Remarks on a Stefan Problem

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1. Introduction. This paper is in two parts and is concerned with Stefan problems of the form

$$(1.1)\qquad \begin{cases} Lu \equiv u_{xx} - u_t = 0 & \text{in } 0 < x < s(t) \text{ for } 0 < t < \infty \\ u(0, t) = f(t) \geq 0 & \text{and } u(s(t), t) = 0 \text{ for } 0 < t < \infty \\ u(x, 0) = \phi(x) \geqq 0 & \text{for } 0 \leqq x \leqq b, \quad s(0) = b \geqq 0 \end{cases}$$

$$(1.2)\qquad \dot{s}(t) = -u_x(s(t), t) \quad \text{for} \quad 0 < t < \infty.$$

In Part II we obtain precise results on the asymptotic behavior of the free boundary $x = s(t)$ as $t \rightarrow \infty$. Theorem 4 gives a necessary and sufficient condition on $f(t)$ in order that $s(t)$ approach a finite limit. Theorem 6 describes the asymptotic behavior of $s(t)$ for any $f(t)$ which is bounded and goes to zero as $t \rightarrow \infty$. Theorem 7 shows, in particular, that $s(t)$ is asymptotic to a parabola whenever $f(t)$ is asymptotic to a positive constant.

When $b = 0$ the above Stefan problem contains a difficulty not present if $b > 0$. This is due to the fact that when $b = 0$ the free boundary $x = s(t)$ actually emanates from a point at which $u(x, t)$ may be discontinuous (unless $f(t)$ has a zero at $t = 0$).

In Part I we prove an existence theorem for the case where $b = 0$. That existence theorem is actually used in Part II in treating the asymptotic behavior of $s(t)$. In Part I we also give a self-contained proof of the monotone dependence of the free boundary on the data which uses nothing more than the parabolic version of Hopf's lemma [4]. From the monotone dependence follows an especially elegant proof of uniqueness that covers the cases $b = 0$ and $b > 0$ simultaneously.

The results of Part I and Part II settle some problems posed as open questions by Friedman in [2] and [3].

Although existence and uniqueness in the case $b > 0$ has been widely discussed (for references, see [4]), the case $b = 0$ does not seem to have been thoroughly treated. In [1] the case $b = 0$ was treated under the hypothesis that, at $t = 0$, $f(t)$ had a zero of a prescribed order. However, Stefan himself, in his original

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work [6], gave an explicit solution for the case where $b = 0$ and $f \equiv$ const. > 0.

The identity

$$(1.3) \qquad s(t)^2 = b^2 + 2\int_0^b \xi\,\phi(\xi)\,d\xi + 2\int_0^t f(\tau)\,d\tau - 2\int_0^{s(t)} \xi\,u(\xi, t)\,d\xi$$

derived from Stoke's theorem plays a key role throughout this work. It is actually an equivalent formulation of the free boundary condition (1.2) (see [1]). Upper bounds on $s(t)$ follow immediately from (1.3). Lower bounds may be obtained by the technique of Part II which uses comparison problems for which $b = 0$ and the following lemma.

Lemma 1. *Let* (s_1, u_1) *and* (s_2, u_2) *be solutions to the Stefan problem* (1.1), (1.2) *corresponding, respectively, to the data* (b_1, ϕ_1, f_1) *and* (b_2, ϕ_2, f_2). *If* $b_1 < b_2$, $\phi_1 \leqq \phi_2$, *and* $f_1 \leqq f_2$ *then* $s_1 < s_2$.

Proof. First we rule out the trivial case where $\phi_1 \equiv 0$ for $0 \leqq x \leqq b_1$ and $f \equiv 0$ in some neighborhood of $t = 0$.

Assume the contrary to the assertion of the lemma and let t_0 ($t_0 > 0$) be the first t for which $s_1(t) = s_2(t)$. Then $\dot{s}_1(t_0) \geqq \dot{s}_2(t_0)$. By the strong maximum principle, $u_2(s_1(t), t) > 0$ for $0 < t < t_0$. Hence $u_2 - u_1 > 0$ in the region $0 < x < s_1(t)$, $0 < t < t_0$. Since both u_1 and u_2 vanish at the point $(s_1(t_0), t_0)$ it follows from the parabolic version of Hopf's lemma that $\dot{s}_1(t_0) - \dot{s}_2(t_0) = u_{2x}(s_1(t_0), t_0) - u_{1x}(s_1(t_0), t_0) < 0$. This is a contradiction. Q.E.D.

