

## REMARKS ON A TRIPLE OF $K$ -CONTACT STRUCTURES

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**Abstract.** We show that any  $K$ -contact 3-structure on a 7-dimensional Riemannian manifold is a Sasakian 3-structure. By this we see that Konishi's construction of a 3-Sasakian manifold over a quaternionic Kähler manifold works for dimension  $\geq 4$ . We also study the case of quaternionic Kähler manifolds of negative scalar curvature by defining a triple of  $K$ -contact structures of  $nS$ -type.

**1. Introduction.** A quaternionic Kähler manifold of dimension  $4r \geq 8$  is defined as a Riemannian manifold  $(M, g)$  whose holonomy group is a subgroup of  $Sp(r) \cdot Sp(1)$ . If the dimension of  $M$  is  $4r=4$ , then the condition that the holonomy group is a subgroup of  $Sp(1) \cdot Sp(1) = SO(4)$  is nothing but the orientability of  $M$ . After the work of Ward [33], in 1991 LeBrun [24] defined a quaternionic Kähler manifold of dimension 4 to be a half-conformally flat Einstein manifold with nonzero scalar curvature. In the same year, noticing a property of a 4-dimensional quaternionic submanifold of a quaternionic Kähler manifold obtained by Marchiafava, Swann [29] gave an analogous definition.

On the other hand, a Riemannian manifold  $(\tilde{M}, \tilde{g})$  admitting a Sasakian 3-structure is called a 3-Sasakian manifold (cf. [9], [10]). In their papers [8], [9] and [10] Boyer-Galicki-Mann completed the classification of homogeneous 3-Sasakian manifolds and gave many examples of strongly inhomogeneous 3-Sasakian manifolds among other results.

It is known that a complete and regular 3-Sasakian manifold has the canonical fibering with fiber  $Sp(1)$  or  $SO(3)$  and the base manifold is a quaternionic Kähler manifold (Ishihara [16], etc.). Konishi [20] studied the converse (for  $\dim M = 4r \geq 8$ ).

Now we obtain the following:

**THEOREM A.** *Any  $K$ -contact 3-structure  $\{\eta_1, \eta_2, \eta_3\}$  on a 7-dimensional Riemannian manifold  $(\tilde{M}, \tilde{g})$  is a Sasakian 3-structure.*

By Theorem A we see that Konishi's theorem works for  $\dim M \geq 4$ .

Concerning Konishi's theorem, if the scalar curvature of a quaternionic Kähler manifold is negative, then the induced 3-structure  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  on the principal

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$SO(3)$ -bundle is a pseudo-Sasakian one such that the signature of  $\tilde{g}$  is  $(3, 4r)$  (cf. [20]), and the fibering can not be a Riemannian submersion. So, in §6, we define a 3-structure  $\{\eta_1, \eta_2, \eta_3, g^*\}$  (which we call a triple of  $K$ -contact structures of  $nS$ -type) associated to a pseudo-Sasakian 3-structure  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  of signature  $(3, 4r)$ .

Any 3-Sasakian manifold is an Einstein manifold and the scalar curvature is positive. Contrary to this, a Riemannian manifold admitting a triple of  $K$ -contact structures of  $nS$ -type has negative scalar curvature and may be considered as a 3-Sasakian-like structure with negative scalar curvature.

**THEOREM B.** *Let  $(M, g)$  be a quaternionic Kähler (not hyperkähler) manifold of dimension  $4r \geq 4$  with negative scalar curvature  $S$ . We normalize  $S$  so that  $S = -16r(r+2)$ . Then there exists a canonically associated principal  $SO(3)$ -bundle  $P(M)$  over  $(M, g)$  which admits a triple  $\{\eta_1, \eta_2, \eta_3, g^*\}$  of  $K$ -contact structures of  $nS$ -type and the fibering  $\pi: (P(M), g^*) \rightarrow (M, g)$  is a Riemannian submersion.*

As for examples of quaternionic Kähler manifolds of negative scalar curvature, see for example, [3], etc. In §7 as examples we calculate the sectional curvatures of  $(P(M), \tilde{g})$  and  $(P(M), g^*)$  over a simply connected complex 2-dimensional space forms  $(M, g)$  with constant holomorphic sectional curvature  $H=8$  and  $-8$ , respectively.

**2.  $K$ -contact 3-structure and Sasakian 3-structure.** Let  $(\tilde{M}, \eta, \tilde{g})$  be a contact Riemannian manifold of dimension  $m=2n+1$  (cf. [7], [21], [31], [32], etc.). Then we have structure tensors  $\xi$  and  $\phi$  satisfying the following:

$$\begin{aligned}\eta(X) &= \tilde{g}(\xi, X), & \eta(\xi) &= \tilde{g}(\xi, \xi) = 1, \\ \phi\xi &= 0, & \phi^2 X &= -X + \eta(X)\xi, & \eta(\phi X) &= 0, \\ d\eta(X, Y) &= 2\tilde{g}(X, \phi Y)\end{aligned}$$

for vector fields  $X$  and  $Y$  on  $\tilde{M}$ . If  $\xi$  is a Killing vector field, then  $\{\eta, \tilde{g}\}$  is called a  $K$ -contact structure. In this case, we have  $\phi = -\nabla\xi$  and  $L_\xi\phi = 0$ . Moreover, the Ricci curvature tensor satisfies  $R_{j\mu}\xi^l = (m-1)\eta_j$  and the sectional curvature for a 2-plane containing  $\xi$  is 1.

Furthermore, if the structure tensors of a  $K$ -contact structure satisfy

$$(2.1) \quad \tilde{\nabla}_k \phi_j^i = \xi^i \tilde{g}_{jk} - \eta_j \delta_k^i, \quad (\text{or } \tilde{R}_{jkl}^i \xi^l = -\xi^i \tilde{g}_{jk} + \eta_j \delta_k^i),$$

then it is called a Sasakian structure (cf. [7], etc.). It is known that any  $K$ -contact structure on a 3-dimensional Riemannian manifold is a Sasakian structure.

Let  $\{\eta_\alpha, \tilde{g}\}$ ,  $\alpha=1, 2, 3$ , be three  $K$ -contact structures. Then,  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  is called a triple of  $K$ -contact structures, if  $\{\xi_1, \xi_2, \xi_3\}$  are orthonormal and  $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$  holds for  $\varepsilon(\alpha, \beta, \gamma) = 1$ , where  $\varepsilon(\alpha, \beta, \gamma) = 1$  means that  $(\alpha, \beta, \gamma)$  is a cyclic permutation of  $(1, 2, 3)$ . In this case we have  $\xi_\alpha = \phi_\beta \xi_\gamma = -\phi_\gamma \xi_\beta$  and  $\eta_\alpha = \eta_\beta \phi_\gamma = -\eta_\gamma \phi_\beta$  for  $\varepsilon(\alpha, \beta, \gamma) = 1$ .

