# Remarks on Cohomological Hall algebras and their representations

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*to the memory of F. Hirzebruch*

# Contents

# 1 Introduction

The aim of this paper is to define and discuss a class of representations of the Cohomological Hall algebras.

## 1.1 Cohomological Hall algebras

The notion of Cohomological Hall algebra (COHA for short) for quivers with potential was introduced in [KoSo5][1]. Since quiver with potential defines a 3-dimensional Calabi-Yau category ($3CY$ category for short), it was expected that COHA should exist for "good" abelian subcategories of 3-dimensional Calabi-Yau categories endowed with additional data (most notably, with orientation data introduced in [KoSo1]). Below we will give an informal description of the construction.

Let $\mathbf{k}$ be a perfect field. Suppose $\mathcal{C}$ is a $\mathbf{k}$-linear triangulated $A_\infty$-category, which is ind-constructible and locally regular in the sense of Section 3 of [KoSo1].[2]. It is explained in Section 3.2 of loc.cit that one can associate with $\mathcal{C}$ the ind-constructible stack $\mathcal{M}_\mathcal{C}$ of objects of $\mathcal{C}$. Local regularity implies that $\mathcal{M}_\mathcal{C}$ is locally presented as an ind-Artin stack over $\mathbf{k}$. Let $\mathcal{A} \subset \mathcal{C}$ be an abelian subcategory. Then we have an ind-constructible substack $\mathcal{M}_\mathcal{A} \subset \mathcal{M}_\mathcal{C}$ of objects of $\mathcal{A}$, which is locally ind-Artin.

The definition of COHA depends on an ind-constructible sheaf $\Phi$ on $\mathcal{M}_\mathcal{C}$. In the case of $3CY$ categories one takes $\Phi = \phi_W$, which is the sheaf of vanishing cycles of the potential $W$ (we recall the definition of potential in Section 2.1). For the sheaf of vanishing cycles to be well-defined on $\mathcal{M}_\mathcal{C}$, the $3CY$-category $\mathcal{C}$ has to be endowed with an *orientation data*. The latter is an ind-constructible super line bundle $\mathcal{L}$ over $\mathcal{M}_\mathcal{C}$, such that for the fiber over a point of $\mathcal{M}_\mathcal{C}$ corresponding to an object $E \in Ob(\mathcal{C})$ one has $\mathcal{L}_E^{\otimes 2} = sdet(Ext^\bullet(E,E))$. Furthermore, it is required that $\mathcal{L}_E$ behaves naturally on exact triangles (see [KoSo1], Section 5 for the details). It follows from local regularity that $\mathcal{L}$ is (locally) a line bundle over an ind-Artin stack.

Let $i : \mathcal{M}_\mathcal{A} \subset \mathcal{M}_\mathcal{C}$ be the natural embedding. Then the pull-back $i^*(\Phi)$ is an ind-constructible sheaf on $\mathcal{M}_\mathcal{A}$. Let $\mathcal{Z} \subset \mathcal{M}_\mathcal{A} \times \mathcal{M}_\mathcal{A}$ be the "Hecke correspondence", which is the stack consisting of pairs of objects $(E,F)$ such that $E \subset F$. There are projections $p_n : \mathcal{Z} \to \mathcal{M}_\mathcal{A}, n = 1,2,3$ such that $p_1(E,F) = E, p_2(E,F) = F, p_3(E,F) = E/F$. We say that the abelian category $\mathcal{A}$ is *good* if the projection $p_1$ is locally a proper morphism of ind-Artin stacks.

As a vector space COHA of $\mathcal{M}_\mathcal{A}$ is defined as $\mathcal{H} := \mathcal{H}_\mathcal{A} = H^\bullet(\mathcal{M}_\mathcal{A}, i^*(\Phi))$. For that one chooses an appropriate cohomology theory of Artin stacks with coefficients in constructible sheaves. The product $m : \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{A}$ is defined by the formula $p_{1*} \circ (p_2^* \otimes p_3^*)(i^*(\Phi) \otimes i^*(\Phi))$. It is expected to be well-defined in general due to the properness of $p_1$.

[1] In fact we considered in the loc.cit. formally smooth algebras with potential.

[2] We can work over any field $\mathbf{k}$, but Calabi-Yau categories which we will discuss later require $char(\mathbf{k}) = 0$. For simplicity we will often assume that $\mathbf{k} = \mathbf{C}$.

Associativity of the product depends in general on the conditions we impose on the sheaf $\Phi$ and the cohomology theory. We do not know those conditions in general. For the sheaf of vanishing cycles $\Phi = \phi_W$ the condition is the Thom-Sebastiani theorem for the chosen cohomology theory.[3]

We will fix an abstract version of Chern character (called the class map in [KoSo1]) i.e. a homomorphism of abelian groups $cl : K_0(\mathcal{C}) \to \Gamma$, where $\Gamma \simeq \mathbf{Z}^n$ is a free abelian group, such that connected components of $\mathcal{M}_{\mathcal{C}}$ are parametrized by $\Gamma$, while classes $cl(E), E \in Ob(\mathcal{A})$ form an additive submonoid $\Gamma_+ \subset \Gamma$. Then COHA of $\mathcal{A}$ will be $\Gamma_+$-graded algebra.

In the case of smooth algebras with potential considered in [KoSo5] the stack $\mathcal{M}_{\mathcal{A}}$ is a countable union of smooth quotient stacks, and the foundational questions are resolved positively. For some ideas about general case one can look at [DyKap].

