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REMARKS ON ARTIN APPROXIMATION WITH CONSTRAINTS

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Abstract

We study various approximation results of solutions of equations $f(x,Y)=0$ where $f(x,Y)\in\mathbb{K}[\![x]\!][Y]^r$ and x and Y are two sets of variables, and where some components of the solutions $y(x)\in\mathbb{K}[\![x]\!]^m$ do not depend on all the variables x_j. These problems were highlighted by M. Artin.

1. Introduction

Let $(R,\mathfrak{m})$ be a Henselian excellent Noetherian local ring, $f=(f_1,\dots,f_r)$ a system of polynomials in $Y=(Y_1,\dots,Y_m)$ over R and $\hat{y}$ a zero of f in the completion $\hat{R}$ of R.

Theorem 1 (Popescu [14], [15], Swan [17]). *For every $c\in\mathbb{N}$ there exists a zero y of f in R such that $y\equiv\hat{y}$ modulo $\mathfrak{m}^c$.*

M. Artin proved in [1, Theorem 1.10] the most important case of this theorem, that is when R is the algebraic power series ring in $x=(x_1,\dots,x_n)$ over a field $\mathbb{K}$. Usually we rewrite Theorem 1 saying that excellent Henselian local rings have the Artin approximation property.

Now suppose that $\hat{R}$ is the formal power series ring in $x=(x_1,\dots,x_n)$ over a field $\mathbb{K}$ and some components of $\hat{y}$ have some constraints, that is they depend only on some of the variables x_j. M. Artin asked if it is possible to find $y\in R^m$ such that the corresponding components depend on the same variables x_j (see [2, Question 4]). More precisely, we have the following question. For a set $J\subset[n]$ we denote by $\mathbb{K}[\![x_J]\!]$ the ring of formal power series in the x_j for $j\in J$.

Question 2 (Artin Approximation with constraints [16, Problem 1, page 68]). Let R be an excellent local subring of $\mathbb{K}[\![x]\!]$, $x=(x_1,\dots,x_n)$ such that the completion of R is $\mathbb{K}[\![x]\!]$ and $f\in R[Y]^r$, $Y=(Y_1,\dots,Y_m)$. Assume that there exists a formal solution $\hat{y}\in\mathbb{K}[\![x]\!]^m$ of $f=0$ such that $\hat{y}_i\in\mathbb{K}[\![x_{J_i}]\!]$ for some subset $J_i\subset[n]$, $i\in[m]$. Is it possible to approximate $\hat{y}$ by a solution $y\in R^m$ of $f=0$ such that $y_i\in R\cap\mathbb{K}[\![x_{J_i}]\!]$, $i\in[m]$?

If R is the algebraic power series ring in $x=(x_1,x_2,x_3)$ over $\mathbb{C}$ then Becker [4] gave a counterexample. If the set (J_i) is totally ordered by inclusion, that is the so called Nested Artin Approximation then this question has a positive answer in [14], [15, Corollary 3.7] (see also [6, Theorem 3.1] for an easy proof in the linear case). However, when R is the con-

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vergent power series ring in $x = (x_1, x_2, x_3)$ over $\mathbb{C}$ then Gabrielov [9] gave a counterexample (see also [10] for a general account on this problem).

A field extension $\mathbb{K} \subset \mathbb{K}'$ is algebraically pure (see [13], [3]) if every finite system of polynomial equations has a solution in $\mathbb{K}$ if it has one in $\mathbb{K}'$. Any field extension of an algebraically closed field is algebraically pure [13]. In connection with Question 2 the following theorem was proved.

Theorem 3 (Kosar-Popescu [11, Theorem 9])**.** *Let $\mathbb{K} \to \mathbb{K}'$ be an algebraically pure morphism of fields and $x = (x_1, \dots, x_n)$. Let J_i, $i \in [m]$ be subsets of $[n]$, and $A_i = \mathbb{K}\langle x_{J_i}\rangle$, resp. $A'_i = \mathbb{K}'\langle x_{J_i}\rangle$, $i \in [m]$ be the algebraic power series in x_{j_l} over $\mathbb{K}$ resp. $\mathbb{K}'$. Set $\mathcal{N} = A_1 \times \cdots \times A_m$ and $\mathcal{N}' = A'_1 \times \cdots \times A'_m$. Let f be a system of polynomials from $\mathbb{K}\langle x\rangle[Y]$, $Y = (Y_1, \dots, Y_m)$, and $\hat{y} \in \mathcal{N}'$, such that $f(\hat{y}) = 0$. Then there exist $y \in \mathcal{N}$ such that $f(y) = 0$ and $ord(y_i) = ord(\hat{y}_i)$ for $i \in [m]$.*