Furthermore, a triple of  $K$ -contact structures  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  is called a  $K$ -contact

3-structure, if the following is satisfied:

$$(2.2) \quad \phi_\alpha = \phi_\beta \phi_\gamma - \xi_\beta \otimes \eta_\gamma = -\phi_\gamma \phi_\beta + \xi_\gamma \otimes \eta_\beta, \quad \varepsilon(\alpha, \beta, \gamma) = 1.$$

A  $K$ -contact 3-structure is called a Sasakian 3-structure, if each  $\{\eta_\alpha, \tilde{g}\}$  is a Sasakian structure. A Riemannian manifold admitting a  $K$ -contact (or Sasakian) 3-structure is of dimension  $m = 4r + 3$ . A Riemannian manifold  $(\tilde{M}, \tilde{g})$  admitting a Sasakian 3-structure is called a 3-Sasakian manifold (cf. [9], [10]). Furthermore, if  $\tilde{g}$  is a pseudo-Riemannian metric, then  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  is called a pseudo- $K$ -contact (or pseudo-Sasakian) 3-structure.

The curvature operator  $R$  on a Riemannian manifold  $(M, g)$  is a linear operator  $R: \bigwedge^2 M \rightarrow \bigwedge^2 M$  defined by  $w = (w_{rs}) \rightarrow R w = (R_{ijrs} w^{rs})$ , where  $\bigwedge^2 M$  denotes the space of 2-forms on  $M$ .

Kashiwada obtained the following:

PROPOSITION 2.1 (cf. [18]). *Let  $(\tilde{M}, \eta_1, \eta_2, \eta_3, \tilde{g})$  be a 3-Sasakian manifold.*

- (i) *It is an Einstein manifold such that  $\text{Ric}^\sim = 2(2r+1)\tilde{g}$  and  $\tilde{S} = 2(2r+1) \cdot (4r+3)$ .*
- (ii)  *$d\eta_\alpha$  are eigen 2-forms of the curvature operator;  $\tilde{R}d\eta_\alpha = 2d\eta_\alpha$ ,  $\alpha = 1, 2, 3$ .*

The 3-dimensional distribution defined by  $\{\xi_1, \xi_2, \xi_3\}$  on  $(\tilde{M}, \tilde{g})$  admitting a triple of  $K$ -contact structures is completely integrable and integral submanifolds are spaces of constant curvature 1. If one of  $\eta_\alpha$ 's is regular, then  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  is called regular, and we have the fibering  $\pi: (\tilde{M}, \eta_1, \eta_2, \eta_3, \tilde{g}) \rightarrow (M, g) = (\tilde{M}, \tilde{g})/\{\xi_1, \xi_2, \xi_3\}$  (cf. [31, §11, §13]). In the (local) fibering of a 3-Sasakian manifold;  $\pi: (\tilde{M}, \eta_1, \eta_2, \eta_3, \tilde{g}) \rightarrow (M, g)$ , we have

$$(2.3) \quad (R(X, Y)Z)^* = \tilde{R}(X^*, Y^*)Z^* - \sum_\alpha [\tilde{g}(Y^*, \phi_\alpha Z^*)\phi_\alpha X^* - \tilde{g}(X^*, \phi_\alpha Z^*)\phi_\alpha Y^* - 2\tilde{g}(X^*, \phi_\alpha Y^*)\phi_\alpha Z^*]$$

for vector fields  $X, Y$  and  $Z$  on  $M$ , where  $X^*$  denotes the horizontal lift of  $X$  (cf. [31, p. 328],  $\xi$ -term is vanishing in Sasakian case). Next, let  $\sigma_U$  be a cross section of  $\tilde{M}$  over  $U$  and define  $\{\hat{I}_U, \hat{J}_U, \hat{K}_U\}$  by  $\hat{I}_U X_x = \pi \phi_1 X_{\sigma_U x}^*$ ,  $x \in U$ , etc. Then we have  $\hat{I}_U^2 = -\text{Id}$  and  $\hat{I}_U \hat{J}_U = \hat{K}_U = -\hat{J}_U \hat{I}_U$ , etc. Furthermore, we define 2-forms  $w_{\hat{I}_U}$ ,  $w_{\hat{J}_U}$  and  $w_{\hat{K}_U}$  on  $U$  by  $w_{\hat{I}_U}(X, Y) = g(X, \hat{I}_U Y)$ , etc., where  $X$  and  $Y$  are vector fields on  $U$ .

Translating Kashiwada's Proposition 2.1 to  $(M, g)$ , we obtain the following:

PROPOSITION 2.2. *In the fibering  $\pi: (\tilde{M}, \eta_1, \eta_2, \eta_3, \tilde{g}) \rightarrow (M, g)$  of a complete and regular 3-Sasakian manifold of dimension  $4r+3 \geq 7$ , we have the following:*

- (i)  *$(M, g)$  is an Einstein manifold such that  $\text{Ric} = 4(r+2)g$  and  $S = 16r(r+2)$ .*
- (ii) *The curvature operator  $R$  of  $(M, g)$  satisfies  $Rw = 8rw$  for  $w = aw_{\hat{I}_U} + bw_{\hat{J}_U} + cw_{\hat{K}_U}$  where  $a, b$  and  $c$  are real numbers.*

PROOF. (i) follows from [31, p. 329]. To show (ii), we put  $\hat{I}_U = \psi_1$ ,  $\hat{J}_U = \psi_2$  and  $\hat{K}_U = \psi_3$  on  $U$ . Projecting the relation (2.3) to  $U$  we obtain

$$(2.4) \quad R_{wzxy} = \tilde{R}_{wzxy} - \sum_{\alpha} [\psi_{xyz} \psi_{\alpha wx} - \psi_{\alpha xz} \psi_{\alpha wy} - 2\psi_{\alpha xy} \psi_{\alpha wz}],$$

where the components are the ones with respect to an orthonormal frame  $\{u_x\}$ ,  $1 \leq x, y, z, w \leq 4r$ , and  $\tilde{R}_{wzxy} = \tilde{g}(\tilde{R}(u_x^*, u_y^*)u_z^*, u_w^*)$  along the section  $\sigma_U U$ . Here, note that we have  $\hat{I}_U = (\psi_{1y}^x)$  and  $w_{\hat{I}_U} = (\psi_{1xy})$ , etc. Let  $\phi_{\alpha j}^i$  be the components of  $\phi_{\alpha}$  with respect to the frame  $\{a_x = u_x^*, a_{4r+\alpha} = \xi_{\alpha}\}$  along  $\sigma_U U$ . Since  $\tilde{R}(u_x^*, u_y^*)\xi_{\alpha} = 0$  by (2.1), we obtain  $\tilde{R}_{wzkl}\phi_1^{kl} = \tilde{R}_{wzxy}\psi_1^{xy} = 2\psi_{1wz}$  by Proposition 2.1 (ii). Next, contracting (2.4) with  $\psi_1^{xy}$  we get  $R_{wzxy}\psi_1^{xy} = 8r\psi_{1wz}$ , and  $Rw_{\hat{I}_U} = 8rw_{\hat{I}_U}$ . Similarly, we obtain the relation for  $w_{\hat{J}_U}$  and  $w_{\hat{K}_U}$ . q.e.d

We notice here that Proposition 2.1 is valid for pseudo-Sasakian 3-structure, and Proposition 2.2 is valid for local fibering of a 3-Sasakian manifold. As is explained in the next section, Proposition 2.2 (ii) was proved for  $4r+3 \geq 11$  as a property of the quaternionic Kähler structure by Ishihara [17].