The formula

$$(1.4) \qquad v(x, t) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{\partial^n}{\partial t^n} [x - s(t)]^{2n}$$

provides, whenever it makes sense, an explicit solution to the inverse problem associated with (1.1), (1.2). That is if the free boundary $s(t)$ is given, then $v(x, t)$ is a solution to the heat equation such that $v(s(t), t) = 0$ and $v_x(s(t), t) = -\dot{s}(t)$ [5]. Formula (1.4) can be used to generate an assortment of barriers and comparison functions which when used in conjunction with the maximum principle and Lemma 1 yield sharp *a priori* bounds for the solution (s, u) of (1.1), (1.2). In particular, the choice $s(t) = ct^{1/2}$ gives an alternate representation of Stefan's solution.

PART I

2. Existence. The problem is the following: Given the data f(t), find two functions $s = s(t)$ and $u = u(s, t)$ such that the pair (s, u) satisfies

$$(2.1) \qquad \begin{cases} Lu \equiv u_{xx} - u_t = 0 & \text{in} \quad 0 < x < s(t) \quad \text{for} \quad 0 < t < \infty \\ u(0, t) = f(t) \geqq 0 \quad \text{and} & u(s(t), t) = 0 \quad \text{for} \quad 0 < t < \infty \\ s(0) = 0 & \end{cases}$$

and

$$\dot{s}(t) = -u_x(s(t), t) \quad \text{for} \quad 0 < t < \infty. \tag{2.2}$$

The assumptions on f are: $f(0) \neq 0$, $f(t) \geqq 0$, and f is continuous except possibly for a finite number of bounded jumps on each compact interval $[0, T]$. For uniqueness we assume u is bounded.

Theorem 1. *Under the assumptions listed above on f, there exists a solution (s, u) to the Stefan problem (2.1), (2.2). The free boundary $s(t) \in C^1(0, \infty)$ is monotonically increasing, and in a neighborhood of zero satisfies an inequality of the form*

$$c_1 t^{1/2} \leqq s(t) \leqq c_2 t^{1/2} \tag{2.3}$$

with c_1, $c_2 > 0$.

In what follows we shall often use the results of [1]. According to the assumptions made on f there is an interval $[0, \eta]$ on which f is continuous and strictly positive. Let M_1 and M_2, $0 < M_1 \leqq M_2$, denote, respectively, the minimum and the maximum of f on $[0, \eta]$. It will suffice to confine our attention to the interval $[0, \eta]$ since, as we shall show, the results of [1] will apply in the interval $[\eta, \infty]$.

Consider problem (1.1), (1.2) for the case in which $\phi \equiv 0$. For any $b > 0$ it has, according to [1], a unique solution $s \equiv s^b$, $u \equiv u^b$. Since $\phi \equiv 0$ the equivalent free boundary condition (1.3) becomes

$$s^b(t)^2 = b^2 + 2 \int_0^t f(\tau)\, d\tau - 2 \int_0^{s^b(t)} \xi\, u^b(\xi, t)\, d\xi. \tag{2.4}$$

Lemma 2. *There are positive constants c_1 and c_2 such that*

$$c_1 t^{1/2} < s^b(t) < b + c_2 t^{1/2} \tag{2.5}$$

for $0 \leqq t \leqq \eta$.

Proof. Consider, for $c > 0$, the function

$$v(x, t) = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{\partial^n}{\partial t^n} [x - ct^{1/2}]^{2n} \tag{2.6}$$

obtained from (1.4) by setting $s(t) = ct^{1/2}$. Then $Lv = 0$, $v(ct^{1/2}, t) = 0$, $-v_x(ct^{1/2}, t) = \frac{1}{2}ct^{-1/2}$, and it is easy to check that

$$\exp\left\{\frac{c^2}{4}\right\} - 1 \leqq v(0, t) \leqq \exp\{c^2\} - 1. \tag{2.7}$$

Moreover, one can check that v is bounded on $0 \leqq x \leqq ct^{1/2}$, $0 \leqq t \leqq \eta$.

