$. Since $M,s\models\varphi$, therefore $M_s,(s)\models\varphi$. Let M'' be (W'',R'',V'') where

- $W''=W_s\cup W'$,
- $R''=R_s\cup R'\cup\{((s,t_1,u_1,\ldots,t_k,u_k),s')\}$,
- $V''=V_s\cup V'$.

The reader may easily verify that for all $(s_0,\ldots,s_n)\in W_s$ and for all formulas χ, if $\deg(\chi)+n\leq 2k$ then $M_s,(s_0,\ldots,s_n)\models\chi$ iff $M'',(s_0,\ldots,s_n)\models\chi$. Since $M_s,(s)\models\varphi$, therefore $M'',(s)\models\varphi$. The reader may also easily verify that for all $t'\in W'$ and for all formulas χ, $M',t'\models\chi$ iff $M'',t'\models\chi$. Since $M',s'\models q$ and $M',s'\not\models\psi$, therefore $M'',s'\models q$ and $M'',s'\not\models\psi$. Since $t_1,\ldots,t_k\in V(q)$, $u_1,\ldots,u_k\in V(p)$, $(s)R_s(s,t_1)R_s(s,t_1,u_1)\ldots R_s(s,t_1,u_1,\ldots,t_k)R_s(s,t_1,u_1,\ldots,t_k,u_k)$ and $M'',(s)\models\varphi$, therefore $M'',(s)\not\models\varphi\rightarrow[qp]^k[q]\psi$. Thus, $\varphi\rightarrow[qp]^k[q]\psi\notin K$. □

PROPOSITION 7.16
The normal modal logic *KD* is adequate.

PROOF. Similar to the proof of Proposition 7.15, seeing that the unravelling of M around s is serial when M is serial. □

PROPOSITION 7.17
The normal modal logic *KT* is adequate.

PROOF. Similar to the proof of Proposition 7.15, defining this time, in order to ensure that M'' is in C_{KT}, $M_s=(W_s,R_s,V_s)$ to be the reflexive closure of the unravelling of M around s. Note that this reflexive closure is obtained by forcing each possible world to be related to itself. □

PROPOSITION 7.18
Let $n \geq 1$. The normal modal logic K_n^{anc} is not adequate.

PROOF. Suppose the normal modal logic K_n^{anc} is adequate. Let $\varphi = \top$, $\psi = \Diamond\top$. Obviously, $deg(\varphi) \leq 2n$. Moreover, $\varphi \rightarrow [qp]^n[q]\psi \in K_n^{anc}$. Since K_n^{anc} is adequate and $deg(\varphi) \leq 2n$, therefore $\varphi \rightarrow [qp]^n\bot \in K_n^{anc}$ or $q \rightarrow \psi \in K_n^{anc}$. In the former case, $[qp]^n\bot \in K_n^{anc}$. Let F be the frame consisting of a single reflexive element. Obviously, $F \models K_n^{anc}$ and $F \not\models [qp]^n\bot$. Hence, $[qp]^n\bot \notin K_n^{anc}$: a contradiction. In the latter case, $q \rightarrow \Diamond\top \in K_n^{anc}$. Let F be the frame consisting of a single irreflexive element. Obviously, $F \models K_n^{anc}$ and $F \not\models q \rightarrow \Diamond\top$. Hence, $q \rightarrow \Diamond\top \notin K_n^{anc}$: a contradiction. □

PROPOSITION 7.19
Let $n \geq 1$. The normal modal logic Alt_n is adequate.

PROOF. Similar to the proof of Proposition 7.15, defining this time, in order to ensure that M'' is in C_n^{det}, $M_s = (W_s, R_s, V_s)$ to be the restriction of the unravelling of M around s to the paths of length at most $2k$. □

PROPOSITION 7.20
If $KB \subseteq L$ then L is not adequate.

PROOF. By Proposition 7.11. □

PROPOSITION 7.21
The normal modal logic $K4$ is not adequate.