The goal of our paper is to replace somehow in Theorem 3 the algebraic power series by formal power series (see Theorem 14) and to state a certain Artin strong approximation with constraints property of the formal power series ring in x over a field $\mathbb{K}$ which is so-called $\aleph_0$-complete (see Corollary 16). This condition on $\mathbb{K}$ is necessary (see Remarks 15, 17). Finally we apply these results to extend approximation results due to J. Denef and L. Lipshitz for differential equations with coefficients in the ring of univariate polynomials to the case of several indeterminates (see Corollaries 18 and 20).

Finite fields, uncountable algebraically closed fields and ultraproducts of fields over $\mathbb{N}$ are $\aleph_0$-complete (see Theorem 5). If $(\mathbb{K}_n)_n$ is a sequence of fields and $\mathcal{F}$ is an ultrafilter of $\mathbb{N}$ we denote by $(\mathbb{K}_n)^*$ the ultraproduct (over the natural numbers) defined as $\left(\prod_{n\in\mathbb{N}} \mathbb{K}_n\right)/\mathcal{F}$, that is the factor of $\left(\prod_{n\in\mathbb{N}} \mathbb{K}_n\right)$ by the ideal $\{(x_n)_{n\in\mathbb{N}} \in \left(\prod_{n\in\mathbb{N}} \mathbb{K}_n\right) : \{n \in \mathbb{N} : x_n = 0\} \in \mathcal{F}\}$. When $\mathbb{K}$ is a single field, $\mathbb{K}^*$ denotes the ultrapower $\left(\prod_{n\in\mathbb{N}} \mathbb{K}\right)/\mathcal{F}$.

2. Solutions of countable systems of polynomial equations

Definition 4. Let $\mathbb{K}$ be a field. We say that $\mathbb{K}$ is $\aleph_0$-*complete* if every countable system $\mathcal{S}$ of polynomial equations (in a countable number of indeterminates) has a solution in $\mathbb{K}$ if and only if every finite sub-system of $\mathcal{S}$ has a solution in $\mathbb{K}$.

Theorem 5. *The following fields are $\aleph_0$-complete:*

a) *Every finite field.*
b) *Every uncountable algebraically closed field.*
c) *Every ultraproduct of fields over the natural numbers.*

Remark 6. Every ultraproduct is either finite or uncountable. So every algebraically closed field which is an ultraproduct is necessarily uncountable.

Proof. Let $\mathcal{S}$ be a system of countably many polynomial equations with coefficients in a field $\mathbb{K}$. We list the polynomial equations of $\mathcal{S}$ as $P_1, \dots, P_n, \dots$ which depends on the variables $x_1, \dots, x_l, \dots$.
For any $N \in \mathbb{N}$ let D_N be an integer such that the polynomials P_i, for $i \leq N$, depend only on the x_j for $j \leq D_N$.

Let us define the canonical projection maps:

$$\pi_{l,k} : \mathbb{K}^l = \mathbb{K}^k \times \mathbb{K}^{l-k} \longrightarrow \mathbb{K}^k \ \ \forall l \geq k \geq 1$$

that sends the vector $(x_1, \ldots, x_l)$ onto $(x_1, \ldots, x_k)$. We also define the projection maps

$$\pi_k : \mathbb{K}^{\mathbb{N}} \longrightarrow \mathbb{K}^k \ \ \forall k \geq 1$$

that send the sequence $(x_1, \ldots, x_n, \ldots)$ onto $(x_1, \ldots, x_k)$.
Let

$$V_\infty := \{x = (x_n)_n \in \mathbb{K}^{\mathbb{N}} \mid P_i(x) = 0 \ \forall i \in \mathbb{N}\}$$

and

$$V_N := \{x = (x_n)_n \in \mathbb{K}^{\mathbb{N}} \mid P_1(x) = \ldots = P_N(x) = 0\} \ \ \forall N \in \mathbb{N}.$$

Then we have that $V_\infty = \cap_{N\in\mathbb{N}} V_N$. By assumption, for every integer $N \geq 1$ we have that

$$V_N = \pi_{D_N}(V_N) \times \mathbb{K}^{\mathbb{N}\setminus\{1,\ldots,D_N\}}.$$