First set $c = c_1 \equiv [\log(1 + M_1)]^{1/2}$ in (2.6). For t sufficiently small, $c_1 t^{1/2} < b \leqq s^b(t)$. Hence Lemma 1 applies and we conclude that $c_1 t^{1/2} < s^b(t)$.

For the second inequality in (2.5) we set $c = c_2 \equiv 2[\log(1 + M_2)]^{1/2}$ in (2.6) and replace x by $x - b$. It follows from [1] that $s^b(t) < b + c_2 t^{1/2}$ for t sufficiently small. Applying Lemma 1 we conclude that $s^b(t) < b + c_2 t^{1/2}$. Q.E.D.

Lemma 3. Let u be any solution of (1.1) *with* $\phi \equiv 0$ *and* $s(t)$ *nondecreasing. Then*

$$0 \leqq \rho^{-1}u(s(t) - \rho, t) \leqq M_2c_1^{-1}t^{-1/2} \quad \textit{for all} \quad 0 < t \leqq \eta \tag{2.8}$$

and $0 < \rho < s(t)$.

Proof. Fix $t_0 > 0$ and set $\omega(x, t) \equiv M_2c_1^{-1}t_0^{-1/2}[s(t_0) - x]$. Then $L\omega = 0$, $\omega(s(t), t) > 0$ for $0 \leqq t \leqq t_0$, $\omega(x, 0) \geqq 0$, and $\omega(0, t) \geqq M_2 \geqq f(t)$ by Lemma 2. Hence by the maximum principle $u(x, t) \leqq \omega(x, t)$ in the region $0 \leqq x \leqq s(t)$, $0 \leqq t \leqq t_0$. Therefore $\rho^{-1}u(s(t_0) - \rho, t_0) \leqq \rho^{-1}M_2c_1^{-1}t_0^{-1/2}\rho = M_2c_1^{-1}t_0^{-1/2}$. Q.E.D.

Lemma 4. For $0 < t \leqq \eta$ *we have*

$$0 < \dot{s}^b(t) \leqq M_2c_1^{-1}t^{-1/2}. \tag{2.9}$$

Proof. Lemma 4 is an immediate consequence of Lemma 3.

Proof of Theorem 1. Let $0 < b \leqq 1$ and set $\sigma^b(t) \equiv ts^b(t)$. Then $0 \leqq \sigma^b(t) \leqq \eta(1 + c_2\eta^{1/2})$ for $0 \leqq t \leqq \eta$ by Lemma 2. Since $\dot{\sigma}^b(t) = s^b(t) + t\dot{s}^b(t)$ it follows from Lemmas 2 and 4 that $0 \leqq \dot{\sigma}^b(t) \leqq 1 + (c_2 + M_2c_1^{-1})\eta^{1/2}$ on $[0, \eta]$. Thus the $\sigma^b(t)$ form an equicontinuous uniformly bounded family on $[0, \eta]$. Choose a sequence of b's tending monotonically to zero. By Arzela's theorem there is a subsequence, denote it by $\sigma_n(t)$, that converges uniformly to a monotonic Lipschitz continuous function $\sigma(t)$.

Now set $s_n(t) = t^{-1}\sigma_n(t)$, $s(t) = t^{-1}\sigma(t)$. It follows from Lemma 2 that $c_1t^{1/2} \leqq s(t) \leqq c_2t^{1/2}$. Moreover, by Lemma 1, $s_{n+1}(t) \leqq s_n(t)$. Let $u(x, t)$ be the solution of (2.1) for this choice of $s(t)$. If $u_n(x, t)$ denotes the solution of (1.1), (1.2) corresponding to $s_n(t)$ then, by Lemma 3, $0 \leqq u_n(s(t), t) \leqq M_2c_1^{-1}t^{-1/2}(s_n(t) - s(t))$. Hence, using the maximum principle, $|u - u_n| = o(t_0^{-3/2})$ on the region $0 \leqq x \leqq s(t)$, $t_0 < t \leqq \eta$, $t_0 > 0$, as n tends to infinity.

