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REMARKS ON CHACON'S BITING LEMMA

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ABSTRACT. Chacon's Biting Lemma states roughly that any bounded sequence in L^1 possesses a subsequence converging weakly in L^1 outside a decreasing family E_k of measurable sets with vanishingly small measure. A simple new proof of this result is presented that makes explicit which sets E_k need to be removed. The proof extends immediately to the case when the functions take values in a reflexive Banach space. The limit function is identified via the Young measure and approximations. The description of concentration provided by the lemma is discussed via a simple example.

1. INTRODUCTION AND MAIN RESULT

The purpose of this note is to give an elementary proof of the following result.

Lemma. *Let* $(\Omega, \mathscr{F}, \mu)$ *be a finite positive measure space,* X *a reflexive Banach space, and let* $\{f^{(j)}\}$ *be a bounded sequence in* $L^1(\Omega; X)$, *i.e.*

$$\sup_j \int_\Omega \|f^{(j)}\|_X \, d\mu = C_0 < \infty.$$

Then there exist a function $f \in L^1(\Omega; X)$, *a subsequence* $\{f^{(\nu)}\}$ *of* $\{f^{(j)}\}$, *and a nonincreasing sequence of sets* $E_k \in \mathscr{F}$ *with* $\lim_{k\to\infty} \mu(E_k) = 0$, *such that*

$$f^{(\nu)} \rightharpoonup f \qquad \text{weakly in } L^1(\Omega \setminus E_k; X)$$

as $\nu \longrightarrow \infty$ *for every fixed* k.

In the above $L^1(\Omega; X)$ denotes the Banach space of (equivalence classes of) strongly measurable mappings $g\colon \Omega \longrightarrow X$ with finite norm

$$\|g\|_1 = \int_\Omega \|g\|_X \, d\mu.$$

Since X is reflexive, the dual $L^1(\Omega; X)^*$ of $L^1(\Omega; X)$ can be identified with the space $L^\infty(\Omega; X^*)$ of strongly measurable mappings $h\colon \Omega \longrightarrow X^*$ such that

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$\|h\|_\infty = \operatorname{ess\,sup}_\Omega \|h\|_{X^*} < \infty$ (cf. Diestel and Uhl [6, pp. 98, 76], A. and C. Ionescu Tulcea [10, p. 95]).

For $X = \mathbf{R}$ the lemma is stated and proved in Brooks and Chacon [5]; another proof, due to Thomsen and Plachky, appears in Plachky [14, pp. 201–202], and is reproduced in Balder [2] but, as pointed out by M. Valadier and E. Balder after the publication of [2], the argument seems to be incomplete. The extension to the case where X is a (separable) reflexive Banach space has been independently given by Balder [3]. This extension is not difficult to obtain and is not the main goal of the present paper.

The result is a useful tool in some variational problems where there is only an L^1 bound on minimizing sequences. One such use has recently been made by Lin [11] in a study of the pure traction problem of nonlinear thermoelasticity; he observed that for $X = \mathbf{R}$ the lemma could easily be deduced from a related lemma of Acerbi and Fusco [1].

Our purpose is providing yet another proof of the lemma here is that our proof is based on different principles and seems to us simpler and more constructive; in particular it makes rather explicit which sets E_k need to be removed from Ω to recover the weak L^1 convergence. The only nontrivial result necessary for the proof is the Dunford-Pettis criterion for weak compactness in L^1. Provided an appropriate Banach space valued version of this criterion is used, the proof for the case when X is a reflexive Banach space is no harder than that for $X = \mathbf{R}$.

To illustrate some features of the lemma, consider now the case $X = \mathbf{R}$. Since $\|f^{(\nu)}\|_1 \le C_0$ it follows that, up to the extraction of a further subsequence, $f^{(\nu)}$ converges weak* to some limit, β say, in the sense of measures. In general there is no connection between f and β, even if β is an L^1 function (see Example 2, page 661, which also shows that the sets E_k cannot in general be chosen to be closed). The difference between f and β measures the amount of concentration in the sequence (cf. P.-L. Lions [12, 13]), provided the $f^{(j)}$ are nonnegative (for general $f^{(j)}$ there is the possibility of cancellation of positive and negative concentrations, so that a suitable measure of the amount of concentration is obtained by considering $|f^{(j)}|$ in place of $f^{(j)}$).