For every positive integers N and k we define

$$C_N^k = \pi_k(V_N).$$

Now set

$$C^k := \bigcap_{N\in\mathbb{N}} C_N^k.$$

We claim that, if for every k, $C_k \neq \emptyset$, then $\mathcal{S}$ has a solution; indeed, by construction $(x_1, \ldots, x_k) \in C^k$ if and only if for every N and k there exists $(x_{k+1}, \ldots,) \in \mathbb{K}^{\mathbb{N}}$ such that $(x_1, \ldots, x_k, x_{k+1}, \ldots) \in V_N$. In particular $\pi_{k+1,k}(C^{k+1}) = C^k$ for every k.
Now let $x_1 \in C^1$. Then there exists $x_2 \in \mathbb{K}$ such that $(x_1, x_2) \in C^2$. By induction we can find a sequence of elements $x_n \in \mathbb{K}$ such that for every k

$$(x_1, \ldots, x_k) \in C^k.$$

Thus the sequence $x = (x_n)_n \in V_N$ for every N so it belongs to V_∞. Hence $\mathcal{S}$ has a solution.

a) Let us assume that $\mathbb{K}$ is a finite field.
Then the C_N^k are finite subsets of $\mathbb{K}^k$. Since $V_{N+1} \subset V_N$ for every N, the sequence $(C_N^k)_N$ is decreasing so it stabilizes. Therefore $C_k \neq \emptyset$ and $\mathcal{S}$ has a solution.

b) Now let us assume that $\mathbb{K}$ is an uncountable algebraically closed field. We have that

$$C_N^k = \pi_k(V_N) = \pi_{D_N,k}\Big(\{x = (x_1, \ldots, x_{D_N}) \in \mathbb{K}^{D_N} \mid P_1(x) = \ldots = P_N(x) = 0\}\Big).$$

Thus the C_N^k are constructible subsets of $\mathbb{K}^k$ since $\mathbb{K}$ is algebraically closed (by Chevalley's Theorem). Let us recall that a constructible set is a finite union of sets of the form $X\setminus Y$ where X and Y are Zariski closed subsets of $\mathbb{K}^k$.
Thus the sequence $(C_N^k)_N$ is a decreasing sequence of constructible subsets of $\mathbb{K}^k$. Let F_N^k denote the Zariski closure of C_N^k. Then the sequence $(F_N^k)_N$ is a decreasing sequence of Zariski closed subsets of $\mathbb{K}^k$. By Noetherianity this sequence stabilizes, i.e. $F_N^k = F_{N_0}^k$ for

every $N \geq N_0$ and some positive integer N_0. By assumption $C^k_{N_0} \neq \emptyset$ so $F^k_{N_0} \neq \emptyset$. Let F be an irreducible component of $F^k_{N_0}$.
Since C^k_N is constructible, $C^k_N = \cup_i \left(X^N_i \backslash Y^N_i\right)$ for a finite number of Zariski closed sets X^N_i and Y^N_i with $X^N_i \backslash Y^N_i \neq \emptyset$ and X^N_i is assumed irreducible. Since X^N_i is irreducible the Zariski closure of $X^N_i \backslash Y^N_i$ is X^N_i. Therefore for $N \geq N_0$ we have that

$$F^k_{N_0} = F^k_N = \cup_i X^N_i.$$

But F being irreducible, for every $N \geq N_0$ one of the X^N_i has to be equal to F. Thus for every $N \geq N_0$ there exists a closed proper subset $Y_N \subset F$ such that

$$F \backslash Y_N \subset C^k_N \ \ \forall N \geq N_0.$$

Since $\mathbb{K}$ is uncountable

$$\bigcup_{N \geq N_0} Y_N \subsetneq F.$$

This is a well known fact (see for instance Exercice 5.10, [12] p. 76). This implies that $C^k \neq \emptyset$ and $\mathcal{S}$ has a solution.

Finally c) is given as in Lemma 2.17 [13]. □

Remark 7. It is quite straightforward to prove that a field $\mathbb{K}$ that is $\aleph_1$-saturated is $\aleph_0$-complete (for the definition of a saturated model see [7, Section 2.3]). One can prove that the three fields of Theorem 5 are $\aleph_1$-saturated providing an alternative proof of the fact that these fields are $\aleph_0$-complete.