## 1.2 Stable framed objects and modules over COHA

Since the above approach to of COHA is somehow similar to the Nakajima's approach to construction of Kac-Moody algebras, it is natural to ask whether one can realize representations of COHA in the cohomology groups of some natural schemes (or stacks), which might also depend on a choice of stability condition on $\mathcal{A}$. Let us explain how it can be achieved.[4]

First we define the moduli space $\mathcal{M}_{\gamma}^{fr,st}, \gamma \in \Gamma_+$ of " stable framed objects of class $\gamma$" (those can be framed sheaves, framed representations of quivers, framed Lagrangian submanifolds, etc.). This notion depends on a choice of stability condition on $\mathcal{A}$. It is expected (see [KoSo7]) that the constructible sheaf $\Phi$ "descends" to each stack $\mathcal{M}_{\gamma}^{fr,st}$.

For a pair of "Chern classes" $\gamma_1, \gamma_2$ let us consider the Hecke correspondence $\mathcal{Z}_{\gamma_1,\gamma_2}$ of pairs $(E_{\gamma_1+\gamma_2}, E_{\gamma_2})$ (the subscripts denote the Chern classes) of framed stable objects such that $E_{\gamma_2}$ is a quotient of $E_{\gamma_1+\gamma_2}$. Let us denote the cohomology theory we used in the definition of COHA by $\mathbf{H}^\bullet$. It descends to each $\mathcal{M}_{\gamma}^{fr,st}$. Furthermore, similarly to the definition of COHA we have three projections of $\mathcal{Z}_{\gamma_1,\gamma_2}$:

a) to $\mathcal{M}_{\gamma_2}^{fr,st}$;

b) to the moduli space $\mathcal{M}_{\gamma_1}$ of all (not framed) objects with fixed $\gamma_1$;

c) to $\mathcal{M}_{\gamma_1+\gamma_2}^{fr,st}$.

Using the pull-back and pushforward construction as in the previous subsection, we obtain a structure of $\mathcal{H}_{\mathcal{A}} = \oplus_\gamma \mathbf{H}^\bullet(\mathcal{M}_\gamma), \Phi$-module over COHA of $\mathcal{A}$ on the space $\oplus_\gamma \mathbf{H}^\bullet(\mathcal{M}_{\gamma}^{fr,st}, \Phi)$.

We will show in Section 3 that moduli stacks of stable framed objects are in fact schemes. Hence graded components of our modules are finite-dimensional vector spaces. Dropping the stability assumption we will still obtain a representation of COHA, but this time in the spaces with infinite-dimensional graded components.

[3]As explained in Section 7 of [KoSo5], it is more convenient to work with compactly supported cohomology and then apply the duality functor.

[4]We warn the reader that our moduli spaces are **not** Nakajima's quiver varieties.

## 1.3 Modules over COHA motivated by physics

COHA can be thought of as a mathematical incarnation of the notion of algebra of (closed) BPS states envisioned in [HaMo1,2]. Motivated by the ideas of S. Gukov (see e.g. [GuSto]) we would like to think about representations of COHA described in the previous subsection as of representations of the algebra of closed BPS states on the vector space of open BPS states. We are going to speculate about applications of this point of view in the last section of the paper. We plan to discuss this relationship more systematically in separate projects jointly with E. Diaconescu, S.Gukov, N. Saulina.

Here we just mention three interesting classes of modules over COHA which have geometric origin and should have interesting applications to gauge theory and knot invariants:

a) Modules over COHA of the resolved conifold $X = tot(\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1))$ realized in the cohomology of moduli spaces of $C$-framed stable sheaves in the sense of [DiHuSo]. Those modules should have applications in the knot theory.

b) Modules over COHAs of the Fukaya categories of local Calabi-Yau 3-folds associated with spectral curves of Hitchin integrable systems. Those should serve as BPS algebras of the gauge theories from class $S$.

c) Modules over COHA realized in the cohomology of the moduli spaces of (framed, possibly ramified) instantons on $\mathbf{P}^2$. We expect such representations being motivated by the idea of geometric engineering. We also expect a link between the algebras of Hecke operators proposed in [Nak1] and those proposed in [So1]. The actual transformation should explain the appearance of COHA on the "Calabi-Yau side" of geometric engineering and conventional ("motivic") Hall algebra on the "instanton side" (cf. also [SchV], [Sz2]). The relationship between various classes of gauge theories should give non-trivial results about corresponding COHAs and their representations (including the relationships between our classes a), b), c)).

## 1.4 Contents of the paper

Section 2 is a reminder on COHA in the framework of quivers with potential. Section 3 is devoted to stable framed objects in triangulated and abelian categories. In Section 4 we discuss representations of COHA realized in the cohomology of the moduli spaces of stable framed representations. We also discuss a possibility to define "full COHA" which consists of two copies of ordinary COHAs. In Section 5 we speculate about representations of COHA motivated by knot theory and physics.

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# 2 Cohomological Hall algebra: reminder

This section is a reminder of some basic facts about the notion of Cohomological Hall algebra. Most of the material is borrowed from [KoSo5], and we refer the reader to loc.cit. for more details and proofs.

## 2.1 COHA and $3CY$ categories

Suppose we are given an ind-constructible 3-dimensional ind-constructible Calabi-Yau category $\mathcal{C}$ over the field $\mathbf{k}, char(\mathbf{k}) = 0$ which is assumed to be locally regular (see [KoSo1]). As explained in Section 3.2 of loc.cit. (and was already mentioned in the Introduction), one can associate with such a category the stack of objects, which is a countable union of schemes over $\mathbf{k}$ of finite type acted by affine algebraic groups. For simplicity of the exposition we take the ground field $\mathbf{k} = \mathbf{C}$.

Some examples of such categories are listed in the Introduction of [KoSo1]. They include various categories of $D$-branes popular in string theory (e.g. the Fukaya category of a compact or local Calabi-Yau 3-fold, the category of perfect sheaves on such a 3-fold, the category of finite-dimensional representations of a quiver with potential, etc.).