For each (s_n, u_n) we have, from (1.3), that

$$s_n(t)^2 = b_n^2 + 2\int_0^t f(\tau)\, d\tau - 2\int_0^{s_n(t)} \xi\, u_n(\xi, t)\, d\xi, \tag{2.10}$$

and taking the limit as $n \to \infty$ we obtain

$$s(t)^2 = 2\int_0^t f(\tau)\, d\tau - 2\int_0^{s(t)} \xi\, u(\xi, t)\, d\xi, \quad t > 0. \tag{2.11}$$

Now using Lemma 2 and Lemma 1 of [1], it follows from the Lipschitz continuity of $s(t)$ on regions $0 < \delta \leqq t \leqq \eta$ that $u_x(s(t), t)$ exists and is continuous for $0 < t \leqq \eta$ and, finally, that (2.11) is equivalent to (2.2).

This completes the existence proof in the interval $[0, \eta]$. But, taking note of Lemma 3, it is seen that Theorem 1 of [1] applies in the interval $[\eta, \infty]$. Q.E.D.

3. Monotone dependence. We now return to the general situation of problem (1.1), (1.2).

If $b > 0$ we assume that $f(t) \geqq 0$ is continuous except possibly for a finite number of bounded jumps on each compact interval $[0, T]$, that $\phi(x)$ is similarly piecewise continuous, and that there is a positive constant N such that $0 \leqq \phi(x) \leqq N(b - x)$ for $0 \leqq x \leqq b$.

If $b = 0$ there is no ϕ and it suffices to make one additional assumption on f: either that $f(0) \neq 0$ (as in section 2) or that $f(t)$ has a zero at $t = 0$ of a prescribed order (as in [1]).

Under the above assumptions we have, combining the results of [1] and those of the previous section, that solutions of (1.1), (1.2) exist. Their uniqueness will follow from the monotonicity theorem proved in this section.

Let (s_1, u_1) be a solution of (1.1), (1.2) corresponding to the data (b_1, ϕ_1, f_1) and let (s_2, u_2) be a solution corresponding to the data (b_2, ϕ_2, f_2).

Theorem 2. *If $b_1 \leqq b_2$, $\phi_1 \leqq \phi_2$, and $f_1 \leqq f_2$ then $s_1 \leqq s_2$.*

Proof. For $\delta > 0$ let (s_2^δ, u_2^δ) be a solution of (1.1), (1.2) corresponding to the data $(b_2 + \delta, \phi_2, f_2)$. Here it is understood that $\phi_2 \equiv 0$ outside its original domain of definition. Using Lemma 1 we have $s_1(t) < s_2^\delta(t)$.

Now (s_2, u_2) and (s_2^δ, u_2^δ), being solutions of (1.1), (1.2), must each satisfy their respective version of (1.3). We subtract them to obtain

$$s_2^\delta(t)^2 - s_2(t)^2 = (b_2 + \delta)^2 - b_2^2 - 2\int_0^{s_2(t)} \xi[u_2^\delta(\xi, t) - u_2(\xi, t)]\, d\xi - 2\int_{s_2(t)}^{s_2^\delta(t)} \xi\, u_2^\delta(\xi, t)\, d\xi.$$

But by the maximum principle we have $u_2^\delta \geqq 0$ and $u_2^\delta - u_2 \geqq 0$. Hence $s_2^\delta(t)^2 \leqq s_2(t)^2 + (b_2 + \delta)^2 - b_2^2$.

Therefore $s_1(t)^2 < s_2(t)^2 + (b_2 + \delta)^2 - b_2^2$. Since $\delta > 0$ can be picked as small as desired we obtain $s_1(t)^2 \leqq s_2(t)^2$ and thus $s_1(t) \leqq s_2(t)$. Q.E.D.

4. Uniqueness.

Theorem 3. *Under the assumptions listed in section 3 the Stefan problem* (1.1), (1.2) *has a unique solution.*

Proof. If (s_1, u_1) and (s_2, u_2) are two solutions, then $s_1 \equiv s_2$ by the monotonicity theorem. Hence $u_1 \equiv u_2$ as well. Q.E.D.

Part II

5. Asymptotic behavior of the free boundary. Throughout this section we shall be concerned with the asymptotic behavior, as $t \to \infty$, of the free boundary $x = s(t)$ for the Stefan problem (1.1), (1.2). Under the assumptions listed at the beginning of section 3 we are assured of the existence and uniqueness of the solution (s, u) to (1.1), (1.2). Moreover, the free boundary condition (1.2) is equivalent to the identity (1.3).