Example 8. Let $\mathbb{K} = \overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$. Since $\overline{\mathbb{Q}}$ is countable we may list its elements as $\alpha_1, \alpha_1, \ldots, \alpha_l, \ldots$. Let $\mathcal{S}$ be the system of equations:

$$P_1 = 0, P_l = (x_1 - \alpha_l)\, x_l - 1 = 0 \ \ \forall l \geq 2.$$

For every integer $N \geq 1$ the vector

$$\left(\alpha_N, \frac{1}{\alpha_N - \alpha_2}, \ldots, \frac{1}{\alpha_N - \alpha_{N-1}}\right) \in \mathbb{K}^{N-1}$$

is a solution of

$$P_1 = \cdots = P_{N-1} = 0.$$

But $\mathcal{S}$ has no solution. Indeed if $x = (x_1, \ldots, x_n, \ldots) \in \mathbb{K}^{\mathbb{N}}$ was a solution of $\mathcal{S}$ then we would have that

$$(x_1 - \alpha_l)x_l = 1 \quad \forall l \geq 2. \tag{2.1}$$

But $x_1 \in \overline{\mathbb{Q}}$ so $x_1 = \alpha_{l_0}$ for some $l_0 \geq 0$. Thus (3.2) for $l = l_0$ would give

$$0 = (x_1 - \alpha_{l_0})x_{l_0} = 1$$

which is impossible. So $\overline{\mathbb{Q}}$ is not an $\aleph_0$-complete field.

EXAMPLE 9. Let $\mathbb{K} = \mathbb{R}$ be the field of real numbers. Let $\mathcal{S}$ be the system of equations:

$$P_1 = 0, P_l = x_l^2 - (x_1 - l) = 0 \;\; \forall l \geq 2.$$

Then $P_1 = \cdots = P_l = 0$ has a solution $x = (x_1, \ldots, x_n)$ if and only if $x_1 - l \geq 0$.
In particular $\mathcal{S}$ has no solution. So $\mathbb{R}$ is not an $\aleph_0$-complete field.

3. Approximation with constraints

We recall some elementary facts on algebraically pure field extensions, referring to [13] and [3, (2.3)] for details.

REMARK 10. (1) If $\mathbb{K} \longrightarrow \mathbb{L}$ is a field extension of real closed fields then it is algebraically pure.

(2) If $\mathbb{K}$ is an infinite field and $x = (x_1, \ldots, x_n)$ then $\mathbb{K} \longrightarrow \mathbb{K}(x)$ is algebraically pure. [13]

(3) If $\mathbb{K}$ is a field and $x = (x_1, \ldots, x_n)$, we denote by $\mathbb{K}\langle\langle x\rangle\rangle$ the field of algebraic power series, and by $\mathbb{K}\{\{x\}\}$ the field of convergent power series (when $\mathbb{K}$ is a complete valued field). Then $\mathbb{K}\langle\langle x\rangle\rangle \longrightarrow \mathbb{K}\{\{x\}\}$ and $\mathbb{K}\{\{x\}\} \longrightarrow \mathbb{K}((x))$ are algebraically pure by Artin approximation theorem. [1]

(4) If $\mathbb{K}_1 \longrightarrow \mathbb{K}_2$ and $\mathbb{K}_2 \longrightarrow \mathbb{K}_3$ are algebraically pure then $\mathbb{K}_1 \longrightarrow \mathbb{K}_3$ is algebraically pure. [13]

Lemma 11. [3] *Let $\mathbb{K}$ be a field and let $\mathbb{K}^*$ be an ultrapower of $\mathbb{K}$. Then the morphism $\mathbb{K} \longrightarrow \mathbb{K}^*$ sending every element $a \in \mathbb{K}$ onto the constant sequence $(a, \ldots, a, \ldots)$ is algebraically pure.*

Proof. Let $\mathcal{S} = (P_i)_{i\in I}$ be a finite system of polynomial equations with coefficients in $\mathbb{K}$ in the indeterminates $Y_1, \ldots, Y_m$. Let us assume that there exists $y^* \in (\mathbb{K}^*)^m$ such that

$$P_i(y^*) = 0 \quad \forall i \in I.$$

Let $(y_n)_{n\in\mathbb{N}} \in (\mathbb{K}^m)^{\mathbb{N}}$ be a sequence whose image in $(\mathbb{K}^*)^m$ is y^*. Therefore for every $i \in I$ there exists $\mathcal{U}_i \in \mathcal{F}$ (here $\mathcal{F}$ denotes the ultrafilter such that $\mathbb{K}^* = \mathbb{K}^{\mathbb{N}}/\mathcal{F}$) such that

$$\forall n \in U_i, \;\; P_i(y_n) = 0.$$

Since I is finite the intersection $\mathcal{U} := \cap_{i\in I}\mathcal{U}_i \in \mathcal{F}$. Thus for every $n \in \mathcal{U}$ we have that

$$P_i(y_n) = 0 \;\; \forall i \in I.$$

Hence $\mathcal{S}$ has a solution in $\mathbb{K}^m$. Therefore $\mathbb{K} \longrightarrow \mathbb{K}^*$ is algebraically pure. □