Recall the approach to the construction of COHA already mentioned in the Introduction. In order to define COHA one has to choose *orientation data* (see [KoSo1], Section 5) on $\mathcal{C}$ as well as a "good" $t$-structure with the ind-Artin heart. Let us denote it by $\mathcal{A}$. The existence of mutation-invariant orientation data is known for a class of $3CY$ categories, e.g. for those associated with a quiver without potential (see [Dav]). There are partial existence results for the derived category of coherent sheaves on a compact Calabi-Yau 3-fold (see e.g. [Hu]). But the general case is still open. In present paper we will assume the existence of the orientation data as a part of the "foundational questions" package. Also, we do not discuss in detail the meaning of the notion of "good" $t$-structure. As we mentioned in the Introduction, the latter includes properness of the morphisms which appear in the definition of the product on COHA.

We assume the existence of the "class map" $cl : K_0(\mathcal{C}) \to \Gamma$ (see [KoSo1]), where $\Gamma \simeq \mathbf{Z}^n$ is a free abelian group endowed with integer skew-symmetric form $\langle \bullet, \bullet \rangle$ (Poisson lattice). We assume that the class map respects the Euler form $\chi(E, F) = \sum_i (-1)^i dim\, Ext^i(E, F)$ on $K_0(\mathcal{C})$ and the form $\langle \bullet, \bullet \rangle$ on $\Gamma$. The lattice $\Gamma$ plays a role of topological $K$-theory of the category $\mathcal{C}$. Finally, we assume that we have fixed an additive submonoid $\Gamma_+ \subset \Gamma$ generated by $cl(E), E \in Ob(\mathcal{A})$.

When the above choices are made, one can define COHA of $\mathcal{A}$ as an associative algebra graded by $\Gamma_+$. Graded components are given by the cohomology of the moduli stacks of objects with the given class $\gamma \in \Gamma$ with the coefficients in the sheaf of vanishing cycles of the potential of $\mathcal{C}$ restricted to $\mathcal{A}$.

For completeness we recall here the notion of potential of a $3CY$ category. Using the $A_\infty$-structure on $\mathcal{C}$ as well as the Calabi-Yau pairing $(\bullet, \bullet)$ (see [KoSo1]) one defines the potential of an object $E$ as a formal series:

$$W_E(a) = \sum_{n\geq 1} \frac{(m_n(a,...,a),a)}{n+1},$$

where $m_n$ are higher composition maps, and the element $a$ belongs to $Hom^1(E,E)$ which is the subspace in the graded space $Hom(E,E)$ consisting of elements of degree 1. By our assumptions the potential $W_E$ is a locally regular function with respect to $E$. Hence we have a partialy formal function $W$ defined by the family of series $W_E$.

**Remark 2.1.1.** *If $\mathcal{C}$ is "minimal on the diagonal" (see [KoSo1]), we can replace $Hom(E,E)$ by its cohomology with respect to the differential $m_1$. In this case we may assume that $a \in Ext^1(E,E)$, which can be thought of as the "tangent space to the moduli stack of formal deformations of $E$". Hence one can think of the potential as a function on the moduli stack of objects which is locally regular along the stack of objects (this follows from the "locally regular" assumption) and formal in the transversal direction.*

Then COHA is a $\Gamma$-graded vector space

$$\mathcal{H} := \oplus_{\gamma\in\Gamma}\mathcal{H}_\gamma\ ,$$

where $\mathcal{H}_\gamma = H^\bullet_{\mathsf{G}_\gamma}(S_\gamma, W_\gamma)$, and $S_\gamma$ is the stack of objects $E$ such that $cl(E) = \gamma$. Recall that we use an appropriate stack version of the cohomology theory $H^\bullet(X,f)$ of a scheme $X$ endowed with a regular function $f$. There are several choices for such theory. They are discussed in [KoSo5], where the above approach made rigorous in the case of $3CY$ categories arising from quivers (more generally, formally smooth algebras) with potential. A version of the cohomology theory which is suitable in the framework of categories is called "critical cohomology" in loc. cit. It is defined by means of the compactly supported cohomology of $X$ with coefficients in the sheaf of vanishing cycles of $f$. Sometimes (e.g. for quivers with potential) the function $f := W$ is regular. In such a case one can use de Rham cohomology defined via the twisted de Rham differential $d + dW \wedge (\bullet)$ or Betti cohomology which is generated by "integration cycles" for the exponential differential forms of the type $exp(W)\nu$. More generally, one can define "motivic" version of COHA. In that case COHA $\mathcal{H}$ is an object of the tensor category of exponential mixed Hodge structures, and the concrete choice of the cohomology theory corresponds to a tensor functor to graded vector spaces ("realization"). It is explained in [KoSo5] that in all cases $\mathcal{H}$ carries an associative algebra structure with "structure constants" defined by means of the cohomology of Hecke correspondences with coefficients in the sheaves of vanishing cycles for the potential $W = (W_\gamma)_{\gamma\in\Gamma}$ (cf. Introduction).

Let us illustrate the above considerations in the case of a quiver $Q$ with potential $W$, which is the main example in [KoSo5]. We set $\mathbf{k} = \mathbf{C}$. If $I$ is the set of vertices of $Q$ then $\Gamma = \mathbf{Z}^I, \Gamma_+ = \mathbf{Z}^I_{\geq 0}$. For any $\gamma = (\gamma^i)_{i\in I} \in \Gamma_+$ we consider $\gamma$-dimensional representations of $Q$ in coordinate vector spaces $(\mathbf{C}^{\gamma^i})_{i\in I}$. It is an affine scheme $\mathsf{M}_\gamma$ naturally acted by the affine algebraic group

$\mathsf{G}_\gamma = \prod_{i\in I} GL(\gamma^i, \mathbf{C})$. Then the corresponding stack of objects is a countable union (over all dimension vectors $\gamma \in \Gamma_+$) of algebraic varieties $Crit(W_\gamma)$ of the critical points of the functions $W_\gamma = Tr(W) : \mathsf{M}_\gamma \to \mathbf{C}$. As a graded vector space COHA is the direct sum $\oplus_{\gamma\in\Gamma_+} H^{bullet}_{\mathsf{G}_\gamma}(\mathsf{M}_\gamma, W_\gamma)$.