PROOF. Suppose the normal modal logic $K4$ is adequate. Let $\varphi = \Box r$ and $k = 1$. Obviously, $deg(\varphi) \leq 2k$. Moreover, $\varphi \rightarrow [qp]^k[q][p]\varphi \in K4$. Since $K4$ is adequate and $deg(\varphi) \leq 2k$, therefore $\varphi \rightarrow [qp]^k[q]\bot \in K4$ or $p \rightarrow \varphi \in K4$. In the former case, $\Box r \rightarrow [qp]^k[q]\bot \in K4$. Let F be the frame consisting of a single reflexive element. Obviously, $F \models K4$ and $F \not\models \Box r \rightarrow [qp]^k[q]\bot$. Hence, $\Box r \rightarrow [qp]^k[q]\bot \notin K4$: a contradiction. In the latter case, $p \rightarrow \Box r \in K4$. Let F be the frame consisting of a single reflexive element. Obviously, $F \models K4$ and $F \not\models p \rightarrow \Box r$. Hence, $p \rightarrow \Box r \notin K4$: a contradiction. □

PROPOSITION 7.22
Let $m, n \geq 1$. If $m \leq n$ then the normal modal logic K_m^n is adequate.

PROOF. Suppose $m \leq n$. Hence, $(m, n) \neq (2, 1)$ and by Proposition 7.13, it suffices to prove that for all formulas φ, ψ, for all distinct constants p, q and for all $k \geq 0$, if $deg(\varphi) \leq 2k$ then

- if $\varphi \rightarrow [qp]^k[q][p]\varphi \in K_m^n$ then $\varphi \rightarrow [qp]^k[q]\bot \in K_m^n$ or $p \rightarrow \varphi \in K_m^n$,
- if $\varphi \rightarrow [qp]^k[q]\psi \in K_m^n$ then $\varphi \rightarrow [qp]^k\bot \in K_m^n$ or $q \rightarrow \psi \in K_m^n$.

We will consider the case when '$m = 1$ and $n = 2$', the reader being invited to adapt the following line of reasoning to the other cases. Remark that K_1^2 is the least normal modal logic containing all formulas of the form $\Diamond\chi \rightarrow \Diamond\Diamond\chi$. As well, remark that C_1^2 is the class of all dense frames, i.e. those frames $F = (W, R)$ such that for all $s, t \in W$, if sRt then there exists $u \in W$ such that sRu and uRt. Let φ, ψ be formulas and $k \geq 0$ be such that $deg(\varphi) \leq 2k$.

Suppose $\varphi \rightarrow [qp]^k[q]\bot \notin K_1^2$ and $p \rightarrow \varphi \notin K_1^2$. Since K_1^2 is complete with respect to C_1^2, therefore let $F = (W, R)$ be a frame in C_1^2 such that $F \not\models \varphi \rightarrow [qp]^k[q]\bot$ and $F' = (W', R')$ be a frame in C_1^2 such that $F' \not\models p \rightarrow \varphi$. Let $M = (W, R, V)$ be a model based on F such that $M \not\models \varphi \rightarrow [qp]^k[q]\bot$ and $M' = (W', R', V')$ be a model based on

F' such that and $M' \not\models p \to \varphi$. Let $s \in W$ be such that $M, s \not\models \varphi \to [qp]^k[q]\bot$ and $s' \in W'$ be such that $M', s' \not\models p \to \varphi$. Hence, $M, s \models \varphi$, $M, s \not\models [qp]^k[q]\bot$, $M', s' \models p$ and $M', s' \not\models \varphi$. Let $t_1, \ldots, t_k, t_{k+1} \in V(q)$ and $u_1, \ldots, u_k \in V(p)$ be such that $sRt_1Ru_1 \ldots Rt_kRu_kRt_{k+1}$. Let $M_s = (W_s, R_s, V_s)$ be the dense closure of the unravelling of M around s. Note that this dense closure is obtained as the limit of the process consisting in adding an intermediate possible world between any two related possible worlds. In particular, (s), (s, t_1), (s, t_1, u_1), ..., $(s, t_1, u_1, \ldots, t_k)$, $(s, t_1, u_1, \ldots, t_k, u_k)$ and $(s, t_1, u_1, \ldots, t_k, u_k, t_{k+1})$ are in W_s. Moreover, $(s)R_s(s, t_1)R_s(s, t_1, u_1) \ldots R_s(s, t_1, u_1, \ldots, t_k)R_s(s, t_1, u_1, \ldots, t_k, u_k)\ R_s(s, t_1, u_1, \ldots, t_k, u_k, t_{k+1})$. The reader may easily verify that for all $(s_0, \ldots, s_n) \in W_s$ and for all formulas χ, if $\deg(\chi) + n \leq 2k$ then $M, s_n \models \chi$ iff $M_s, (s_0, \ldots, s_n) \models \chi$. Since $M, s \models \varphi$, therefore $M_s, (s) \models \varphi$. Let M'' be the dense closure of (W'', R'', V'') where

- $W'' = W_s \cup W'$,
- $R'' = R_s \cup R' \cup \{((s, t_1, u_1, \ldots, t_k, u_k, t_{k+1}), s')\}$,
- $V'' = V_s \cup V'$.