Proposition 12. *Let $\mathbb{K}$ be a $\aleph_0$-complete field. Let $x = (x_1, \ldots, x_n)$, $Y = (Y_1, \ldots, Y_m)$, $f = (f_1, \ldots, f_r) \in \mathbb{K}[[x]][Y]^r$ and $J_i \subset [n]$, $i \in [m]$.*
If for every $c \in \mathbb{N}$ there exists $y^{(c)} \in \mathbb{K}[[x]]^m$, with $y_i^{(c)} \in \mathbb{K}[[x_{J_i}]]$ for every i, such that

$$f(y^{(c)}) \equiv 0 \text{ modulo } (x)^c$$

then there exists $y \in \mathbb{K}[[x]]^m$, with $y_i \in \mathbb{K}[[x_{J_i}]]$ for every i, such that

$$f(y) = 0.$$

Proof. Let us set

$$B_i := \mathbb{N}^{\varepsilon_{1,i}} \times \cdots \times \mathbb{N}^{\varepsilon_{m,i}}$$

where $\varepsilon_{k,i} = 1$ if $k \in J_i$, $\varepsilon_{k,i} = 0$ if $k \notin J_i$, and

$$Y_i = \sum_{\alpha \in B_i} Y_{i,\alpha} x^\alpha \quad \forall i = 1, \dots, m.$$

We denote by $P_{k,\beta}$ the coefficient of x^β in $f_k(\sum_{\alpha \in B_1} Y_{1,\alpha} x^\alpha, \dots, \sum_{\alpha \in B_m} Y_{m,\alpha} x^\alpha)$. Let us denote by $\mathcal{S}$ the system of polynomial equations

$$P_{k,\beta} = 0, \ k \in [p], \ \beta \in \mathbb{N}^n. \tag{3.1}$$

depending on the variables $Y_{i,\alpha}$ for $i \in [m]$ and $\alpha \in B_i$.
Since $\mathbb{K}$ is a $\aleph_0$-complete field and every finite sub-system of $\mathcal{S}$ has a solution, $\mathcal{S}$ has a solution $(y_{i,\alpha})_{i \in [m], \alpha \in B_i}$ with coefficients in $\mathbb{K}$. Thus if $y = (y_1, \dots, y_m)$ with

$$y_i = \sum_{\alpha \in B_i} y_{i,\alpha} x^\alpha$$

then we have that $f(y) = 0$. □

Example 13. In [5] two examples are given that show that this statement is no longer true without the condition of $\mathbb{K}$ being $\aleph_0$-complete: the first one is a system of polynomial equations over the algebraic closure of $\mathbb{F}_p$ (see Example (i) p. 200 [5]) and the second one is an example of polynomial equations over $\mathbb{Q}$ (see Example (ii) p. 200 [5]).

Theorem 14. *Let $\mathbb{K} \subset \mathbb{K}'$ be an algebraically pure field extension where $\mathbb{K}$ is $\aleph_0$-complete. We set $x = (x_1, \dots, x_n)$ and $f \in \mathbb{K}[[x]][Y]^r$, $Y = (Y_1, \dots, Y_m)$.*
Assume that there exists a solution $\hat{y} \in \mathbb{K}'[[x]]^m$ of $f = 0$ such that

$$\hat{y}_i \in \mathbb{K}'[[x_{J_i}]]$$

for some subsets $J_i \subset [n]$, $i \in [m]$. Then there is a solution $y \in \mathbb{K}[[x]]^m$ of $f = 0$ such that $y_i \in \mathbb{K}[[x_{J_i}]]$ and $ord(y_i) = ord(\hat{y}_i)$, $i \in [m]$.