## 2.2 COHA for quivers without potential

COHA is non-trivial even if $W = 0$. In the latter case

$$\mathcal{H}_\gamma := H^\bullet_{\mathsf{G}_\gamma}(\mathsf{M}_\gamma).$$

Since $\mathsf{M}_\gamma$ is equivariantly contractible, and $\mathsf{G}_\gamma$ is homotopy equivalent to its maximal torus, one can use the toric localization and obtain an explicit formula for the product which expresses COHA as a shuffle algebra. In the formula below we identify equivariant cohomology of a point with respect to the trivial action of the torus $(\mathbf{C}^*)^n$ with the space of symmetric polynomials in $n$ variables.

**Theorem 2.2.1.** *The product $f_1 \cdot f_2$ of elements $f_i \in \mathcal{H}_{\gamma_i}$, $i = 1, 2$ is given by the symmetric function $g((x_{i,\alpha})_{i\in I, \alpha\in\{1,\dots,\gamma^i\}})$, where $\gamma := \gamma_1 + \gamma_2$, obtained from the following function in variables $(x'_{i,\alpha})_{i\in I,\alpha\in\{1,\dots,\gamma_1^i\}}$ and $(x''_{i,\alpha})_{i\in I,\alpha\in\{1,\dots,\gamma_2^i\}}$:*

$$f_1((x'_{i,\alpha}))\, f_2((x''_{i,\alpha}))\, \frac{\prod_{i,j\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=1}^{\gamma_2^j}(x''_{j,\alpha_2} - x'_{i,\alpha_1})^{a_{ij}}}{\prod_{i\in I}\prod_{\alpha_1=1}^{\gamma_1^i}\prod_{\alpha_2=1}^{\gamma_2^i}(x''_{i,\alpha_2} - x'_{i,\alpha_1})}\,,$$

*by taking the sum over all shuffles for any given $i \in I$ of the variables $x'_{i,\alpha}, x''_{i,\alpha}$ (the sum is over $\prod_{i\in I}\binom{\gamma^i}{\gamma_1^i}$ shuffles).*

Here $a_{ij}$ is the number of arrows in $Q$ from the vertex $i$ to vertex $j$.

For example, let $Q = Q_d$ be a quiver with just one vertex and $d \geq 0$ loops. Then the product formula specializes to

$$(f_1 \cdot f_2)(x_1, \dots, x_{n+m}) :=$$

$$\sum_{i_1,\dots,j_m} f_1(x_{i_1}, \dots, x_{i_n})\, f_2(x_{j_1}, \dots, x_{j_m}) \left(\prod_{k=1}^{n}\prod_{l=1}^{m}(x_{j_l} - x_{i_k})\right)^{d-1}$$

for symmetric polynomials, where $f_1$ has $n$ variables, and $f_2$ has $m$ variables. The sum is taken over all $\{i_1 < \dots < i_n, j_1 < \dots < j_m, \{i_1, \dots, i_n, j_1, \dots, j_m\} = \{1, \dots, n+m\}$. The product $f_1 \cdot f_2$ is a symmetric polynomial in $n+m$ variables. One can show that for even $d$ the algebra is isomorphic to the infinite Grassmann algebra, while for odd $d$ one gets an infinite symmetric algebra.

We introduce a double grading on algebra $\mathcal{H}$, by declaring that a homogeneous symmetric polynomial of degree $k$ in $n$ variables has bigrading $(n, 2k+(1-d)n^2)$. Equivalently, one can shift the cohomological grading in $H^\bullet(\mathrm{BGL}(n, \mathbf{C}))$ by $[(d-1)n^2]$. In general, even for quivers without potential each component $\mathcal{H}_\gamma$ has also the grading by cohomological degree. Total $\Gamma \times \mathbf{Z}$-grading can be further refined, since $\mathcal{H}_\gamma$ carries the weight filtration (as an object of the category of exponential mixed Hodge structures, see [KoSo5]). Hence typically COHA has $\Gamma \times \mathbf{Z} \times \mathbf{Z}$-grading (which is not compatible with the product). More precisely, it is shown in [KoSo5] that for $W = 0$ COHA is graded by the Heisenberg group.

Finally, we remark that in the case of Dynkin quivers there are other interesting explicit formulas for the product in COHA (see [Rim]).

## 2.3 COHA for quiver $A_2$

The quiver $A_2$ has two vertices $\{1, 2\}$ and one arrow $1 \leftarrow 2$. The Cohomological Hall algebra $\mathcal{H}$ of this quiver contains two subalgebras $\mathcal{H}_L$, $\mathcal{H}_R$ corresponding to representations supported at the vertices 1 and 2 respectively. Clearly each subalgebra $\mathcal{H}_L$, $\mathcal{H}_R$ is isomorphic to the Cohomological Hall algebra for the quiver $A_1 = Q_0$. Hence it is an infinite Grassmann algebra. Let us denote the generators by $\xi_i$, $i = 0, 1, \ldots$ for the vertex 1 and by $\eta_i$, $i = 0, 1, \ldots$ for the vertex 2. Each generator $\xi_i$ or $\eta_i$ corresponds to an additive generator of the group $H^{2i}(BGL(1, \mathbb{C})) \simeq \mathbb{Z} \cdot x^i$. Then one can check that $\xi_i, \eta_j$, $i, j \geqslant 0$ satisfy the relations

$$\xi_i\xi_j + \xi_j\xi_i = \eta_i\eta_j + \eta_j\eta_i = 0\,, \quad \eta_i\,\xi_j = \xi_{j+1}\eta_i - \xi_j\eta_{i+1}\,.$$