The reader may easily verify that for all $(s_0, \ldots, s_n) \in W_s$ and for all formulas χ, if $\deg(\chi) + n \leq 2k$ then $M_s, (s_0, \ldots, s_n) \models \chi$ iff $M'', (s_0, \ldots, s_n) \models \chi$. Since $M_s, (s) \models \varphi$, therefore $M'', (s) \models \varphi$. The reader may also easily verify that for all $t' \in W'$ and for all formulas χ, $M', t' \models \chi$ iff $M'', t' \models \chi$. Since $M', s' \models p$ and $M', s' \not\models \varphi$, therefore $M'', s' \models p$ and $M'', s' \not\models \varphi$. Since $t_1, \ldots, t_k, t_{k+1} \in V(q)$, $u_1, \ldots, u_k \in V(p)$, $(s)R_s(s, t_1)R_s(s, t_1, u_1) \ldots R_s(s, t_1, u_1, \ldots, t_k)R_s(s, t_1, u_1, \ldots, t_k, u_k)R_s(s, t_1, u_1, \ldots, t_k, u_k, t_{k+1})$ and $M'', (s) \models \varphi$, therefore $M'', (s) \not\models \varphi \to [qp]^k[q][p]\varphi$. Thus, $\varphi \to [qp]^k[q][p]\varphi \notin K_1^2$.

Suppose $\varphi \to [qp]^k\bot \notin K_1^2$ and $q \to \psi \notin K_1^2$. Since K_1^2 is complete with respect to C_1^2, therefore let $F = (W, R)$ be a frame in C_1^2 such that $F \not\models \varphi \to [qp]^k\bot$ and $F' = (W', R')$ be a frame in C_1^2 such that $F' \not\models q \to \psi$. Let $M = (W, R, V)$ be a model based on F such that $M \not\models \varphi \to [qp]^k\bot$ and $M' = (W', R', V')$ be a model based on F' such that and $M' \not\models q \to \psi$. Let $s \in W$ be such that $M, s \not\models \varphi \to [qp]^k\bot$ and $s' \in W'$ be such that $M', s' \not\models q \to \psi$. Hence, $M, s \models \varphi$, $M, s \not\models [qp]^k\bot$, $M', s' \models q$ and $M', s' \not\models \psi$. Let $t_1, \ldots, t_k \in V(q)$ and $u_1, \ldots, u_k \in V(p)$ be such that $sRt_1Ru_1 \ldots Rt_kRu_k$. Let $M_s = (W_s, R_s, V_s)$ be the dense closure of the unravelling of M around s. In particular, (s), (s, t_1), (s, t_1, u_1), ..., $(s, t_1, u_1, \ldots, t_k)$ and $(s, t_1, u_1, \ldots, t_k, u_k)$ are in W_s. Moreover, $(s)R_s(s, t_1)R_s(s, t_1, u_1) \ldots R_s(s, t_1, u_1, \ldots, t_k)R_s(s, t_1, u_1, \ldots, t_k, u_k)$. The reader may easily verify that for all $(s_0, \ldots, s_n) \in W_s$ and for all formulas χ, if $\deg(\chi) + n \leq 2k$ then $M, s_n \models \chi$ iff $M_s, (s_0, \ldots, s_n) \models \chi$. Since $M, s \models \varphi$, therefore $M_s, (s) \models \varphi$. Let M'' be the dense closure of (W'', R'', V'') where

- $W'' = W_s \cup W'$,
- $R'' = R_s \cup R' \cup \{((s, t_1, u_1, \ldots, t_k, u_k), s')\}$,
- $V'' = V_s \cup V'$.