2. Proof of the lemma

Let $\{f^{(j)}\}$ satisfy (1) and for $l \ge 0$ define

$$\varphi_j(l) = \int_{\{\|f^{(j)}\|_X \ge l\}} \|f^{(j)}\|_X \, d\mu.$$

Then

(i) $\varphi_j(0) = \|f^{(j)}\|_1 \le C_0$;

(ii) for each j, $\varphi_j(\cdot)$ is nonincreasing and upper semicontinuous (the upper semicontinuity follows, for example, by considering the points x_0 where

$\|f^{(j)}(x_0)\|_X \geq l$ and the points x_1 where $\|f^{(j)}(x_1)\|_X < l$, and applying Lebesgue's Dominated Convergence Theorem;

(iii) $\varphi_j(l) \longrightarrow 0$ as $l \longrightarrow \infty$, for each fixed j.

By these properties and the Helly Selection Theorem we can extract a subsequence, again denoted $f^{(j)}$, such that

$$\alpha(l) \stackrel{\text{def}}{=} \lim_{j\to\infty} \varphi_j(l)$$

exists for all $l \geq 0$. Clearly $\alpha(\cdot)$ is nonincreasing. Let $\alpha_\infty = \lim_{l\to\infty} \alpha(l)$.

Case 1. $\alpha_\infty = 0$. In this case the subsequence $\{f^{(j)}\}$ is sequentially weakly relatively compact in $L^1(\Omega; X)$. In fact given $\varepsilon > 0$ we can choose l_0 sufficiently large so that $\alpha(l_0) < \varepsilon$, then j_0 sufficiently large so that $\varphi_j(l_0) < \varepsilon$ for all $j \geq j_0$, and then $l_1 > l_0$ sufficiently large so that $\varphi_j(l_1) < \varepsilon$ for all $j < j_0$. Thus $\varphi_j(l_1) < \varepsilon$ for all j, so that by a Banach space valued version of the Dunford-Pettis Theorem (A. and C. Ionescu Tulcea [10, p. 117], Diestel & Uhl [6, pp. 101, 76]; the reader interested only in the case $X = \mathbf{R}$ can consult, for example, Edwards [9, p. 274]) there exists a further subsequence $\{f^{(\nu)}\}$ which converges weakly in $L^1(\Omega; X)$ to some $f \in L^1(\Omega; X)$, so that the conclusion of the lemma holds with all the sets E_k empty.

Case 2. $\alpha_\infty > 0$. In this case we claim that there exists a subsequence $l_j \longrightarrow \infty$ such that $\varphi_j(l_j) \longrightarrow \alpha_\infty$. Indeed, we can define $l_j = \sup\{l \geq 0 \colon \varphi_j(l) \geq \alpha_\infty - l^{-1}\}$. The supremum is attained because $\varphi_j(l) \to 0$ as $l \to \infty$ and φ_j is upper semicontinuous. If $\{l_j\}$ contained a bounded subsequence $\{l_\gamma\}$ then we would have $\varphi_\gamma(l') < \alpha_\infty - (l')^{-1}$ for any $l' > \sup_\gamma l_\gamma$; letting γ tend to ∞ gives a contradiction since $\alpha(\cdot)$ is nonincreasing. Hence $l_j \to \infty$. Also, for any $m \geq 0$,

$$\alpha_\infty - l_j^{-1} \leq \varphi_j(l_j) \leq \varphi_j(m) \qquad \text{for } j \text{ sufficiently large.}$$

Hence $\alpha_\infty \leq \varliminf_{j\to\infty} \varphi_j(l_j) \leq \varlimsup_{j\to\infty} \varphi_j(l_j) \leq \alpha(m)$, and letting $m \to \infty$ gives $\varphi_j(l_j) \to \alpha_\infty$.