Let us introduce the elements $\nu_i^1 = \xi_0\eta_i$, $i \geqslant 0$ and $\nu_i^2 = \xi_i\eta_0$, $i \geqslant 0$. It is easy to see that $\nu_i^1\nu_j^1 + \nu_j^1\nu_i^1 = 0$, and similarly the generators $\nu_i^2$ anticommute. Thus we have two infinite Grassmann subalgebras in $\mathcal{H}$ corresponding to these two choices: $\mathcal{H}^{(1)} \simeq \bigwedge(\nu_i^1)_{i\geqslant 0}$ and

$\mathcal{H}^{(2)} \simeq \bigwedge(\nu_i^2)_{i\geqslant 0}$. One can directly check the following result.

**Proposition 2.3.1.** *The multiplication (from the left to the right) induces isomorphisms of graded vector spaces:*

$$\mathcal{H}_L \otimes \mathcal{H}_R \xrightarrow{\sim} \mathcal{H}, \quad \mathcal{H}_R \otimes \mathcal{H}^{(i)} \otimes \mathcal{H}_L \xrightarrow{\sim} \mathcal{H}\,,\ i = 1, 2\,.$$

## 2.4 COHA for Jordan quiver with polynomial potential

Let us consider the quiver $Q_1$ which has one vertex and one loop $l$ (Jordan quiver), and choose as the potential $W = \sum_{i=0}^N c_i l^i$, $c_N \neq 0$ an arbitrary polynomial of degree $N \in \mathbf{Z}_{\geqslant 0}$ in one variable.

In the case $N = 0$, the question about COHA reduces to the quiver $Q_1$ without potential. This case was considered before. The algebra $\mathcal{H}$ is the symmetric algebra of infinitely many variables.

In the case $N = 1$ COHA is one-dimensional.

In the case $N = 2$ we may assume without loss of generality that $W = -l^2$. Then COHA $\mathcal{H} = \mathcal{H}^{(Q_1,W)}$ is the exterior algebra with infinitely many generators (infinite Grassmann algebra). This can be shown directly.

In the case when the degree $N \geqslant 3$, one can show that the bigraded algebra $\mathcal{H}$ is isomorphic to the $(N-1)$-st tensor power of the infinite Grassmann algebra of the case $N = 2$.

Basically the above examples are the only cases in which we know COHA explicitly. On the other hand, generating functions for the dimensions of its graded components (we call them *motivic DT-series* in [KoSo1,5]) are known in many cases.

## 2.5 Stability conditions and motivic DT-invariants

Definition of COHA depends on the abelian category $\mathcal{A}$ (e.g. on the $t$-structure) but does not depend on the central charge, which is a homomorphism of groups $Z : \Gamma \to \mathbf{C}$. This raises the question about the role of stability condition.

Having a central charge $Z : \Gamma \to \mathbf{C}$ we can define a full subcategory $\mathcal{A}$ of our category $\mathcal{C}$ generated by semistable objects with a central charge sitting in a given strict sector $V \subset \mathbf{R}^2$ which has the vertex in the origin. For example, we can take $V = l$ to be a ray with the vertex at the origin. Taking $V$ to be the upper-half plane we arrive to a category which is the heart of a $t$-structure of $\mathcal{C}$. Only in these two cases the categories generated by semistables are abelian.

As explained in [KoSo5], for a fixed strict sector $V$, one can define a $\Gamma$-graded *vector space*

$$\mathcal{H}(V) := \oplus_{\gamma\in\Gamma}\mathcal{H}_\gamma(V) \ .$$

But this space cannot be endowed in general with a structure of an associative algebra (except of the case when $V = l$ or $V$ being an upper-half plane). The problem is with properness of morphisms of the corresponding stacks.

It was observed in [KoSo5], Section 5.2 that the algebras $\mathcal{H}_l := \mathcal{H}(l)$ "look as" universal enveloping algebras of some Lie algebras $\mathfrak{g}_l$ which are analogous to the "positive root" Lie algebras $\mathfrak{g}_\alpha, \alpha > 0$ of Kac-Moody algebras. Then similarly to the isomorphism $U(n_+) \simeq \otimes_{\alpha>0} U(\mathfrak{g}_\alpha)$ (which depends on a chosen order on the set of positive roots) one should expect an isomorphism $\mathcal{H}(V) \simeq \otimes_{l\subset V}\mathcal{H}_l$ where the tensor product is taken in the clockwise order over all rays in the sector $V$. This was demonstrated in [Rim] in the case of Dynkin quivers without potential. In particular, taking $V$ to be the upper-half plane we obtain a factorization of the COHA $\mathcal{H}$ into the tensor product of COHAs for individual rays. COHA for each ray $l$ is typically commutative. It can be computed from the knowledge of space of semistable objects in the fixed $t$-structure whose central charges belong to $l$. For a generic central charge we have two possibilities: either $l$ does not contain $Z(\gamma)$ for $\gamma \in \Gamma$, or $l$ contains only multiples $nZ(\gamma_0), n > 0$ for some primitive vector $\gamma_0$ (an furthermore, only vectors $n\gamma_0, n \in \mathbf{Z}_{>0}$ are mapped by $Z$ to $l$). In this case $\mathcal{H}_l$ is indeed commutative and can be computed explicitly in many cases.