**3. Quaternionic Kähler structure.** We recall the definition of the quaternionic Kähler structure after Ishihara [17] (cf. [1], [2], [5], [6], [14], etc.). Let  $\{U\}$  be an open covering of a Riemannian manifold  $(M, g)$  of dimension  $4r$ . We assume that for each  $U$  there are almost complex structures  $\{\hat{I}_U, \hat{J}_U, \hat{K}_U\}$  on  $U$  satisfying

$$(3.1) \quad \hat{I}_U \hat{J}_U = \hat{K}_U = -\hat{J}_U \hat{I}_U, \quad (\hat{J}_U \hat{K}_U = \hat{I}_U = -\hat{K}_U \hat{J}_U, \hat{K}_U \hat{I}_U = \hat{J}_U = -\hat{I}_U \hat{K}_U)$$

$$(3.2) \quad g(X, Y) = g(\hat{I}_U X, \hat{I}_U Y) = g(\hat{J}_U X, \hat{J}_U Y) = g(\hat{K}_U X, \hat{K}_U Y)$$

and for any  $U$  and  $V$  with nonempty intersection we have

$$(3.3) \quad \begin{aligned} \hat{I}_V &= s_{11} \hat{I}_U + s_{12} \hat{J}_U + s_{13} \hat{K}_U, \\ \hat{J}_V &= s_{21} \hat{I}_U + s_{22} \hat{J}_U + s_{23} \hat{K}_U, \\ \hat{K}_V &= s_{31} \hat{I}_U + s_{32} \hat{J}_U + s_{33} \hat{K}_U, \end{aligned}$$

on  $U \cap V$ , where  $s_{UV} = (s_{ij})$  is an  $SO(3)$ -valued function on  $U \cap V$ . So, we have a 3-dimensional subbundle  $\mathcal{S}$  of  $\text{End}(TM)$  and  $\{U, \hat{I}_U, \hat{J}_U, \hat{K}_U\}$  is called an almost quaternionic Kähler structure on  $(M, g)$ . Furthermore, we consider the following condition:

$$(3.4) \quad \begin{aligned} \nabla_X \hat{I}_U &= 2r_U(X) \hat{J}_U - 2q_U(X) \hat{K}_U, \\ \nabla_X \hat{J}_U &= -2r_U(X) \hat{I}_U + 2p_U(X) \hat{K}_U, \\ \nabla_X \hat{K}_U &= 2q_U(X) \hat{I}_U - 2p_U(X) \hat{J}_U, \end{aligned}$$

where  $p_U, q_U$  and  $r_U$  are 1-forms on  $U$  and  $X$  is a vector field on  $U$ . For the case of  $\dim M \geq 8$ , if (3.4) holds on each  $U$ , then  $\{U, \hat{I}_U, \hat{J}_U, \hat{K}_U\}$  is called a quaternionic Kähler structure on  $(M, g)$ . Here, we assume that not all  $p_U, q_U$  and  $r_U$  are vanishing. If all  $p_U, q_U$  and  $r_U$  are vanishing, then the structure is called hyperkähler. We define 2-forms

$w_{\hat{I}_U}$ ,  $w_{\hat{J}_U}$  and  $w_{\hat{K}_U}$  on  $U$  as before. Corresponding to  $\mathcal{S}$ , we have the 3-dimensional bundle  $\mathcal{S}^b$  defined by  $\{U, w_{\hat{I}_U}, w_{\hat{J}_U}, w_{\hat{K}_U}\}$  as its covariant form.

Two facts that  $g$  of a quaternionic Kähler manifold  $(M, g)$  is an Einstein metric and that the curvature operator satisfies  $R = (S/2(r+1))\text{Id}$  for  $\mathcal{S}^b$  follow from (3.4) under the assumption  $\dim M \geq 8$  (cf. [17]).

Applying the Ricci identity to (3.4), Ishihara [17] obtained the following:

$$(3.5) \quad \begin{aligned} R w_{\hat{I}_U} &= 4r(dp_U + 2q_U \wedge r_U), & R w_{\hat{J}_U} &= 4r(dq_U + 2r_U \wedge p_U), \\ R w_{\hat{K}_U} &= 4r(dr_U + 2p_U \wedge q_U) \end{aligned}$$

for  $\dim M = 4r \geq 4$ .

**4. Oriented Riemannian 4-manifold.** Let  $(M, g)$  be a 4-dimensional oriented Riemannian manifold and let  $U$  be an open set with orthonormal frame field  $\{u_1, u_2, u_3, u_4\}$ , which is compatible with the given orientation. First we define the action of almost complex structures  $\{\hat{I}_U, \hat{J}_U, \hat{K}_U\}$  to  $u_1$  by  $u_2 = \hat{I}_U u_1$ ,  $u_3 = \hat{J}_U u_1$  and  $u_4 = \hat{K}_U u_1$ . Next we assume (3.1). Then  $\{\hat{I}_U, \hat{J}_U, \hat{K}_U\}$  is a set of almost complex structures satisfying (3.1) and (3.2). We cover  $M$  by these  $\{U, \hat{I}_U, \hat{J}_U, \hat{K}_U\}$ . Let  $U$  and  $V$  be two such open sets with nonempty intersection. Let  $p \in U \cap V$  and let  $Y$  be a unit vector at  $p$ . Then  $\{\hat{I}_V Y, \hat{J}_V Y, \hat{K}_V Y\}$  is expressed by  $\{\hat{I}_U Y, \hat{J}_U Y, \hat{K}_U Y\}$  using an element  $(s_{ij})$  of  $SO(3)$ . By the algebraic relations satisfied by  $\hat{I}_V, \hat{J}_V, \dots, \hat{K}_V$ , we can verify that they satisfy (3.3) on  $U \cap V$ . So we have a 3-dimensional subbundle  $\mathcal{S}$  of  $\text{End}(TM)$ . Next we show (3.4) on  $U$ . Let  $C = \{x(t); 0 \leq t \leq t_1\}$  be an integral curve of  $X$  passing through a point  $p$  of  $U$ ,  $x(0) = p$ . By  $\tau_t$  for  $t$  ( $0 < t \leq t_1$ ) we denote the parallel translation from  $p$  to  $x(t)$  along  $C$ . We fix a unit tangent vector  $Y$  at  $p$  and define  $\hat{I}'_U$  by  $\hat{I}'_U Y = \tau_t^{-1} \hat{I}_U(x(t)) \tau_t Y$ . Then we have the relation as (3.3) with respect to  $\{\hat{I}'_U, \hat{J}'_U, \hat{K}'_U\}$  and  $\{\hat{I}_U, \hat{J}_U, \hat{K}_U\}$  at  $p$ . So, differentiating these with respect to  $t$  we have (3.4). The above means that any 4-dimensional oriented Riemannian manifold is an almost quaternionic Kähler manifold satisfying (3.4).