We next claim that

$$\lim_{m\to\infty} \sup_{j\geq 1} \int_{\{m \leq \|f^{(j)}\|_X < l_j\}} \|f^{(j)}\|_X \, d\mu = 0. \tag{1}$$

To see this, note firstly that

$$S(m) \stackrel{\text{def}}{=} \sup_{j\geq 1} \int_{\{m \leq \|f^{(j)}\|_X < l_j\}} \|f^{(j)}\|_X \, d\mu$$

is nonincreasing, and secondly that

$$S(m) = \sup_{j \geq 1,\ l_j > m} [\varphi_j(m) - \varphi_j(l_j)].$$

Given any $\varepsilon > 0$, there exists m_1 such that $\alpha(m_1) < \alpha_\infty + \varepsilon$. Then there exists j_0 such that if $j \geq j_0$ then $\varphi_j(m_1) \leq \alpha(m_1) + \varepsilon$ and $\varphi_j(l_j) \geq \alpha_\infty - \varepsilon$, and hence

$$\varphi_j(m_1) - \varphi_j(l_j) \le \alpha(m_1) + \varepsilon - \alpha_\infty + \varepsilon \le 3\varepsilon.$$

Choosing m_2 such that $m_2 \ge m_1$ and $m_2 \ge \max_{j<j_0} l_j$ we deduce that

$$S(m_2) \le 3\varepsilon,$$

which proves (1).

Given $\delta > 0$, choose a new subsequence, again denoted $\{f^{(j)}\}$, such that $(\sum_j l_j^{-1})C_0 \le \delta$. Let $E = \cup_j\{\|f^{(j)}\|_X \ge l_j\}$. Then since

$$l_j \mu(\{\|f^{(j)}\|_X \ge l_j\}) \le \int_{\{\|f^{(j)}\|_X \ge l_j\}} \|f^{(j)}\|_X \, d\mu \le C_0,$$

we have $\mu(E) \le \delta$, and

$$\lim_{m\to\infty} \sup_{j\ge 1} \int_{\{\|f^{(j)}\|_X \ge m\}\setminus E} \|f^{(j)}\|_X \, d\mu \le \lim_{m\to\infty} \sup_{j\ge 1} \int_{\{m\le \|f^{(j)}\|_X < l_j\}} \|f^{(j)}\|_X \, d\mu = 0.$$

Hence by the Banach space valued Dunford-Pettis Theorem $\{f^{(j)}\}$ is sequentially weakly relatively compact in $L^1(\Omega \setminus E; X)$. Repeating this procedure for $\delta = k^{-1}$, $k = 1, 2, \ldots$, and taking successive subsequences, we obtain a diagonal subsequence $\{f^{(\nu)}\}$, a nonincreasing sequence E_k of μ-measurable sets with $\lim_{k\to\infty} \mu(E_k) = 0$, and a strongly μ-measurable function $f\colon \Omega \longrightarrow X$, such that $f^{(\nu)} \rightharpoonup f$ weakly in $L^1(\Omega \setminus E_k; X)$ for every k. Since each E_k differs from a set in $\mathscr{F}$ by a set of measure zero we can suppose that $E_k \in \mathscr{F}$ for each k. Finally, we have that

$$\int_{\Omega\setminus E_k} \|f\|_X \, d\mu \le \varliminf_{\nu\to\infty} \int_{\Omega\setminus E_k} \|f^{(\nu)}\|_X \, d\mu \le C_0,$$

so that letting $k \to \infty$ we deduce that $f \in L^1(\Omega; X)$. This completes the proof.

Remarks. 1. The use of Helly's Theorem in the proof is not essential; it suffices to extract a subsequence of the φ_j which converges for each positive integer l.

2. The function f is unique in the sense that if there exists a subsequence $\{f^{(\nu)}\}$ and two families E_k, $\widetilde{E}_k$ of measurable sets as in the lemma such that, for each k, $f^{(\nu)} \rightharpoonup f$ in $L^1(\Omega \setminus E_k; X)$ and $f^{(\nu)} \rightharpoonup \tilde{f}$ in $L^1(\Omega \setminus \widetilde{E}_k; X)$, then $f = \tilde{f}$. This follows since, by a suitable choice of test function, $f = \tilde{f}$ a.e. in $\Omega \setminus (E_k \cup \widetilde{E}_k)$ for each k.