Proof. Let us write $\hat{y}_i = \sum_{\alpha \in B_i} \hat{y}_{i,\alpha} x^\alpha$ where $B_i \subset \mathbb{N}^n$ denotes the support of $\hat{y}_i$.
We have that

$$\begin{aligned} f(\hat{y}) = 0 &\Longleftrightarrow f_k(\hat{y}) = 0 \ \ \forall k = 1, \dots, r \\ &\Longleftrightarrow \forall k, \ \forall \beta \in \mathbb{N}^n \text{ the coefficient of } x^\beta \text{ in } f_k(\hat{y}) \text{ is } 0. \end{aligned}$$

Let us denote by $P_{k,\beta}$ the coefficient of x^β in f_k after replacing each Y_i by the term $\sum_{\alpha \in B_i} Y_{i,\alpha} x^\alpha$, and let $\mathcal{S}$ be the system of equations

$$P_{k,\beta} = 0 \ \ \forall k \in \mathbb{N}, \ \forall \beta \in \mathbb{N}^n$$

in the indeterminates $Y_{i,\alpha}$ for $i = 1, \dots, m$ and $\alpha \in B_i$. Since $\mathcal{S}$ has a solution in $\mathbb{K}'$ every finite sub-system of $\mathcal{S}$ has a solution in $\mathbb{K}'$ and, since $\mathbb{K} \longrightarrow \mathbb{K}'$ is algebraically pure, every finite sub-system of $\mathcal{S}$ has a solution in $\mathbb{K}$. Then, since $\mathbb{K}$ is a $\aleph_0$-complete field the system $\mathcal{S}$

has a solution $(y_{i,\alpha})_{i\in[m],\alpha\in B_i}$ with coefficients in $\mathbb{K}$. This means that if $y=(y_1,\dots,y_m)$ with

$$y_i=\sum_{\alpha\in B_i} y_{i,\alpha}x^\alpha$$

then $f(y)=0$. Since B_i is the support of $\hat{y}_i$,the support of y_i is included in the support of $\hat{y}_i$ for every i. In particular we have that $\mathrm{ord}(\hat{y}_i)\leq \mathrm{ord}(y_i)$ for every i.
Now let us assume moreover that $\mathrm{ord}(\hat{y}_i)=c_i$ and that, for every $i=1,\dots,m$, $\hat{y}_{i,\alpha_i}\neq 0$ with $|\alpha_i|=c_i$ (here for $\beta=(\beta_1,\dots,\beta_n)$ we set $|\beta|:=\beta_1+\cdots+\beta_n$). Then there exists, for $i=1,\dots,m$, an element $\hat{z}_i\in\mathbb{K}'$ such that

$$\hat{y}_{i,\alpha_i}\hat{z}_i=1,\ \forall i=1,\dots,m.$$

By adding the equations

$$Y_{i,\alpha_i}Z_i=1,\ \forall i=1,\dots,m \tag{3.2}$$

to the system $\mathcal{S}$ we can suppose that there exists $z_i\in\mathbb{K}$ for every i such that Equations (3.2) are satisfied. Thus

$$\mathrm{ord}(y_i)=c_i=\mathrm{ord}(\hat{y}_i)\ \ \forall i=1,\dots,m$$

and the theorem is proven. □

REMARK 15. By Lemmas 5.1 and 5.2 [16] every system $\mathcal{T}$ of partial polynomial differential equations with coefficients in $\mathbb{K}[\![x]\!]$ (with $x=(x_1,\dots,x_n)$) and indeterminates Y_1, $\dots$, Y_m, provides a system $\mathcal{S}$ of polynomial equations with coefficients in $\mathbb{K}[\![x]\!][t]$ (with $t=(t_1,\dots,t_l)$) and indeterminates $Y_1,\dots,Y_m,Z_1,\dots,Z_k$ such that $y\in\mathbb{K}[\![x]\!]^m$ is a solution of $\mathcal{T}$ if and only if there exists $z\in\mathbb{K}[\![x,t]\!]^k$ such that (y,z) is a solution of $\mathcal{S}$ and z satisfies some constraints conditions as in Proposition 12.
By Corollary 4.7 [8] there exists a system of partial differential equations $\mathcal{T}$ defined over $\overline{\mathbb{Q}}$ having a solution whose components are in $\mathbb{C}[\![x]\!]$ but no solution whose components are in $\overline{\mathbb{Q}}[\![x]\!]^m$. So it shows that there exists a system of polynomial equations $\mathcal{S}$ with coefficients in $\overline{\mathbb{Q}}[x]$ which has no solution $y\in\overline{\mathbb{Q}}[\![x]\!]^m$ such that $y_i\in\overline{\mathbb{Q}}[\![x_{J_i}]\!]$ for every i for some $J_i\subset[n]$, but has a solution $y'\in\mathbb{C}[\![x]\!]^m$ such that $y'_i\in\mathbb{C}[\![x_{J_i}]\!]$ for every i.
This shows that Theorem 14 is no longer true in general if $\mathbb{K}$ is not $\aleph_0$-complete.
Moreover since this system $\mathcal{S}$ has a solution with coefficients in $\mathbb{C}$ satisfying the constraints conditions and since $\overline{\mathbb{Q}}\longrightarrow\mathbb{C}$ is algebraically pure, for every $c\in\mathbb{N}$ there exists $y^{(c)}\in\overline{\mathbb{Q}}[\![x]\!]^m$ (satisfying the constraints conditions) such that $f(y^{(c)})\in(x)^c$. But there is no $y\in\overline{\mathbb{Q}}[\![x]\!]^m$ (satisfying the constraints conditions) such that $f(y)=0$. This also provides an example showing that Proposition 12 is not true if $\Bbbk=\overline{\mathbb{Q}}$.