Let  $\bigwedge^2 M = \bigwedge_+^2 M \oplus \bigwedge_-^2 M$  be the decomposition of the space  $\bigwedge^2 M$  by the Hodge star operator  $*$ . Since the Weyl conformal curvature tensor  $W \in \text{End}(\bigwedge^2 M)$  leaves  $\bigwedge_+^2 M$  and  $\bigwedge_-^2 M$  invariant, we have the decomposition  $W = W^+ + W^-$ . If  $W^+ = 0$  ( $W^- = 0$ , resp.),  $(M, g)$  is called anti-self-dual (self-dual, resp.).

Let  $\{\theta_1, \theta_2, \theta_3, \theta_4\}$  be the dual coframe of  $\{u_1, u_2 = \hat{I}_U u_1, u_3 = \hat{J}_U u_1, u_4 = \hat{K}_U u_1\}$ . Then we obtain  $w_{\hat{I}_U} = -(\theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4)$ ,  $w_{\hat{J}_U} = -(\theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4)$ , and  $w_{\hat{K}_U} = -(\theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3)$ . So, we have  $\bigwedge_+^2 M = \Gamma(\mathcal{S}^b)$ .

**LEMMA 4.1.** *Let  $(M, g)$  be a 4-dimensional oriented Riemannian manifold.*

(i) *If  $(M, g)$  is an anti-self-dual Einstein manifold, then the curvature operator  $R$  satisfies  $Rw = (S/6)w$  for any  $w \in \bigwedge_+^2 M = \Gamma(\mathcal{S}^b)$ .*

(ii) *If the curvature operator  $R$  of  $(M, g)$  satisfies  $Rw = \lambda w$  for any  $w \in \bigwedge_+^2 M = \Gamma(\mathcal{S}^b)$ , then  $\lambda = S/6$  and  $(M, g)$  is an anti-self-dual Einstein manifold.*

PROOF. We put  $G = \text{Ric} - (S/4)g$ . Then the curvature operator  $R$  is expressed as

$$(4.1) \quad R = (S/6)\text{Id} + (1/2)G \wedge g + W$$

for  $\bigwedge^2 M = \bigwedge_+^2 M \oplus \bigwedge_-^2 M$ , where  $(1/2)G \wedge g$  corresponds to the cross term of the action of  $R$  (cf. for example, [6, p. 48, p. 51]; the difference of the constant factor comes from the definition of  $R$ ). If  $G=0$  and  $W^+=0$  we obtain  $Rw = (S/6)w$  for any  $w \in \Gamma(\mathcal{S}^b)$ .

Next we show (ii). By (4.1) we see that  $(M, g)$  is an Einstein manifold, and so it suffices to show that  $W^+w = \mu w$  for any  $w \in \bigwedge_+^2 M$  implies  $W^+ = 0$ . This is clear by  $\text{Tr } W^+ = 0$  (cf. [11, p. 409], [28]). q.e.d.

By Proposition 2.2 and Lemma 4.1, we obtain the following:

**COROLLARY 4.2.** *Let  $(\tilde{M}, \eta_1, \eta_2, \eta_3, \tilde{g})$  be a complete and regular 3-Sasakian manifold of dimension 7. Then, the base manifold  $(M, g)$  of the canonical fibering is an anti-self-dual Einstein manifold with  $S=48$ .*

By Corollary 4.2 and after LeBrun-Swann, we define a 4-dimensional quaternionic Kähler manifold as an anti-self-dual Einstein manifold.

Now we have the following: Let  $(\tilde{M}, \eta_1, \eta_2, \eta_3, \tilde{g})$  be a complete and regular 3-Sasakian manifold of dimension  $m \geq 7$ . Then it is a principal  $Sp(1)$ - or  $SO(3)$ -bundle over a quaternionic Kähler manifold (cf. [16], etc.).

**5. Sasakian 3-structure on the principal  $SO(3)$ -bundle  $P(M)$ .** First we give the statement of Konishi's theorem adding the 4-dimensional case. We need the case (ii) of the theorem later in §6.

**THEOREM 5.1** (cf. [20]). *Let  $(M, g)$  be a quaternionic Kähler (not hyperkähler) manifold of dimension  $4r \geq 4$  with nonzero scalar curvature  $S$ . Then, there exists a canonically associated principal  $SO(3)$ -bundle  $P(M)$  over  $(M, g)$ ;*

(i) *If  $S$  is positive, then  $P(M)$  admits a Sasakian 3-structure  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$ . Furthermore, if  $S=16r(r+2)$ , then the fibering is a Riemannian submersion.*

(ii) *If  $S$  is negative, then  $P(M)$  admits a pseudo-Sasakian 3-structure  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$ . In this case, the signature of  $\tilde{g}$  is  $(3, 4r)$  and the fibering is not a Riemannian submersion.*

Konishi's construction of a  $K$ -contact 3-structure  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  on  $P(M)$  works for  $\dim M = 4r \geq 4$  (under our definition of quaternionic Kähler structure of dimension 4). However, the proof for  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  to be a Sasakian 3-structure is effective for  $\dim M \geq 8$ . (Note that Konishi claims that “(3.5) and (3.6) lead us to  $c_d=0$ ” in [19, p. 5] (as for the notation see [19]). This is valid for  $m \neq 1$ , i.e.,  $\dim M \geq 8$ . Moreover, (2.5) of [19] is trivial for  $\dim M = 4$ .) Furthermore, Konishi's proof for  $\dim M \geq 8$  is not so simple. So, we give a proof which is simpler and effective for  $\dim M \geq 4$  in Lemma 5.2. As a preliminary to this lemma we explain Konishi's construction of a  $K$ -contact 3-structure on  $P(M)$ .