The notion of motivic DT-series (i.e. virtual Poincaré series of $\mathcal{H}$) does not depend on the central charge. On the other hand, motivic *DT-invariants* $\Omega^{mot}(\gamma)$ (they correspond in physics to refined BPS invariants) can be defined only after a choice of stability condition (i.e. the central charge in case of quivers). Definition of DT-invariants is based on the theory of factorization systems developed in [KoSo5]. It follows from loc. cit. that the motivic DT-series factorizes as a product of the powers of shifted quantum dilogarithms.

Those powers are motivic DT-invariants.

As a side remark we mention that factorizations systems appear in different disguises when mathematicians try to make sense of the operator product expansion in physics (we can mention e.g. the work of Beilinson and Drinfeld on chiral algebras or Costello's work on OPE in QFT). From this point of view it is not clear why factorization systems appear in our story.

## 2.6 Generators of COHA

Definition of COHA depends on a choice of a $t$-structure. This means that for different $t$-structures their COHAs are not necessarily isomorphic. For example, if we start with a pair $(Q, W)$ consisting of a quiver $Q$ with potential $W$ and make a mutation at a vertex $i_0 \in I$, then COHA for the mutated pair $(Q', W')$ is different from the one for $(Q, W)$. On the other hand we can compute motivic DT-series for the mutated pair. As was explained in [KoSo1] and [KoSo5], if we make a mutation at the vertex $i_0 \in I$ then the motivic DT-series for $(Q, W)$ and $(Q', W')$ are related by the conjugation by the motivic DT-series corresponding to the ray $l_0 = \mathbf{R}_{>0} \cdot Z(\gamma_{i_0})$ (which is essentially the quantum dilogarithm). Thus the question arises: what should be called COHA for a *triangulated* $3CY$ category $\mathcal{C}$? We do not know the answer to this question, but we can see some structures which should be incorporated in the definition.

For example, let us consider all COHAs corresponding to all possible mutations. Let $M$ be the orbit of the pair $(Q, W)$ under the action of the group of mutations. Then to any $m \in M$ we can assign COHA $\mathcal{H}_m$. More generally, we can consider rotations $Z \mapsto Ze^{i\theta}$ of the central charge and get the corresponding COHA $\mathcal{H}_{e^{i\theta}}$. This defines a structure of cosheaf of algebras over $S^1$. Each stalk is the COHA for the corresponding $t$-structure.

Next question is about the space of generators of COHA. Recall the following conjecture from [KoSo5] which was proved by Efimov (see [Ef]). It is formulated for symmetric quivers. Such quivers arise naturally in relation to 2-dimensional Calabi-Yau categories and Kac-Moody algebras.

**Theorem 2.6.1.** *Let $\mathcal{H}$ be the COHA (considered as an algebra over $\mathbf{Q}$) for the abelian category of finite-dimensional representations of a symmetric quiver $Q$. Then $\mathcal{H}$ is a free supercommutative algebra generated by a graded vector space $V$ over $\mathbf{Q}$ of the form $V = V' \otimes \mathbf{Q}[x]$, where $x$ is an even variable of bidegree $(0, 2) \in \mathbf{Z}^I_{\geq 0} \times \mathbf{Z}$, and for any given $\gamma$ the space $V'_{\gamma,k} \neq 0$ is non-zero (and finite-dimensional) only for* **finitely many** *$k \in \mathbf{Z}$.*

In general we expect (see [KoSo5] for the precise question) that $\mathcal{H}$ is isomorphic to the universal enveloping algebra of a graded Lie algebra $V := V' \otimes \mathbf{C}[x]$ which satisfies the conditions of the Theorem 2.6.1. Mutations act on $V$, hence we obtain a collection of vector spaces $V_m$ (one for each $t$-structure $m$). From the point of view the chamber structure of the space of stability conditions, we can say that with every chamber we associate its own COHA. Change of the chamber corresponds to the wall-crossing, which at the level of COHA is a conjugation (with a shift of grading).

# 3 Framed and stable framed objects

In this section we present a definition of stable framed objects following [KoSo7] as well as a related construction of modules over COHA of the same authors (unpublished).

## 3.1 Stable framed objects in triangulated categories

We recall the definition of stable framed object from [KoSo7] in the case of triangulated categories. Then we discuss some versions in the case of abelian categories.

Let $\mathcal{C}$ be a triangulated $A_\infty$-category over the ground field $\mathbf{k}$, which we assume to be an algebraically closed of characteristic zero. We fix a stability condition $\tau \in Stab(\mathcal{C})$. Let $\Phi : \mathcal{C} \to D^b(Vect_{\mathbf{k}})$ be an exact functor to the triangulated category of bounded complexes of $\mathbf{k}$-vector spaces.

For a fixed ray $l$ in the upper-half plane with the vertex at the origin, we denote by $\mathcal{C}_l := \mathcal{C}_l^{ss}$ the abelian category of $\tau$-semistable objects having the central charge in $l$. We will impose the following assumption: $\Phi$ *maps* $\mathcal{C}_l$ *to the complexes concentrated in non-negative degrees.*

**Definition 3.1.1.** *Framed object (or $\Phi$-framed object, if we want to stress dependence on the framing functor) is a pair $(E, f)$ where $E \in Ob(\mathcal{C}_l)$ and $f \in H^0(\Phi(E))$.*

Let $(E_1, f_1)$ and $(E_2, f_2)$ be two framed objects. We define a morphism $\phi : (E_1, f_1) \to (E_2, f_2)$ as a morphism $E_1 \to E_2$ such that the induced map $H^0(\Phi(E_1)) \to H^0(\Phi(E_2))$ maps $f_1$ to $f_2$. Framed objects naturally form a category, and hence there is a notion of isomorphic framed objects.