The reader may easily verify that for all $(s_0, \ldots, s_n) \in W_s$ and for all formulas χ, if $\deg(\chi) + n \leq 2k$ then $M_s, (s_0, \ldots, s_n) \models \chi$ iff $M'', (s_0, \ldots, s_n) \models \chi$. Since $M_s, (s) \models \varphi$, therefore $M'', (s) \models \varphi$. The reader may also easily verify that for all $t' \in W'$ and for all formulas χ, $M', t' \models \chi$ iff $M'', t' \models \chi$. Since $M', s' \models q$ and $M', s' \not\models \psi$, therefore $M'', s' \models q$ and $M'', s' \not\models \psi$. Since $t_1, \ldots, t_k \in V(q)$, $u_1, \ldots, u_k \in V(p)$, $(s)R_s(s, t_1)R_s(s, t_1, u_1) \ldots R_s(s, t_1, u_1, \ldots, t_k)R_s(s, t_1, u_1, \ldots, t_k, u_k)$ and $M'', (s) \models \varphi$, therefore $M'', (s) \not\models \varphi \to [qp]^k[q]\psi$. Thus, $\varphi \to [qp]^k[q]\psi \notin K_1^2$. □

PROPOSITION 7.23
Let $m, n \geq 1$. If $m > n$ and $(m, n) \neq (2, 1)$ then the normal modal logic K_m^n is not adequate.

PROOF. Similar to the proof of Proposition 7.21, defining this time $\varphi = \Box^n r$ and $k = \lceil \frac{n+2}{2(m-n)} \rceil (m-n) - 1$. □

8 Unification in adequate normal modal logics

Let L be a normal modal logic. In this section, we shall prove that if L is adequate then L is nullary. Let φ be the formula considered in Section 6. Let $\sigma_\bot$, $\sigma_\top$ and for all $k \geq 0$, σ_k be the substitutions considered in Section 6. Remind that $\sigma_\bot$, $\sigma_\top$ and for all $k \geq 0$, σ_k are L-unifiers of φ.

LEMMA 8.1
Let σ be a substitution and $k \geq 0$. If L is adequate, σ is an L-unifier of φ, $\deg(\sigma(x)) \leq 2k$ and $\deg(\sigma(y)) \leq 2k$ then $\sigma_\top \preceq_L \sigma$ or $\sigma_k \preceq_L \sigma$.

PROOF. Suppose L is adequate, σ is an L-unifier of φ, $\deg(\sigma(x)) \leq 2k$, $\deg(\sigma(y)) \leq 2k$, $\sigma_\top \not\preceq_L \sigma$ and $\sigma_k \not\preceq_L \sigma$. Hence, by Lemma 6.3, $\sigma(x) \leftrightarrow p \notin L$ or $\sigma(y) \leftrightarrow q \notin L$. Since σ is an L-unifier of φ, therefore $\sigma(\varphi) \in L$. Since $\sigma(\varphi) = (\sigma(x) \rightarrow p) \wedge (\sigma(y) \rightarrow q) \wedge (\sigma(x) \rightarrow [q]\sigma(y)) \wedge (\sigma(y) \rightarrow [p]\sigma(x))$, therefore $(\sigma(x) \rightarrow p) \wedge (\sigma(y) \rightarrow q) \wedge (\sigma(x) \rightarrow [q]\sigma(y)) \wedge (\sigma(y) \rightarrow [p]\sigma(x)) \in L$. Thus, $\sigma(x) \rightarrow p \in L$, $\sigma(y) \rightarrow q \in L$, $\sigma(x) \rightarrow [q]\sigma(y) \in L$ and $\sigma(y) \rightarrow [p]\sigma(x) \in L$. Since σ is an L-unifier of φ and $\sigma_k \not\preceq_L \sigma$, therefore by Lemma 6.5, $\sigma(x) \rightarrow [qp]^k[q]\bot \notin L$ or $\sigma(y) \rightarrow [pq]^k[p]\bot \notin L$.

Case '$\sigma(x) \leftrightarrow p \notin L$ and $\sigma(x) \rightarrow [qp]^k[q]\bot \notin L$'. Since $\sigma(x) \rightarrow p \in L$, therefore $p \rightarrow \sigma(x) \notin L$. Since L is adequate, $\deg(\sigma(x)) \leq 2k$ and $\sigma(x) \rightarrow [qp]^k[q]\bot \notin L$, therefore $\sigma(x) \rightarrow [qp]^k[q][p]\sigma(x) \notin L$. Hence, $\sigma(x) \rightarrow [q]\sigma(y) \notin L$ or $\sigma(y) \rightarrow [p]\sigma(x) \notin L$: a contradiction.