Theorem 4. If

$$\int_0^\infty f(\tau)\, d\tau = +\infty \quad \textit{then} \quad \lim_{t\to\infty} s(t) = +\infty.$$

If

$$\int_0^\infty f(\tau)\, d\tau < \infty \quad \textit{then} \quad \lim_{t\to\infty} s(t) = l^{1/2},$$

where

$$l = b^2 + 2\int_0^b \xi\, \phi(\xi)\, d\xi + 2\int_0^\infty f(\tau)\, d\tau. \tag{5.1}$$

Proof. Consider first the case where f has compact support. Since $u \geqq 0$ it follows from (1.3) that $s(t) \leqq l^{1/2}$. But $0 \leqq u \leqq w$ where w is the solution of

$$\begin{cases} Lw = 0 \quad \text{in} \quad 0 < x < l^{1/2} \\ u(0, t) = f(t) \quad \text{and} \quad u(l^{1/2}, t) = 0 \quad \text{for} \quad 0 \leqq t \leqq \infty \\ u(x, 0) = \begin{cases} \phi(x), & 0 \leqq x \leqq b \\ 0, & b \leqq x \leqq l^{1/2}. \end{cases} \end{cases} \tag{5.2}$$

Since f has compact support it is clear that $w(x, t) \to 0$ uniformly in x as $t \to \infty$. Hence

$$0 \leqq \int_0^{s(t)} \xi\, u(\xi, t)\, d\xi \leqq \int_0^{l^{1/2}} \xi\, w(\xi, t)\, d\xi \to 0$$

as $t \to \infty$. This proves $\lim_{t\to\infty} s(t) = l^{1/2}$ for all f with compact support.

For general f set

$$f_n(t) = \begin{cases} f(t), & 0 \leqq t \leqq n \\ 0, & n < t < \infty \end{cases}$$

and define the corresponding l_n and s_n. By the monotonicity theorem $s_n(t) \leqq s(t) \leqq l^{1/2}$. According to the previous paragraph $l_n^{1/2} = \lim_{t\to\infty} s_n(t) \leqq \underline{\lim}_{t\to\infty} s(t) \leqq \overline{\lim}_{t\to\infty} s(t) \leqq l^{1/2}$. Now let $n \to \infty$. Since $l_n \to l$ the desired results follow. Q.E.D.

Theorem 5. Assume $\int_0^\infty f(\tau)\, d\tau = +\infty$ *and let* (s, u) *be the solution of* (1.1), (1.2). *For any* $t_0 \geqq 0$ *such that* $f(t_0) \neq 0$, *let* (σ, v) *be the solution of*

$$\begin{cases} Lv = 0 \quad \textit{in} \quad 0 < x < \sigma(t), \quad t_0 < t < \infty \\ v(0, t) = f(t) \quad \textit{and} \quad v(\sigma(t), t) = 0 \quad \textit{for} \quad t_0 < t < \infty \\ \sigma(t_0) = 0 \end{cases} \tag{5.3}$$

and

$$\dot{\sigma}(t) = -v_x(\sigma(t), t) \quad \textit{for} \quad t_0 < t < \infty. \tag{5.4}$$

Then as $t \to \infty$,

$$\frac{s(t)}{\sigma(t)} = 1 + O\left(\frac{1}{\sigma(t)}\right). \tag{5.5}$$

In particular, $s(t) \sim \sigma(t)$.

Proof. Using once again Lemma 1, we have $\sigma(t) \leqq s(t)$. Hence $v \leqq u$ for $0 \leqq x \leqq \sigma(t)$, $t_0 \leqq t$. Therefore, by (1.3),

$$s(t)^2 - s(t_0)^2 - 2\int_0^{s(t_0)} \xi\, u(\xi, t_0)\, d\xi = 2\int_{t_0}^t f(\tau)\, d\tau - 2\int_0^{s(t)} \xi\, u(\xi, t)\, d\xi$$

$$\leqq 2\int_{t_0}^t f(\tau)\, d\tau - 2\int_0^{\sigma(t)} \xi\, v(\xi, t)\, d\xi = \sigma(t)^2.$$

Hence

$$1 \leqq \frac{s(t)^2}{\sigma(t)^2} \leqq 1 + \left\{s(t_0)^2 + 2\int_0^{s(t_0)} \xi\, u(\xi, t_0)\, d\xi\right\}\sigma(t)^{-2}$$

$$= 1 + O\left(\left(\frac{1}{\sigma(t)}\right)^2\right) \qquad \text{Q.E.D.}$$

For any $0 \leqq t_0 \leqq t \leqq \infty$ set $\|f\|_{t_0}^t \equiv \max_{t_0 \leqq \tau \leqq t} f(\tau)$. In the next theorem we shall consider only those f for which $\|f\|_{t_0}^\infty$ is finite.