**Definition 3.1.2.** *We call the framed object $(E, f)$ stable is there is no exact triangle $E' \to E \to E''$ in $\mathcal{C}$ with $E'$ non-isomorphic to $E$ such that both $E', E'' \in Ob(\mathcal{C}_l)$ and such that there is $f' \in H^0(\Phi(E'))$ which is mapped to $f \in H^0(\Phi(E))$.*

Then one deduces the following result (see [KoSo7]), proof of which we reproduce here for completeness.

**Proposition 3.1.3.** *If $(E, f)$ is a stable framed object then $Aut(E, f) = \{1\}$.*

*Proof.* Let $h \in Aut(E)$ satisfies the property that its image $\Phi(h)$ preserves $f$. We may assume that $h \in Hom^0(E, E)$. We would like to prove that $h = id$. Assume the contrary. Let $h_1 := h - id \neq 0$. Then $\Phi(h_1)(f) = 0$. Since the category $\mathcal{C}_l$ is abelian, the morphism $h_1 \neq 0$ gives rise to a short exact sequence in $\mathcal{C}_l$:

$$0 \to Ker(h_1) \to E \to Im(h_1) \to 0,$$

where $Im(h_1) \neq 0$. Hence there exists an exact triangle $E' \to E \to E''$ in $\mathcal{C}$ with $E' = Ker(h_1)$ non-isomorphic to $E$ and $E'' = Im(h_1)$. Let us consider a short exact sequence in $\mathcal{C}_l$ given by

$$0 \to Ker(h_1) \to E \to E \to Coker(h_1) \to 0,$$

where the morphism $E \to E$ is $h_1$. Since the functor $\Phi$ is exact we get short exact sequence of vector spaces

$$H^0(Ker(\Phi(h_1))) \to H^0(\Phi(E)) \to H^0(\Phi(E)) \to H^0(Ker(\Phi(h_1))) \to H^1(Ker(\Phi(h_1))) \to ....$$

Let us remark that by the assumption that $\Phi$ maps $\mathcal{C}_l$ to complexes with non-negative cohomology, we conclude that if $E' \to E \to E''$ is an exact triangle then in the induced exact sequence

$$H^{-1}(\Phi(E'')) \to H^0(\Phi(E')) \to H^0(\Phi(E)) \to ...$$

the first terms is trivial. Hence the functor $H^0\Phi$ maps monomorphisms in $\mathcal{C}_l$ to monomorphisms in the category $Vect_{\mathbf{k}}$ of $\mathbf{k}$-vector spaces.

Let us decompose $h_1$ into a composition of the morphism $\psi : E \to Im(h_1)$ and the natural embedding $j : Im(h_1) \to E$. Applying $\Phi$, and using $\Phi(h_1)(f) = 0$ and the above remark we conclude that $\Phi(\psi)(f) = 0$.

Finally, applying $\Phi$ to the short exact sequence

$$0 \to Ker(h_1) \to E \to Im(h_1) \to 0,$$

we obtain a short exact sequence in $Vect_{\mathbf{k}}$:

$$H^0(Ker(\Phi(h_1))) \to H^0(\Phi(E)) \to H^0(\Phi(Im(h_1))),$$

where the last arrow is $\Phi(\psi)$. Since $\Phi(\psi)(f) = 0$ we conclude that there exists $f_1 \in H^0(Ker(\Phi(h_1)))$ which is mapped into $f$. This contradicts to the assumption that the pair $(E, f)$ is framed stable. The Proposition is proved. ■

**Corollary 3.1.4.** *The moduli stack of stable framed objects is in fact a scheme.*

In many examples it is a smooth projective scheme.

## 3.2 Stable framed objects and torsion pairs

The above definitions can be repeated almost word-by-word, if we replace an ind-Artin (or locally regular) triangulated category $\mathcal{C}$ by an ind-Artin abelian category $\mathcal{A}$. Then we have a definition of the framed and stable framed objects in the framework of abelian categories. Let us discuss its relation to the notion of torsion pair (see e.g. [H] for a short introduction).

Recall that a torsion pair for the abelian category $\mathcal{A}$ is given by a pairs of two full subcategories $\mathcal{T}, \mathcal{F} \subset \mathcal{A}$ such that $Hom(T, F) = 0$ for any pair $T \in Ob(\mathcal{T}), F \in Ob(\mathcal{F})$ and such that any object $E \in Ob(\mathcal{F})$ admits (a unique) decomposition

$$0 \to T \to E \to F \to 0$$

with the same meaning of $F$ and $T$. Here $T$ is called the *torsion* part of $E$ and $F$ is called the *torsion-free* part of $E$. The origin of the terminology is clear from the theory of abelian groups or theory of coherent sheaves on curves.

Let us assume as before that our abelian category$\mathcal{A}$ is $\mathbf{k}$-linear. Suppose we are given a stability condition on $\mathcal{A}$ with the central charge $Z$. Fix $\theta \in (0, \pi)$. Then the pair of full subcategories $\mathcal{T}_\theta = \{T \in Ob(\mathcal{A}|Arg(Z(T)) > \theta\}$, $\mathcal{F}_\theta = \{F \in Ob(\mathcal{A}|Arg(Z(F)) \leq \theta\}$ defines a torsion pair for $\mathcal{A}$ (one can exchange strict and non-strict inequality signs). Let us fix a non-zero object $P \in Ob(\mathcal{A})$. It defines a functor $\mathcal{F}_\theta \to Vect_{\mathbf{k}}$ given by $\Phi(E) = Hom(P, E)$. Framed objects are pairs $(E, f : P \to E)$. Then we can give the following version of the notion of stable framed object: $(E, f)$ *is stable framed if either $f$ is epimorphism or $Coker(f)$ is a non-zero object of $\mathcal{T}_\theta$.*

Then the above Proposition 3.1.3 still holds, and the proof is much simpler.