Using the transition functions  $\{s_{UV}\}$  in §3 for  $4r \geq 8$  and in §4 for  $4r = 4$ , we have the principal  $SO(3)$ -bundle  $P(M)$  over  $(M, g)$ . We denote by  $so(3)$  the Lie algebra of  $SO(3)$  and fix a basis  $\{e_1, e_2, e_3\}$  such that  $[e_\alpha, e_\beta] = 2e_\gamma$  for  $\varepsilon(\alpha, \beta, \gamma) = 1$ . Next, we define an  $so(3)$ -valued 1-form  $\omega_U$  on  $U$  by

$$(5.1) \quad \omega_U = p_U e_1 + q_U e_2 + r_U e_3,$$

using 1-forms  $p_U, q_U$  and  $r_U$  on  $U$  in (3.4). Then, we see that there is a connection form  $\omega$  in  $P(M)$  such that  $\sigma_U^* \omega = \omega_U$  for a certain cross-section  $\sigma_U$  of  $P(M)$  over  $U$  for each  $U$ . We denote the curvature form of  $\omega$  by  $\Omega$ ;  $\Omega(\tilde{X}, \tilde{Y}) = d\omega(\tilde{X}, \tilde{Y}) + [\omega(\tilde{X}), \omega(\tilde{Y})]$ , where  $\tilde{X}$  and  $\tilde{Y}$  are vector fields on  $P(M)$ . By (5.1) we obtain

$$(5.2) \quad \sigma_U^* \Omega = (dp_U + 2q_U \wedge r_U)e_1 + (dq_U + 2r_U \wedge p_U)e_2 + (dr_U + 2p_U \wedge q_U)e_3.$$

By (3.5),  $R = (S/2(r+2))\text{Id}$  for  $r \geq 2$  and Proposition 4.1 (i) for  $r = 1$ , we obtain

$$(5.3) \quad \sigma_U^* \Omega = 2\theta(w_{\hat{f}_U} e_1 + w_{\hat{f}_U} e_2 + w_{\hat{K}_U} e_3), \quad \theta = S/16r(r+2).$$

Next we express the connection form  $\omega$  on  $P(M)$  as  $\omega = \sum_\alpha \eta_\alpha e_\alpha$ , and we denote the fundamental vector fields corresponding to  $e_\alpha$  by  $\xi_\alpha, \alpha = 1, 2, 3$ . Here, we have  $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$  for  $\varepsilon(\alpha, \beta, \gamma) = 1$ . Finally we define a (pseudo-) Riemannian metric  $\tilde{g}$  on  $P(M)$  by

$$(5.4) \quad \tilde{g} = \theta \pi^* g + \sum_\alpha \eta_\alpha \otimes \eta_\alpha$$

for  $\theta > 0$  ( $\theta < 0$ , resp.). Then  $\xi_\alpha, \alpha = 1, 2, 3$ , are unit Killing vector fields on  $P(M)$ . Furthermore, we have  $\eta_\alpha(X) = \tilde{g}(\xi_\alpha, X)$  and  $\eta_\alpha(\xi_\beta) = \tilde{g}(\xi_\alpha, \xi_\beta) = \delta_{\alpha\beta}$ . Denoting the Riemannian connection with respect to  $\tilde{g}$  by  $\tilde{\nabla}$ , we define (1,1)-tensor fields  $\phi_\alpha$  by  $\phi_\alpha = -\tilde{\nabla} \xi_\alpha$ , for  $\alpha = 1, 2, 3$ . Then we see that  $\phi_\alpha \xi_\alpha = 0$  and  $\phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha = \xi_\gamma$  for  $\varepsilon(\alpha, \beta, \gamma) = 1$ . Since  $\xi_\alpha$  is a unit Killing vector field, we have  $d\eta_\alpha(\tilde{X}, \tilde{Y}) = 2\tilde{g}(\tilde{X}, \phi_\alpha \tilde{Y})$ ,  $\alpha = 1, 2, 3$ , for vector fields  $\tilde{X}$  and  $\tilde{Y}$  on  $P(M)$ . By  $\phi_\alpha^H$  we denote the restriction of  $\phi_\alpha$  to the horizontal subspace at each point of  $P(M)$ , i.e.,  $\phi_\alpha = \phi_\alpha^H + \xi_\gamma \otimes \eta_\beta - \xi_\beta \otimes \eta_\gamma$  for  $\varepsilon(\alpha, \beta, \gamma) = 1$ . Let  $X$  and  $Y$  be vector fields on  $U$  and  $X^*$  and  $Y^*$  be their horizontal lifts on  $P(M)$ . By  $\hat{f}_U^H$  we denote the horizontal lift of  $\hat{f}_U$  along the cross section  $\sigma_U U$ ; that is,  $(\hat{f}_U^H X^*)_{\sigma_U x} = (\hat{f}_U X_x)^*$ ,  $x \in U$ . Then, considering the  $e_1$ -part of (5.3) along  $\sigma_U U$  and using horizontal property of the curvature form, we obtain

$$(5.5) \quad (d\eta_1 + 2\eta_2 \wedge \eta_3)(X^*, Y^*)_{\sigma_U x} = 2\theta w_{\hat{f}_U}(X, Y)_x$$

at  $x \in U$ . The left hand side of the above is equal to

$$d\eta_1(X^*, Y^*)_{\sigma_U x} = 2\tilde{g}(X^*, \phi_1 Y^*)_{\sigma_U x} = 2\tilde{g}(X^*, \phi_1^H Y^*)_{\sigma_U x},$$

and the right hand side is equal to  $2\theta g(X, \hat{f}_U Y)_x = 2\tilde{g}(X^*, \hat{f}_U^H Y^*)_{\sigma_U x}$ . Therefore we get  $\phi_1^H = \hat{f}_U^H$  along  $\sigma_U U$ . Similarly, we get  $\phi_2^H = \hat{f}_U^H$  and  $\phi_3^H = \hat{K}_U^H$  along  $\sigma_U U$ . So, we see that  $\phi_1^H, \phi_2^H$  and  $\phi_3^H$  satisfy the algebraic relations as (3.1) on  $P(M)$ . Furthermore, we obtain  $\phi_\alpha^2 = -\text{Id} + \xi_\alpha \otimes \eta_\alpha$ ,  $\alpha = 1, 2, 3$  and

$$(5.6) \quad \phi_\alpha = \phi_\beta \phi_\gamma - \xi_\beta \otimes \eta_\gamma = -\phi_\gamma \phi_\beta + \xi_\gamma \otimes \eta_\beta, \quad \varepsilon(\alpha, \beta, \gamma) = 1.$$

Therefore,  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  defines a (pseudo-)  $K$ -contact 3-structure on  $P(M)$ , if  $\theta > 0$  (if  $\theta < 0$ , resp.). Here we state the following lemma:

LEMMA 5.2.  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  defines a (pseudo-) Sasakian 3-structure on  $P(M)$ .

Proof will be given after some explanation about Theorems A and A':

THEOREM A'. Any pseudo- $K$ -contact 3-structure  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  of signature  $(3, 4)$  on a 7-dimensional manifold  $\tilde{M}$  is a pseudo-Sasakian 3-structure.