Corollary 16. *Let $\mathbb{K}$ be a $\aleph_0$-complete field. Let us set $x=(x_1,\dots,x_n)$, $f=(f_1,\dots,f_r)\in\mathbb{K}[\![x]\!][Y]^r$, $Y=(Y_1,\dots,Y_m)$ and $J_i\subset[n]$, $i\in[m]$. Then there exists a map $\nu:\mathbb{N}^m\to\mathbb{N}$ such that if $y'=(y'_1,\dots,y'_m)$, $y'_i\in\mathbb{K}[\![x_{J_i}]\!]$, $i\in[m]$ satisfies $f(y')\equiv 0$ modulo $(x)^{\nu(c)}$ for some $c=(c_1,\dots,c_m)\in\mathbb{N}^m$ and $\mathrm{ord}(y'_i)=c_i$, $i\in[m]$ then there exists $y_i\in\mathbb{K}[\![x_{J_i}]\!]$ for all $i\in[m]$ such that $y=(y_1,\dots,y_m)$ is a zero of f and $\mathrm{ord}(y_i)=c_i$ for all $i\in[m]$.*

Proof. Let c be as above. For proof by contradiction suppose that for each $q\in\mathbb{N}$ there exists $\hat{y}_q\in\mathbb{K}[\![x]\!]^m$ with $f(\hat{y})\equiv 0$ modulo x^q, $\hat{y}_{q,i}\in\mathbb{K}[\![x_{J_i}]\!]$, $\mathrm{ord}(\hat{y}_{q,i})=c_i$, but there exists no solution y' in $\mathbb{K}[\![x]\!]$ with $y'_i \in \mathbb{K}[\![x_{J_i}]\!]$, $\mathrm{ord}(y'_i) = c_i$. Then let us define $y^*_i = [(y_{qi})_q] \in \mathbb{K}[\![x_{J_i}]\!]^*$. So we have that $f(y^*) \in \cap_q x^q \mathbb{K}[\![x]\!]^*$. Set $\bar{y} = y^*$ modulo $\cap_q x^q \mathbb{K}[\![x]\!]^*$ which corresponds to an element in $\mathbb{K}^*[\![x]\!]$ with $f(\bar{y}) = 0$ (see Lemma 3.4 [5]), $\mathrm{ord}(\bar{y}_i) = c_i$ and $\bar{y}_i \in \mathbb{K}^*[\![x_{J_i}]\!]$. By Lemma 11 and Theorem 14 there exists $y \in \mathbb{K}[\![x]\!]^m$ with $f(y) = 0$, $\mathrm{ord}(y_i) = c_i$ and $y_i \in \mathbb{K}[\![x_{J_i}]\!]$. We obtain a contradiction, so the theorem is true. □

Remark 17. In Example (iii) p. 201 [5] an example of a system of polynomial equations over $\mathbb{C}$ with constraints is given for which the following is shown: there is no $\nu \in \mathbb{N}$ such that if there exists $\hat{y} \in \mathbb{C}[\![x]\!]^m$ with $f(x, \hat{y}) \in (x)^\nu$ with the given constraints then there exists a solution $y \in \mathbb{C}[\![x]\!]$ of $f = 0$ with same constraints and such that $y \equiv \hat{y}$ modulo (x).