3. Identification of f via the Young measure and approximations

In this section we show how the function f in the lemma can be identified in terms of the Young measure and various approximation procedures such as truncation. For simplicity we restrict attention to the case $X = \mathbf{R}^m$, $\mu = n$-dimensional Lebesgue measure, $\Omega \subset \mathbf{R}^n$ μ-measurable with $\mu(\Omega) < \infty$.

Since $\sup_\nu \|f^{(\nu)}\|_1 < \infty$ there exists a family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbf{R}^m$ (the *Young measure*), depending measurably on x, and a further subsequence, again denoted $\{f^{(\nu)}\}$, with the following property (cf. Ball [4]): if $g : \mathbf{R}^m \longrightarrow \mathbf{R}$ is continuous, if $A \subset \Omega$ is μ-measurable, and if

$$g(f^{(\nu)}) \rightharpoonup z \qquad \text{weakly in } L^1(A;\mathbf{R}),$$

then $g(\cdot) \in L^1(\mathbf{R}^m;\nu_x)$ for a.e. $x \in A$ (where the exceptional set possibly depends on g) and

$$z(x) = \int_{\mathbf{R}^m} g(\lambda)\, d\nu_x(\lambda) \stackrel{\text{def}}{=} \langle \nu_x , g\rangle \qquad \text{a.e. } x \in A.$$

Applying this with $A = \Omega \setminus E_k$ and $g(\lambda) = \lambda_i$, where $\lambda = (\lambda_1, \dots, \lambda_m)$, we deduce that the f defined in the lemma is given by

$$f(x) = \langle \nu_x , \lambda\rangle = \int_{\mathbf{R}^m} \lambda\, d\nu_x(\lambda) \qquad \text{a.e. } x \in \Omega.$$

We now suppose that continuous functions $g_k : \mathbf{R}^m \longrightarrow \mathbf{R}^m$, $k = 1, 2, \dots$, are given satisfying the conditions:

(i) $g_k(\lambda) \longrightarrow \lambda$ as $k \longrightarrow \infty$, for each fixed $\lambda \in \mathbf{R}^m$,
(ii) $|g_k(\lambda)| \le C_1(1 + |\lambda|)$ for all k, all $\lambda \in \mathbf{R}^m$, where C_1 is a constant,
(iii) $\lim_{|\lambda|\to\infty} |\lambda|^{-1}|g_k(\lambda)| = 0$ for each k.

The conditions (i)–(iii) hold in the following important cases:
(a) (truncation at level k)

$$g_k(\lambda) = \psi(k^{-1}|\lambda|)\lambda,$$

where

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ t^{-1} & \text{if } t \ge 1. \end{cases}$$

(b) (approximation by $1/p$-th powers)

$$g_k(\lambda) = \begin{cases} |\lambda|^{-1+p_k^{-1}}\lambda & \text{if } \lambda \ne 0, \\ 0 & \text{if } \lambda = 0, \end{cases}$$

where $p_k > 1$, $\lim_{k\to\infty} p_k = 1$.
Then we have the

Proposition. *For each fixed k there exists $f_k \in L^1(\Omega;\mathbf{R}^m)$ such that as $\nu \longrightarrow \infty$*

$$g_k(f^{(\nu)}) \rightharpoonup f_k \qquad \textit{weakly in } L^1(\Omega;\mathbf{R}^m).$$

As $k \longrightarrow \infty$,

$$f_k \longrightarrow f \qquad \textit{strongly in } L^1(\Omega;\mathbf{R}^m).$$

Proof. Fix k. By (iii),

$$\lim_{l\to\infty} \int_{\{|f^{(\nu)}|>l\}} |g_k(f^{(\nu)}(x))|\, dx = 0$$

uniformly in ν. Thus by the Dunford-Pettis Theorem $g_k(f^{(\nu)})$ is sequentially weakly relatively compact in $L^1(\Omega;\mathbf{R}^m)$. Hence by the properties of the Young measure given above,

$$g_k(f^{(\nu)}) \rightharpoonup f_k \qquad \text{in } L^1(\Omega;\mathbf{R}^m),$$

as $\nu \longrightarrow \infty$, where

$$f_k(x) = \int_{\mathbf{R}^m} g_k(\lambda)\, d\nu_x(\lambda) \qquad \text{a.e. } x \in \Omega. \tag{3}$$