PROOF OF THEOREMS A AND A'. We consider a local fibering of a manifold  $\tilde{M}$  with (pseudo-)  $K$ -contact 3-structure  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$ ;  $\pi: \tilde{U} \rightarrow U = \tilde{U}/\{\xi_1, \xi_2, \xi_3\}$ .  $U$  is 4-dimensional and admits an almost quaternionic Kähler structure  $\{\hat{I}_U, \hat{J}_U, \hat{K}_U, g\}$  which is canonically related with  $\{\phi_1, \phi_2, \phi_3\}$  on  $\tilde{U}$  by a cross section and (5.4), where  $\theta = 1$  ( $-1$ , resp.) if  $\tilde{g}$  is a (pseudo-, resp.) Riemannian metric. So, the proof is completed if Lemma 5.2 is proved.

PROOF OF LEMMA 5.2. It suffices to show (2.1), i.e.,

$$(5.7) \quad (\tilde{\nabla}_V \phi_\alpha) V' = \tilde{g}(V, V') \xi_\alpha - \eta_\alpha(V') V, \quad \alpha = 1, 2, 3,$$

for vector fields  $V$  and  $V'$  on  $P(M)$ . It is enough to show (5.7) for  $\alpha = 1$ , since the other cases are similarly proved.

(i) First, we study the case where  $V$  and  $V'$  are horizontal. In the (local) fibering of  $K$ -contact 3-structure we have

$$(5.8) \quad \tilde{\nabla}_{X^*} Y^* = (\nabla_X Y)^* - \sum_\alpha \tilde{g}(X^*, \phi_\alpha Y^*) \xi_\alpha$$

for vector fields  $X$  and  $Y$  on  $M$  (cf. [31, p. 328]). Here, we notice that (5.8) is valid also for the case where  $\theta = -1$  in (5.4). Let  $\tilde{x}_o$  be an arbitrary point of  $P(M)$  and let  $\sigma_U$  be a cross section of  $P(M)$  over  $U$  such that  $\sigma_U U$  contains the trajectory of  $X^*$  passing through  $\tilde{x}_o$ . Let  $u_1$  be a unit vector field on  $U$  and  $u_1^*$  be its horizontal lift. We define an orthonormal frame field  $\{u_1, u_2, u_3, u_4\}$  on  $U$  by  $(u_{\alpha+1})_y = \pi \phi_\alpha(u_1^*)_{\sigma_U y}$ ,  $y \in U$ ,  $\alpha = 1, 2, 3$ . Then we have  $\{\hat{I}_U, \hat{J}_U, \hat{K}_U\}$  on  $U$  with respect to this frame field, and we have 1-forms  $p_U, q_U$  and  $r_U$  on  $U$ . By our choice of the cross section we obtain

$$r_U(X) = \eta_3(\sigma_U X) = \eta_3(X^*) = 0,$$

and  $q_U(X) = 0$  at  $\pi \tilde{x}_o = x_o \in U$ . Therefore, (3.4) implies  $\nabla_X \hat{I}_U = 0$  at  $x_o$ . We put  $Y = \hat{I}_U Z$  for a vector field  $Z$  on  $U$ . Then, the left hand side of (5.8) is

$$\begin{aligned} \tilde{\nabla}_{X^*}(\hat{I}_U Z)^* &= \tilde{\nabla}_{X^*}(\phi_1 Z^*) = (\tilde{\nabla}_{X^*} \phi_1) Z^* + \phi_1 \tilde{\nabla}_{X^*} Z^* \\ &= (\nabla_X \phi_1) Z^* + \phi_1 (\nabla_X Z)^* - \tilde{g}(X^*, \phi_2 Z^*) \xi_3 + \tilde{g}(X^*, \phi_3 Z^*) \xi_2 \end{aligned}$$

at  $\tilde{x}_o$ , and the first and second terms of the right hand side of (5.8) are



$$(\nabla_X(\hat{I}_U Z))^* = (\hat{I}_U \nabla_X Z)^* = \phi_1(\nabla_X Z)^*,$$

$$-\sum_{\alpha} \tilde{g}(X^*, \phi_{\alpha} \phi_1^H Z^*) \xi_{\alpha} = \tilde{g}(X^*, Z^*) \xi_1 + \tilde{g}(X^*, \phi_3 Z^*) \xi_2 - g(X^*, \phi_2 Z^*) \xi_3$$

at  $\tilde{x}_o$ . So, we obtain  $(\tilde{\nabla}_{X^*} \phi_1) Z^* = \tilde{g}(X^*, Z^*) \xi_1$  at  $\tilde{x}_o$ . This is (5.7) in this case.

(ii) Next, we study the case where  $V$  is horizontal and  $V'$  is vertical. We have  $(\tilde{\nabla}_{X^*} \phi_1) \xi_2 = \tilde{\nabla}_{X^*} \xi_3 - \phi_1 \tilde{\nabla}_{X^*} \xi_2 = -\phi_3 X^* + \phi_1 \phi_2 X^* = 0$ . Similarly we get  $(\tilde{\nabla}_{X^*} \phi_1) \xi_3 = 0$ , and  $(\tilde{\nabla}_{X^*} \phi_1) \xi_1 = -\phi_1 \tilde{\nabla}_{X^*} \xi_1 = \phi_1^2 X^* = -X^* = -\eta_1(\xi_1) X^*$ .

(iii) Finally, we study the case where  $V$  is vertical. By  $L_{\xi_{\alpha}} \xi_{\beta} = [\xi_{\alpha}, \xi_{\beta}] = 2\xi_{\gamma}$ , we have  $L_{\xi_{\alpha}} \tilde{\nabla} \xi_{\beta} = 2\tilde{\nabla} \xi_{\gamma}$ , and  $L_{\xi_{\alpha}} \phi_{\beta} = 2\phi_{\gamma}$ . So, we have  $\tilde{\nabla}_{\xi_{\alpha}} \phi_{\beta} + \phi_{\alpha} \phi_{\beta} - \phi_{\beta} \phi_{\alpha} = 2\phi_{\gamma}$ . Hence,  $\tilde{\nabla}_{\xi_{\alpha}} \phi_1 = \xi_1 \otimes \eta_{\alpha} - \xi_{\alpha} \otimes \eta_1$  for  $\alpha = 1, 2, 3$ . Thus, we have (5.7) at  $\tilde{x}_o$ . q.e.d.