4. Approximation for differential equations

Corollary 18. *Let $\mathbb{K}$ be a $\aleph_0$-complete field. Let F be a system of polynomial equations in $z_1, \ldots, z_q$ and some of their differentials $\partial^{|j_1|} z_{i_1}/\partial x^{j_1}, \ldots, \partial^{|j_s|} z_{i_s}/\partial x^{j_s}$, $i_1, \ldots, i_s \in [q]$, and $j_1, \ldots, j_s \in \mathbf{N}^n$, with coefficients in $\mathbb{K}[\![x]\!]$. If $F = 0$ has approximate solutions up to any order then $F = 0$ has a solution with coefficients in $\mathbb{K}[\![x]\!]$.*

Proof. Exactly as in Remark 15, Lemmas 5.1 and 5.2 [16] show that for such a system $F = 0$ there is a system of polynomial equations $G = 0$ with coefficients in $\mathbb{K}[\![x]\!][t]$ (with $t = (t_1, \ldots, t_l)$) and indeterminates $Y_1, \ldots, Y_m, Z_1, \ldots, Z_k$ such that $y \in \mathbb{K}[\![x]\!]^m$ is a solution of $F = 0$ if and only if there is $z \in \mathbb{K}[\![x, t]\!]^k$ such that (y, z) is a solution of $G = 0$ with constraints.
Moreover $y \in \mathbb{K}[\![x]\!]^m$ is an approximate solution of $F = 0$ up to order c if and only if there is $z \in \mathbb{K}[\![x, t]\!]^k$ such that (y, z) is an approximate solution of $G = 0$ up to degree c with constraints. This shows that Proposition 12 implies Corollary 18. □

Remark 19. This theorem has been proven in [8] in the case of a single indeterminate x under some different hypothesis on $\mathbb{K}$, namely $\mathbb{K}$ has to be a characteristic zero field which is either algebraically closed, a real closed field or a Henselian valued field. Still in [8] they remark that this theorem is quite easy to prove when $\mathbb{K} = \mathbb{C}$.
Again in [8] is given an example of a system of partial differential equations with coefficients in $\mathbb{R}[\![x_1, \ldots, x_n]\!]$ for $n \geq 2$ having approximate solution up to any degree, but no exact solution (see Corollary 4.10 [8]). And Corollary 4.7 [8] provides an analogous example in the case where $\mathbb{K} = \overline{\mathbb{Q}}$. These examples show that the univariate case and the case of several variables x are different.

Corollary 20. *Let $\mathbb{K}$ be a $\aleph_0$-complete field. Let F be a system of differential equations in $z_1, \ldots, z_q$ and some of their differentials $\partial^{|j_1|} z_{i_1}/\partial x^{j_1}, \ldots, \partial^{|j_s|} z_{i_s}/\partial x^{j_s}$, $i_1, \ldots, i_s \in [q]$, and $j_1, \ldots, j_s \in \mathbf{N}^n$ with coefficients in $\mathbb{K}[\![x]\!]$. Then there exists a map $\tau : \mathbf{N}^{q+s} \to \mathbf{N}$ such that if $z' = (z'_1, \ldots, z'_q)$, satisfies*

$$F(z', \partial^{|j_1|} z'_{i_1}/\partial x^{j_1}, \ldots, \partial^{|j_s|} z_{i_s}/\partial x^{j_s}) \equiv 0 \ \textit{modulo} \ (x)^{\tau(c)}$$

for some $c = (c_1, \ldots, c_q, c_{i_1, j_1}, \ldots, c_{i_s, j_s}) \in \mathbf{N}^{q+s}$ and ord$(z'_i) = c_i$, $i \in [q]$,

$$ord\left(\frac{\partial^{|j_k|}z'_{i_k}}{\partial x^{j_k}}\right) = c_{i_k,j_k},$$

$k \in [s]$ then there exists $z = (z_1, \dots, z_q) \in \mathbb{K}[[x]]^q$ a solution of F together with its corresponding differentials such that $ord(z_i) = c_i$ for all $i \in [q]$ and

$$ord\left(\frac{\partial^{|j_k|}z_{i_k}}{\partial x^{j_k}}\right) = c_{i_k,j_k}, \quad k \in [s].$$

Proof. Let $f \in \mathbb{K}[[x]][Y]^r$, $Y = (Y_1, \dots, Y_m)$, $m > q + s$ be the transformation of F in an algebraic system of equations with constraints as done in the proof of Corollary 18. Assume that z_i corresponds to Y_i and $\partial^{|j_k|}z_{i_k}/\partial x^{j_k}$ corresponds to Y_{q+k}. Then applying Corollary 16 to f we get a function $\tau : \mathbf{N}^{q+s} \to \mathbf{N}$ which works also in our case F. □

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