Case '$\sigma(x) \leftrightarrow p \notin L$ and $\sigma(y) \rightarrow [pq]^k[p]\bot \notin L$'. Since $\sigma(x) \rightarrow p \in L$, therefore $p \rightarrow \sigma(x) \notin L$. Since $\sigma(y) \rightarrow [pq]^k[p]\bot \notin L$, therefore $\sigma(y) \rightarrow [pq]^k\bot \notin L$. Since L is adequate, $\deg(\sigma(y)) \leq 2k$ and $p \rightarrow \sigma(x) \notin L$, therefore $\sigma(y) \rightarrow [pq]^k[p]\sigma(x) \notin L$. Hence, $\sigma(x) \rightarrow [q]\sigma(y) \notin L$ or $\sigma(y) \rightarrow [p]\sigma(x) \notin L$: a contradiction.

Case '$\sigma(y) \leftrightarrow q \notin L$ and $\sigma(x) \rightarrow [qp]^k[q]\bot \notin L$'. Since $\sigma(y) \rightarrow q \in L$, therefore $q \rightarrow \sigma(y) \notin L$. Since $\sigma(x) \rightarrow [qp]^k[q]\bot \notin L$, therefore $\sigma(x) \rightarrow [qp]^k\bot \notin L$. Since L is adequate, $\deg(\sigma(x)) \leq 2k$ and $q \rightarrow \sigma(y) \notin L$, therefore $\sigma(x) \rightarrow [qp]^k[q]\sigma(y) \notin L$. Hence, $\sigma(x) \rightarrow [q]\sigma(y) \notin L$ or $\sigma(y) \rightarrow [p]\sigma(x) \notin L$: a contradiction.

Case '$\sigma(y) \leftrightarrow q \notin L$ and $\sigma(y) \rightarrow [pq]^k[p]\bot \notin L$'. Since $\sigma(y) \rightarrow q \in L$, therefore $q \rightarrow \sigma(y) \notin L$. Since L is adequate, $\deg(\sigma(y)) \leq 2k$ and $\sigma(y) \rightarrow [pq]^k[p]\bot \notin L$, therefore $\sigma(y) \rightarrow [pq]^k[p][q]\sigma(y) \notin L$. Hence, $\sigma(x) \rightarrow [q]\sigma(y) \notin L$ or $\sigma(y) \rightarrow [p]\sigma(x) \notin L$: a contradiction. □

LEMMA 8.2
If L is adequate then there exists no L-minimal L-complete set of L-unifiers of φ.

PROOF. Suppose L is adequate and there exists an L-minimal L-complete set of L-unifiers of φ. Let Σ be an L-minimal L-complete set of L-unifiers of φ. By the fact that Σ is an L-complete set of L-unifiers of φ, let $\sigma \in \Sigma$ be such that $\sigma \preceq_L \sigma_\bot$. Let $k \geq 0$ be such that $\deg(\sigma(x)) \leq 2k$ and $\deg(\sigma(y)) \leq 2k$. Since L is adequate and $\sigma \in \Sigma$, therefore by Lemma 8.1 and the fact that Σ is a set of L-unifiers of φ, $\sigma_\top \preceq_L \sigma$ or $\sigma_k \preceq_L \sigma$.

Case '$\sigma_\top \preceq_L \sigma$'. Since $\sigma \preceq_L \sigma_\bot$, therefore by Lemma 5.2, $\sigma_\top \preceq_L \sigma_\bot$. Let τ be a substitution such that $\sigma_\top \circ \tau \simeq_L \sigma_\bot$. Hence, $\tau(\sigma_\top(x)) \leftrightarrow \sigma_\bot(x) \in L$ and $\tau(\sigma_\top(y)) \leftrightarrow \sigma_\bot(y) \in L$. Since $\tau(\sigma_\top(x)) = p$, $\sigma_\bot(x) = \bot$, $\tau(\sigma_\top(y)) = q$ and $\sigma_\bot(y) = \bot$, therefore $p \leftrightarrow \bot \in L$ and $q \leftrightarrow \bot \in L$. Thus, $\neg p \wedge \neg q \in L$. Consequently, $\bot \in L$. Hence, L is not coherent. Thus, L is not adequate: a contradiction.