**6. A triple of  $K$ -contact structures of  $nS$ -type.** Concerning the case (ii) of Theorem 5.1, it is natural to study what is the proper structure of  $P(M)$  so that the fibering is a Riemannian submersion. Let  $\{\eta_1, \eta_2, \eta_3\}$  be a Sasakian 3-structure on a pseudo-Riemannian manifold  $(\tilde{M}, \tilde{g})$  of signature  $(3, 4r)$  modeled on (5.4) with  $\theta = -1$ . We define a Riemannian metric  $g^*$  on  $\tilde{M}$  by

$$(6.1) \quad g^* = -\tilde{g} + 2 \sum_{\alpha} \eta_{\alpha} \otimes \eta_{\alpha}.$$

Then  $\xi_1, \xi_2$  and  $\xi_3$  are orthonormal Killing vector fields with respect to  $g^*$ , too. We have  $g^{*kl} = -\tilde{g}^{kl} + 2 \sum_{\alpha} \xi_{\alpha}^k \xi_{\alpha}^l$ . Furthermore, the difference  $\Theta_{jk}^i$  of the coefficients  $\Gamma_{jk}^{*i}$  and  $\tilde{\Gamma}_{jk}^i$  of the Riemannian connections  $\nabla^*$  and  $\tilde{\nabla}$  is given by

$$(6.2) \quad \Theta_{jk}^i = \Gamma_{jk}^{*i} - \tilde{\Gamma}_{jk}^i = 2 \sum_{\alpha} (\phi_{\alpha k}^i \eta_{\alpha j} + \phi_{\alpha j}^i \eta_{\alpha k}).$$

So, calculating  $\phi_{\alpha}^* = -\nabla^* \xi_{\alpha}$ , we obtain

$$\phi_{\alpha j}^* = -\phi_{\alpha j}^i - 2(\xi_{\beta}^i \eta_{\gamma j} - \xi_{\gamma}^i \eta_{\beta j}), \quad \varepsilon(\alpha, \beta, \gamma) = 1.$$

Hence, we get  $\xi_{\alpha} = \phi_{\beta}^* \xi_{\gamma} = -\phi_{\gamma}^* \xi_{\beta}$  and  $\eta_{\alpha} = \eta_{\beta} \phi_{\gamma}^* = -\eta_{\gamma} \phi_{\beta}^*$ . Furthermore, we have

$$(6.3) \quad \phi_{\alpha}^* = -\phi_{\beta}^* \phi_{\gamma}^* - \xi_{\beta} \otimes \eta_{\gamma} + 2\xi_{\gamma} \otimes \eta_{\beta} = \phi_{\gamma}^* \phi_{\beta}^* + \xi_{\gamma} \otimes \eta_{\beta} - 2\xi_{\beta} \otimes \eta_{\gamma}$$

for  $\varepsilon(\alpha, \beta, \gamma) = 1$ . Each  $\{\phi_{\alpha}^*, \xi_{\alpha}, \eta_{\alpha}, g^*\}$  is a  $K$ -contact structure. However, it is not Sasakian. The important relation corresponding to (2.1) is the following:

$$(6.4) \quad \nabla_k^* \phi_{\alpha j}^* = \xi_{\alpha}^i g_{jk}^* - \eta_{\alpha j} \delta_k^i + A_{\alpha k j}^i,$$

where, denoting  $\Psi_{jk} = \sum_{\alpha} \eta_{\alpha j} \eta_{\alpha k}$  and  $\Psi_j^i = \sum_{\alpha} \xi_{\alpha}^i \eta_{\alpha j}$ , we have put

$$A_{\alpha k j}^i = -6\xi_{\alpha}^i \Psi_{jk} + 6\eta_{\alpha j} \Psi_k^i - 2(\xi_{\beta}^i \phi_{\gamma k}^* + \phi_{\gamma k}^* \eta_{\beta j} + 2\phi_{\gamma j}^* \eta_{\beta k})$$

$$+ 2(\xi_{\gamma}^i \phi_{\beta k}^* + \phi_{\beta k}^* \eta_{\gamma j} + 2\phi_{\beta j}^* \eta_{\gamma k}), \quad \varepsilon(\alpha, \beta, \gamma) = 1.$$

We notice that the restriction of  $A_{\alpha}$  to the horizontal subspace is vanishing.

Using (6.2) we can calculate the Riemannian curvature tensor  $R^*$  as

(6.5)

$$R^{*i}_{jkl} = \tilde{R}^i_{jkl} + 2(\Psi^i_l \tilde{g}_{jk} - \Psi^i_k \tilde{g}_{jl}) - 8(\Psi^i_l \Psi_{jk} - \Psi^i_k \Psi_{jl}) + 2 \sum_{\alpha} [\phi_{\alpha k}^i \phi_{\alpha jl} - \phi_{\alpha l}^i \phi_{\alpha jk} + 2\phi_{\alpha j}^i \phi_{\alpha kl}] \\ + 4 \sum_{\alpha: \varepsilon(\alpha, \beta, \gamma) = 1} [\phi_{\alpha k}^i (\eta_{\beta j} \eta_{\gamma l} - \eta_{\beta l} \eta_{\gamma j}) - \phi_{\alpha l}^i (\eta_{\beta j} \eta_{\gamma k} - \eta_{\beta k} \eta_{\gamma j}) + 2\phi_{\alpha j}^i (\eta_{\beta k} \eta_{\gamma l} - \eta_{\beta l} \eta_{\gamma k})].$$

The Ricci curvature tensor is given by

$$(6.6) \quad R^{*}_{ji} = \tilde{R}_{ji} + 12\tilde{g}_{ji} - 12\Psi_{ji} = -2(2r+7)g^{*}_{ji} + 8(r+2)\Psi_{ji},$$

and the scalar curvature is  $S^{*} = -2(8r^2 + 22r - 3)$ .

**DEFINITION.** A triple  $\{\eta_1, \eta_2, \eta_3, g^{*}\}$  of  $K$ -contact structures is said to be of  $nS$ -type, if (6.3) and (6.4) are satisfied.

Since the procedure from  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  to  $\{\eta_1, \eta_2, \eta_3, g^{*}\}$  is reciprocal, for a triple  $\{\eta_1, \eta_2, \eta_3, g^{*}\}$  of  $K$ -contact structures of  $nS$ -type, we have a pseudo-Sasakian 3-structure  $\{\eta_1, \eta_2, \eta_3, \tilde{g}\}$  by  $\tilde{g} = -g^{*} + 2\sum_{\alpha} \eta_{\alpha} \otimes \eta_{\alpha}$ . So, we have proved the following:

**PROPOSITION 6.1.** *Let  $(\tilde{M}, g^{*})$  be a  $(4r+3)$ -dimensional Riemannian manifold admitting a triple  $\{\eta_1, \eta_2, \eta_3, g^{*}\}$  of  $K$ -contact structures of  $nS$ -type, where  $r \geq 1$ . Then the Ricci curvature tensor is given by  $\text{Ric}^{*} = -2(2r+7)g^{*} + 8(r+2)\sum_{\alpha} \eta_{\alpha} \otimes \eta_{\alpha}$ , and the scalar curvature  $S^{*} = -2(8r^2 + 22r - 3)$  is negative.*

**PROPOSITION 6.2.** *Let  $(\tilde{M}, g^{*})$  be a  $(4r+3)$ -dimensional complete Riemannian manifold admitting a triple  $\{\eta_1, \eta_2, \eta_3, g^{*}\}$  of  $K$ -contact structures of  $nS$ -type, where  $r \geq 1$ . If  $\{\eta_1, \eta_2, \eta_3\}$  is regular, then we have a Riemannian submersion  $\pi: (\tilde{M}, g^{*}) \rightarrow (M, g)$ , where  $(M, g)$  is a quaternionic Kähler manifold of negative scalar curvature  $-16r(r+2)$ .*

**PROOF OF THEOREM B.** In (5.4) we have  $\tilde{g} = -\pi^{*}g + \sum_{\alpha} \eta_{\alpha} \otimes \eta_{\alpha}$  for  $\theta = -1$ . So we have a Riemannian metric  $g^{*} = \pi^{*}g + \sum_{\alpha} \eta_{\alpha} \otimes \eta_{\alpha}$ , and the fibering  $\pi: (P(M), g^{*}) \rightarrow (M, g)$  is a Riemannian submersion.