Theorem 6. *Assume that*

$$\int_0^\infty f(\tau)\, d\tau = +\infty \quad \textit{and} \quad \lim_{t_0 \to \infty} \|f\|_{t_0}^\infty = 0.$$

Let (s, u) be the solution of (1.1), (1.2). Then, as $t \to \infty$,

$$s(t) \sim \left(2\int_0^t f(\tau)\, d\tau\right)^{1/2}. \tag{5.6}$$

Corollary. *If, for $\rho > 0$ and $0 < \delta < 1$, $f(t) \sim \rho t^{-\delta}$, then*

$$s(t) \sim \left(\frac{2\rho}{1-\delta}\right)^{1/2} t^{(1-\delta)/2}.$$

Corollary. *If, for $\rho > 0$, $f(t) \sim \rho t^{-1}$, then $s(t) \sim (2\rho \log t)^{1/2}$.*

Proof of theorem 6. Choose any t_0 such that $f(t_0) \neq 0$ and hold it fixed. Let (σ, v) be the corresponding solution of (5.3), (5.4). By the maximum principle $0 \leqq v(x, t) \leqq \|f\|_{t_0}^t$. Since

$$\sigma(t)^2 = 2\int_{t_0}^t f(\tau)\, d\tau - 2\int_0^{\sigma(t)} \xi\, v(\xi, t)\, d\xi$$

$$\geqq 2\int_{t_0}^t f(\tau)\, d\tau - \|f\|_{t_0}^t\, \sigma(t)^2,$$

we have

$$\left(\frac{1}{1+\|f\|_{t_0}^{t}}\right) 2\int_{t_0}^{t} f(\tau)\,d\tau \leqq \sigma(t)^2. \tag{5.7}$$

Now, as $u \geqq 0$,

$$\begin{aligned}\sigma(t)^2 \leqq s(t)^2 &= b^2 + 2\int_0^b \xi\,\phi(\xi)\,d\xi + 2\int_0^t f(\tau)\,d\tau - 2\int_0^{s(t)} \xi\,u(\xi, t)\,d\xi \\ &\leqq b^2 + 2\int_0^b \xi\,\phi(\xi)\,d\xi + 2\int_0^t f(\tau)\,d\tau.\end{aligned}$$

Upon dividing the above equation by $2\int_0^t f(\tau)\,d\tau$, using (5.7), and taking the limit as $t \to \infty$, we obtain

$$\left(\frac{1}{1+\|f\|_{t_0}^{\infty}}\right) \leqq \varliminf_{t\to\infty} \frac{s(t)^2}{2\int_0^t f(\tau)\,d\tau} \leqq \varlimsup_{t\to\infty} \frac{s(t)^2}{2\int_0^t f(\tau)\,d\tau} \leqq 1.$$

Under the hypothesis on f we may now take t_0 arbitrarily large demonstrating that

$$\lim_{t\to\infty} \frac{s(t)^2}{2\int_0^t f(\tau)\,d\tau} = 1. \qquad \text{Q.E.D.}$$

The next theorem is concerned with those f which do not go to zero as $t \to \infty$.

Theorem 7. *For* $\rho > 0$ *let*

$1^0 \quad f(t) \sim \rho$

$2^0 \quad \log f(t) \sim \rho t^{\gamma} \qquad (0 < \gamma < \infty)$

and (s, u) *be the solution of* (1.1), (1.2). *Then*

$1^0 \quad s(t) \sim \beta t^{1/2}$

$2^0 \quad s(t) \sim \beta t^{(1+\gamma)/2}$

with $\beta = \beta(\rho, \gamma)$ *given, in each respective case, by*

$1^0 \quad \beta$ *is the solution of* $\rho = \sum_{n=1}^{\infty} \frac{n!}{(2n)!}\beta^{2n}$