To prove the $L^1(\Omega;\mathbf{R}^m)$ convergence of f_k to f, we use the dominated convergence theorem. We note first that the function F defined by

$$F(x) = \int_{\mathbf{R}^m} |\lambda|\, d\nu_x(\lambda)$$

belongs to $L^1(\Omega;\mathbf{R})$; this follows by applying the lemma to the sequence $\{|f^{(\nu)}|\}$ and using the properties of the Young measure given above with $g(\lambda) = |\lambda|$. From (ii), (3) we deduce that

$$|f_k(x)| \leq C_1(1 + F(x)) \qquad \text{for a.e. } x \in \Omega.$$

It thus suffices to show that $f_k(x) \longrightarrow f(x)$ for a.e. $x \in \Omega$. But this follows from (i), (ii), (3) by a preliminary application of the dominated convergence theorem to the sequence $\{g_k(\cdot)\}$ in $L^1(\mathbf{R}^m;\nu_x)$ for x fixed; indeed the upper bound $C_1(1 + |\lambda|)$ in (ii) belongs to $L^1(\mathbf{R}^m;\nu_x)$ since $F(x)$ is finite for a.e. $x \in \Omega$.

Remark 3. It is easily shown that in the cases (a), (b) above the convergence of $g_k(f^{(\nu)})$ to f_k holds weak* in $L^\infty(\Omega;\mathbf{R}^m)$ and weakly in $L^{p_k}(\Omega;\mathbf{R}^m)$, respectively.

4. Examples, and discussion about concentrations

Example 1. The following statement is *false*: given any bounded sequence $\{f^{(j)}\}$ in $L^1(\Omega;X)$ and any $\delta > 0$, there exists a subset $E \subset \Omega$ with $\mu(E) < \delta$ such that $\{f^{(j)}\}$ is sequentially weakly relatively compact in $L^1(\Omega\backslash E;X)$. Consider the case $X = \mathbf{R}$, $\Omega = (0,1)$ with Lebesgue measure, and the sequence $\{f^{j,k}\}$, $j, k = 1, 2, \ldots, j \neq k$, defined by

$$f^{j,k}(x) = \begin{cases} q_k^{-1} & \text{if } x \in (q_j - q_k, q_j + q_k), \\ 0 & \text{otherwise}, \end{cases}$$

where $\{q_j\}$ is an enumeration of the rationals in $(0,\infty)$. Note that $\int_\Omega |f^{j,k}|\, dx \leq 2$. Let $A \subset (0,1)$ have positive Lebesgue measure. We show that, for arbitrary $l > 0$, there exist an infinite number of pairs of j, k such that

$$\int_{\{|f^{j,k}| \geq l\} \cap A} |f^{j,k}|\, dx \geq 1.$$

In fact, let $x_0 \in (0,1)$ be a point of density of A. Then there exists $r \in (0, l^{-1})$ such that $\text{meas}\{(x_0 - r, x_0 + r) \cap A\} \geq 2r \cdot \frac{3}{4}$. Let $q_j \longrightarrow x_0$, $q_k \longrightarrow r$, where q_j, q_k are rationals and $q_k < l^{-1}$. Then

$$\int_{\{|f^{j,k}| \geq l\} \cap A} |f^{j,k}|\, dx = q_k^{-1} \text{meas}\{(q_j - q_k, q_j + q_k) \cap A\}$$

$$\longrightarrow r^{-1} \text{meas}\{(x_0 - r, x_0 + r) \cap A\} \geq \frac{3}{2},$$

as $j, k \longrightarrow \infty$.