**7. Examples and remarks.** It is known as Hitchin's theorem that 4-dimensional compact quaternionic Kähler manifolds with positive scalar curvature are sphere  $S^4$  and the complex projective space  $CP^2$  with respective standard Riemannian metric (cf. [6], [13], [15]). Hence, we see that any 7-dimensional complete and regular 3-Sasakian manifold is either  $S^7$ ,  $RP^7$  or  $SU(3)/U(1)$  (cf. [8]), where  $RP^7$  and  $SU(3)/U(1)$  are the principal  $SO(3)$ -bundle  $P(M)$  over  $M = S^4$  and  $CP^2$ , respectively, and  $S^7$  is the principal  $Sp(1)$ -bundle over  $S^4$ .

By the classification of 8-dimensional compact quaternionic Kähler manifolds due to Poon and Salamon [25], the classification of 11-dimensional complete and regular 3-Sasakian manifolds was also completed by Boyer-Galicki-Mann [8]. Furthermore, the classification of homogeneous 3-Sasakian manifolds was completed also by Boyer-Galicki-Mann [10] using Wolf's work [34]. Here, it may be noticed that the

spheres  $S^{4r+3}$  are the only homogeneous 3-Sasakian manifolds with the canonical fibering with fiber  $Sp(1)$ .

For the comparison with example ii, we explain about  $SU(3)/U(1)$  a little.

EXAMPLE i. Let  $(CP^2, J, g)$  be a complex 2-dimensional projective space with constant holomorphic sectional curvature  $H$ . Then the Riemannian curvature tensor  $R$  is expressed as

$$(7.1) \quad R_{ijkl} = (H/4)[g_{ik}g_{jl} - g_{il}g_{jk} + J_{ik}J_{jl} - J_{il}J_{jk} + 2J_{ij}J_{kl}],$$

and the Ricci curvature tensor is given by  $\text{Ric} = (3H/2)g$ . We normalize the scalar curvature  $S$  to  $S=48$ , i.e.,  $H=8$ . With respect to the orientation by an orthonormal frame  $\{v_1, v_2=Jv_1, v_3, v_4=-Jv_3\}$  the curvature operator  $R$  has eigenvalues 8, 8, 8, 24, 0, 0, and  $W^+=0$ . So, we have a quaternionic Kähler structure on  $(CP^2, g)$ . The sectional curvature  $\tilde{K}(\tilde{X}, \tilde{Y})$  of the 3-Sasakian structure on the canonical principal  $SO(3)$ -bundle  $P(CP^2)$  over  $CP^2$  takes all values between  $-1$  and  $5$  (cf. [10]), i.e.,

$$\begin{aligned} \tilde{K}(v_1^*, v_2^*) &= \tilde{K}(v_3^*, v_4^*) = 5, \\ \tilde{K}(v_1^*, v_3^*) &= \tilde{K}(v_1^*, v_4^*) = \tilde{K}(v_2^*, v_3^*) = \tilde{K}(v_2^*, v_4^*) = -1, \\ \tilde{K}(\tilde{v}, \xi_\alpha) &= 1, \quad \text{for } \tilde{v} \perp \xi_\alpha, \end{aligned}$$

where  $v_1^*$  (etc.) denotes the horizontal lift of  $v_1$  (etc.).

EXAMPLE ii. Let  $(CD^2, J, g)$  be a complex 2-dimensional simply connected space form with constant holomorphic sectional curvature  $H=-8$ . Then the Riemannian curvature tensor  $R$  is expressed as (7.1) with  $H=-8$ . We understand the fibering of the  $SO(3)$ -bundle  $P(CD^2)$  in two ways;  $(P(CD^2), g^*) \rightarrow (CD^2, g)$  and  $(P(CD^2), \tilde{g}) \rightarrow (CD^2, g)$ . By  $X^*$ ,  $Y^*$  and  $Z^*$  we denote the horizontal lifts of vector fields  $X$ ,  $Y$  and  $Z$  on  $CD^2$ . Then, by (2.3), (6.5) and (7.1), we obtain

$$\begin{aligned} R^*(X^*, Y^*)Z^* &= -\sum_{\alpha} [\tilde{g}(Y^*, \phi_{\alpha}Z^*)\phi_{\alpha}X^* - \tilde{g}(X^*, \phi_{\alpha}Z^*)\phi_{\alpha}Y^* - 2\tilde{g}(X^*, \phi_{\alpha}Y^*)\phi_{\alpha}Z^*] \\ &\quad - 2[g(Z, Y)X - g(Z, X)Y + g(Z, JY)JX - g(Z, JX)JY \\ &\quad + 2g(X, JY)JZ]^*. \end{aligned}$$

We define an orthonormal frame  $\{v_j^*, \xi_{\alpha}\}$ ,  $1 \leq j \leq 4$ ,  $1 \leq \alpha \leq 3$ , at a point of  $(P(CD^2), g^*)$  by using the horizontal lifts  $\{v_j^*\}$  of  $\{v_1, v_2=Jv_1, v_3, v_4=-Jv_3\}$ . In this case we have  $v_{\alpha+1}^* = \phi_{\alpha}v_1^* = -\phi_{\alpha}^*v_1^*$  for  $\alpha=1, 2, 3$ . Then the sectional curvature of  $(P(CD^2), g^*)$  with a triple of  $K$ -contact structures of  $nS$ -type takes all values between  $-11$  and  $1$ , i.e.,

$$\begin{aligned} K^*(v_1^*, v_2^*) &= K^*(v_3^*, v_4^*) = -11, \\ K^*(v_1^*, v_3^*) &= K^*(v_1^*, v_4^*) = K^*(v_2^*, v_3^*) = K^*(v_2^*, v_4^*) = -5, \\ K^*(\tilde{v}, \xi_{\alpha}) &= 1, \quad \text{for } \tilde{v} \perp \xi_{\alpha}, \end{aligned}$$

where the last one holds for any  $K$ -contact structure.

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