$2^0 \quad \beta = 2\rho^{1/2}\gamma^{\gamma/2}(1+\gamma)^{-(1+\gamma)/2}.$

Proof. Set $\alpha = (1+\gamma)/2$, $0 \leqq \gamma < \infty$, and consider the function

$$V_\beta^\alpha(x, t) = \sum_{n=1}^{\infty} \frac{1}{(2n)!}\frac{\partial^n}{\partial t^n}[x - \beta t^\alpha]^{2n} \tag{5.8}$$

obtained from (1.4) by setting $s(t) = \beta t^\alpha$. Then V_β^α is the solution of (1.1), (1.2) for which the free boundary is βt^α. Since $V_\beta^\alpha(0, t)$ is given explicitly by

$$V_\beta^\alpha(0, t) = \sum_{n=1}^{\infty} \frac{\Gamma(2\alpha n + 1)}{\Gamma(2n + 1)\Gamma(\gamma n + 1)} \beta^{2n} t^{\gamma n},$$

it is not hard to verify that, as $t \to \infty$,

$$1^0 \qquad V_\beta^\alpha(0, t) \sim \rho \qquad (\gamma = 0)$$

$$2^0 \qquad \log V_\beta^\alpha(0, t) \sim \rho t^\gamma \qquad (0 < \gamma < \infty)$$

in each of the two cases provided that β is given in terms of ρ and γ as indicated in the theorem.

Now let ϵ, $0 < \epsilon < \beta$, be given. By the hypothesis on $f(t)$ and according to the asymptotic behavior of $V_\beta^\alpha(0, t)$ as described above, there is a t_0 such that $V_{\beta-\epsilon}^\alpha(0, t) \leqq f(t) \leqq V_{\beta+\epsilon}^\alpha(0, t)$ for $t \geqq t_0$. By choosing t_0 larger, if necessary, we can insure that $V_{\beta-\epsilon}^\alpha(0, t_0)$, $f(t_0)$, and $V_{\beta+\epsilon}^\alpha(0, t_0)$ are all $\neq 0$. Denote by $\sigma_{-\epsilon}(t)$, $\sigma(t)$, and $\sigma_{+\epsilon}(t)$ the free boundaries in the solutions of (5.3), (5.4) which correspond, respectively, to the data $V_{\beta-\epsilon}^\alpha(0, t)$, $f(t)$, and $V_{\beta+\epsilon}^\alpha(0, t)$, for $v(0, t)$. Then by the monotonicity theorem, $\sigma_{-\epsilon}(t) \leqq \sigma(t) \leqq \sigma_{+\epsilon}(t)$ for $t \geqq t_0$. But by Theorem 5, $\sigma_{-\epsilon}(t) \sim (\beta - \epsilon)t^\alpha$, $\sigma_{+\epsilon}(t) \sim (\beta + \epsilon)t^\alpha$, and $\sigma(t) \leqq s(t) \leqq \sigma(t) + O(1)$. Hence

$$\beta - \epsilon = \lim_{t\to\infty} \frac{\sigma_{-\epsilon}(t)}{t^\alpha} \leqq \varliminf_{t\to\infty} \frac{s(t)}{t^\alpha}$$

$$\leqq \varlimsup_{t\to\infty} \frac{s(t)}{t^\alpha} \leqq \lim_{t\to\infty} \frac{\sigma_{+\epsilon}(t) + O(1)}{t^\alpha} = \beta + \epsilon.$$

Since ϵ was arbitrary, $s(t) \sim \beta t^\alpha$. Q.E.D.

Bibliography

[1] J. R. Cannon & C. D. Hill, Existence, uniqueness, stability, and monotone dependence in a Stefan problem for the heat equation, *J. Math. Mech.*, **17** (1967) 1–20.

[2] A. Friedman, Free boundary problems for parabolic equations I. Melting of solids, *J. Math. Mech.*, **8** (1959) 499–518.

[3] A. Friedman, Remarks on Stefan-type free boundary problems for parabolic equations, *J. Math. Mech.*, 9 (1960) 885–904.

[4] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Inc., Englewood Cliffs, N. J., 1964.

[5] C. D. Hill, Parabolic equations in one space variable and the non-characteristic Cauchy problem, *Comm. Pure Appl. Math.*, **20** (1967) 619–633.

[6] J. Stefan, Über einige Probleme der Theorie der Wärmeleitung, *Geb.*, **98** (1889) 473–484.

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