Example 2. We take $X = \mathbf{R}$, $\Omega = (0,1)$ with Lebesgue measure, and define for $j = 2, 3, \ldots$,

$$f^{(j)}(x) = \begin{cases} j^2/2 & \text{for } x \in (k(j+1)^{-1} - j^{-3}, k(j+1)^{-1} + j^{-3}), \\ & k = 1, \ldots, j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f^{(j)}\|_1 = 1$ for each j, and it is easily proved that $f^{(j)} \overset{*}{\rightharpoonup} 1$ in the sense of measures. We now identify the function f and a possible choice of the sets E_k of the lemma. We take

$$E_k = \bigcup_{j \geq k} \{f^{(j)} \neq 0\}, \tag{4}$$

corresponding to the choice $l_j = j^2/2$ in the proof of the lemma. Then, since $\text{meas}\{f^{(j)} \neq 0\} = 2j^{-2}$,

$$\text{meas}\, E_k \leq \sum_{j \geq k} 2j^{-2} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty,$$

and if $x \in \Omega \setminus E_k$, $f^{(j)}(x) = 0$ for all $j \geq k$ (so that in particular $f^{(j)} \longrightarrow 0$ a.e. in $(0,1)$). Hence $f = 0$. In this example we do not need to extract a subsequence.

Since f is unique (see Remark 2, page 658) and since $\lim_{j\to\infty} \int_I f^{(j)}\, dx = \text{meas}\, I$ for any open interval $I \subset (0,1)$, it follows that the sets E_k cannot be chosen to be closed.

In Example 2, the weak* limit of $f^{(j)}$ in the sense of measures (or, more precisely, the difference $1 - 0$ between the weak* limit of $f^{(j)}$ and the f of the lemma) sees the concentrations of $f^{(j)}$ as being in the limit smeared out uniformly throughout Ω. The same is true of the generalized Young measure of DiPerna and Majda [7], which in this example is constant in Ω. The lemma, on the other hand, shows that in general the concentration takes place on progressively smaller and smaller sets. In Example 2 there is even a set of points, whose complement is of arbitrarily small measure, at which the $f^{(j)}$ are for large enough j identically zero, and it does not seem satisfactory to describe these points as being points of concentration.

An attempt to give a precise meaning to concentration sets, in a context different from but related to ours, has been made by DiPerna and Majda [8] for the purpose of applications to the Euler equations of fluid mechanics. They consider, for example, the case of a sequence $v^{(j)}$ converging weakly in $L^2(\Omega;\mathbf{R})$ to v, say, where $\Omega \subset \mathbf{R}^m$ is open, and define the associated 'reduced defect measure' θ as the outer measure

$$\theta(B) = \limsup_{j\to\infty} \int_B |v^{(j)} - v|^2\, dx\,,$$

for any Borel subset B of Ω. They then define the 'concentration sets' for θ as the Borel sets E for which $\Omega \setminus E$ is a countable union of null sets of θ. Thus they are interested in detecting on which sets an L^2 weakly convergent sequence converges strongly, while in this paper our goal has been to isolate the sets where a bounded sequence in L^1 is not weakly convergent in L^1. We may nevertheless try to apply these definitions to Example 2 by setting $v^{(j)} = (f^{(j)})^{1/2}$, $v = 0$. However, the conclusion is unfortunately that any Borel set E (including the empty set) is a concentration set. To prove this we set $G = \cap_{k\geq 2} E_k$, where the E_k are given by (4). Then G is a Borel set of Lebesgue measure zero, and by the definition of θ is hence a null set for θ. Then, since $\Omega \setminus E_k$ is a null set for θ, the equation

$$\Omega = G \cup \bigcup_{k\geq 2} (\Omega \setminus E_k)$$

shows that the empty set is a concentration set, and it is easily proved that any Borel set containing a concentration set is itself a concentration set.

These remarks suggest that the tools presently available do not give as complete a description of concentrations as one might desire.

Added in proof. We are grateful to M. Valadier for having pointed out to us the paper of M. Slaby, *Strong convergence of vector-valued pramarts and subpramarts*, Probability and Mathematics, **5** (1985) 187-196, who proves a result essentially equivalent to the Biting lemma by an argument similar to ours. Some ingredients of the argument also appear in P. L. Lions [12 Lemma 1.1].

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