Case '$\sigma_k \preceq_L \sigma$'. By the fact that Σ is an L-complete set of L-unifiers of φ, let $\tau \in \Sigma$ be such that $\tau \preceq_L \sigma_{k+1}$. By Lemma 4.1, $[qp]^k[q]\bot \to [qp]^{k+1}[q]\bot \in L$ and $[pq]^k[p]\bot \to [pq]^{k+1}[p]\bot \in L$. Hence, $p \wedge [qp]^k[q]\bot \to [qp]^{k+1}[q]\bot \in L$ and $q \wedge [pq]^k[p]\bot \to [pq]^{k+1}[p]\bot \in L$. Thus, by Lemma 6.6, $\sigma_{k+1} \preceq_L \sigma_k$. Since $\sigma_k \preceq_L \sigma$ and $\tau \preceq_L \sigma_{k+1}$, therefore by Lemma 5.2, $\tau \preceq_L \sigma$. Since $\sigma, \tau \in \Sigma$, therefore by the fact that Σ is an L-minimal set of substitutions, $\tau \simeq_L \sigma$. Since $\sigma_k \preceq_L \sigma$ and $\tau \preceq_L \sigma_{k+1}$, therefore by Lemmas 5.1 and 5.2, $\sigma_k \preceq_L \sigma_{k+1}$. Consequently, by Lemma 6.6, $p \wedge [qp]^{k+1}[q]\bot \to [qp]^k[q]\bot \in L$ and $q \wedge [pq]^{k+1}[p]\bot \to [pq]^k[p]\bot \in L$. Hence, L is not coherent. Thus, L is not adequate: a contradiction. □

PROPOSITION 8.3
If L is adequate then L is nullary.

PROOF. By Lemma 8.2. □

COROLLARY 8.4
The following normal modal logics are nullary:

- the normal modal logics K, KD and KT,
- for all $n \geq 1$, the normal modal logic Alt_n,
- for all $m, n \geq 1$, if $m \leq n$ then the normal modal logic K^n_m.

TABLE 1. Known facts and open problems in the determination of the type of unification with constants in some of the most popular normal modal logics

L	Type of L for unif. with constants
K	Nullary—[21]
KD	Nullary—[4]
KT	Nullary—Corollary 8.4
KB	?
KDB	?
KTB	?
$K5$	?
$KD5$	?
$K45$	Unitary or nullary—[5]
$KD45$	Unitary or nullary—[5]
$S5$	Unitary—[1]
$K4$	Finitary—[16]
$S4$	Finitary—[16]
$K4.3$	?
$KD4.3$	?
$S4.3$	Unitary—[14]
GL	Finitary—[16]
K^{bou}_1	Unitary—see Section 5
K^{bou}_n when $n \geq 2$	?
Alt_1	Nullary—[6]
Alt_n when $n \geq 2$	Nullary—see Section 5
K^n_m when $m \leq n$	Nullary—Corollary 8.4
K^n_m when $m > n$ and $(m, n) \neq (2, 1)$	?

TABLE 2. Known facts and open problems in the determination of the type of elementary unification in some of the most popular normal modal logics

L	Type of L for elementary unif.
K	Nullary—[21]
KD	?
KT	?
KB	?
KDB	?
KTB	?
$K5$	?
$KD5$	?
$K45$	Unitary or nullary—[5]
$KD45$	Unitary—[7]
$S5$	Unitary—[1]
$K4$	Finitary—[16]
$S4$	Finitary—[16]
$K4.3$	?
$KD4.3$	?
$S4.3$	Unitary—[14]
GL	Finitary—[16]
K_1^{bou}	Unitary—see Section 5
K_n^{bou} when $n \geq 2$	?
Alt_1	Nullary—[6]
Alt_n when $n \geq 2$	Nullary—see Section 5
K_m^n when $m \leq n$	?
K_m^n when $m > n$ and $(m, n) \neq (2, 1)$	?

PROOF. By Propositions 7.15–7.17, 7.19, 7.22 and 8.3. □

9 Conclusion and open problems

In this paper, we have proved that unification in a normal modal logic L is of nullary type when L is adequate. Remark that in order to present our proof above, we had to assume that the language of modal logic contains at least two distinct variables and two distinct constants. Note that the nullariness of K has been proved by Jeřàbek [21] who only assumed that the language contains at least one variable. This shows that K is nullary both for unification with constants and for elementary unification. As well, note that the nullariness of KD has been proved by Balbiani and Gencer [4] who only assumed that the language contains at least one variable and one constant. This only shows that KD is nullary for unification with constants. As for the nullariness of Alt_1, it has also been proved by Balbiani and Tinchev [6] who only assumed that the language contains at least one variable. This shows that Alt_1 is nullary both for unification with constants and for elementary unification. As mentioned in Section 5, the proof given in [6] that Alt_1 is nullary can be easily adapted for all $n \geq 2$ to a proof that Alt_n is nullary. Much remains to be done. See Tables 1 and 2 for known facts and open problems in the determination of the type of unification with constants and the type of elementary unification in some of the most popular normal modal logics.

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