**Proposition 3.2.1.** *The automorphism group of a stable framed object is trivial.*

*Proof.* Let $h : E \to E$ be an automorphism such that $h \circ f = f$. Then $(h - id)$ vanishes on the image of $f$. If $f$ is an epimorphism, we conclude that $h = id$. Otherwise, assume $h \neq id$. Then $(h - id)$ defines a non-trivial morphism $Coker(f) \to E$ which contradicts to the assumption on $Coker(f)$ and the definition of torsion pair. Hence $h = id$. ■

From this Proposition we again conclude that stable framed objects form a scheme, not a stack.

**Remark 3.2.2.** *Notice that in the proof of the Proposition 3.2.1 we did not really use a fixed slope $\theta$, we rather worked with an individual object $E$. Hence we can give the following version of the notion of stable framed object for the framing functor defined by means of an object $P$: stable framed object is a pair $(E, f)$ such that $E$ is a non-zero object of category $\mathcal{A}$, and $f : P \to E$ is a morphism which is either an epimorphism or a morphism with non-zero cokernel satisfying the condition that $Arg(Coker(f)) > Arg(E)$ (we denote $Arg(Z(E))$ by $Arg(E)$ to simplify the notation). Yet another possibility is to require that all Harder-Narasimhan factors of $E$ belong to $\mathcal{T}_\theta$ (or require that all HN factors of $E$ has arguments strictly bigger than the $Arg(E)$). For all described versions the Corollary 3.1.4 remains true.*

## 3.3 Stable framed representations of quivers

Let $\mathbf{k}$ be an algebraically closed field.

In the case of quivers without potential there is a well-known way (exploited by Nakajima and Reineke among others) to construct framed objects by adding a new vertex $i_0$ and $d_i$ new arrows $i_0 \to i$ for each vertex $i \in I$ of the quiver $Q$. If we denote by $W_i$ the vector space spanned by $d_i$ arrows, then the framing functor $\Phi$ assigns to a representation $E = (E_i)_{i \in I}$ the vector space $\prod_{i \in I} Hom(W_i, E_i)$. Let $\gamma = (\gamma^i) \in \mathbf{Z}^I_{\geq 0}$ be a dimension vector.

Then a framed representation of $Q$ is given by a representation of the extended quiver $\widehat{Q}$ with the set of vertices $I \sqcup \{i_0\}$ of dimension $(\gamma^{i_0} = 1, (\gamma^i)_{i \in I})$, a collection of new $d_i$ arrows $i_0 \to i, i \in I$, and the collection of linear maps $W_i \to E_i$.

Let us fix a central charge $Z : \mathbf{Z}^I \to \mathbf{C}$ and a ray $l := l_\theta = \mathbf{R}_{>0} e^{i\theta}, 0 < \theta \leq \pi$. Then $\mathcal{C}_l$ is the category of semistables with the central charge in $l$. A framed representation is stable framed if the following condition is satisfied (see e.g. [Re1]):

*the representation $E$ of the quiver $Q$ is semistable with central charge in $l$, and satisfies the condition that it does not have a subrepresentation $E'$ which contains the images of all vector spaces $W_i, i \in I$ and has a bigger argument of the central charge.*

There are many versions of the above criterion. For example, one can start with several additional vertices instead of just one. Also, one can restate the above criterion in terms of stable representations of the extended quiver $\widehat{Q}$. The later approach makes it clear why the notion of stable framed representation can be thought of as a generalization of the notion of a cyclic representation.

**Remark 3.3.1.** *For the quiver $Q_2$ with one vertex and 2 loops there are no nontrivial stability conditions. Then stable framed objects is the same as left ideals of finite codimension in the path algebra of $Q_2$. The moduli space of stable framed objects is known as the non-commutative Hilbert scheme of $\mathbf{k}^2$.*

# 4 Modules over COHA from stable framed objects

## 4.1 Quiver case

Let $\mathbf{k}$ be an algebraically closed field.

Let fix a quiver $Q$ with the set of vertices $I$ as well as a central charge $Z : \mathbf{Z}^I \to \mathbf{C}$. We also fix a slope $0 < \theta \leq \pi$ and the corresponding ray $l = l_\theta = \mathbf{R}_{>0} \cdot e^{i\theta}$. In order to specify the framing we fix a collection $(d_i)_{i \in I}$ of non-negative integer numbers. An additional (framing) vertex is denoted by $i_0$. The corresponding extended quiver will be denoted by $\widehat{Q} := Q^{i_0}((d_i)_{i \in I})$.

Given a dimension vector $\gamma \in \mathbf{Z}^I_{\geq 0}$ we denote by $\mathsf{M}^{st}_{\gamma,(d_i)_{i \in I}} := \mathsf{M}^{st,l}_{\gamma,(d_i)_{i \in I}}$ the scheme of stable framed representations of dimension $\gamma$ having $Z(\gamma) \in l$. We denote by $\mathsf{M}_{\gamma,(d_i)_{i \in I}} := \mathsf{M}^{l}_{\gamma,(d_i)_{i \in I}}$ the bigger space of framed representations (no stability conditions is imposed). The group $\mathsf{G}_\gamma = \prod_i GL(\gamma^i, \mathbf{k})$ acts freely on $\mathsf{M}^{st}_{\gamma,(d_i)_{i \in I}}$. We denote by $V^l_{\gamma,(d_i)_{i \in I}} = V^\theta_{\gamma,(d_i)_{i \in I}}$ the graded vector space $H^\bullet_{\mathsf{G}_\gamma}(\mathsf{M}^{st}_{\gamma,(d_i)_{i \in I}}) = H^\bullet(\mathsf{M}^{st}_{\gamma,(d_i)_{i \in I}}/\mathsf{G}_\gamma)$.

Recall that with the ray $l = l_\theta$ we can associate COHA

$$\mathcal{H}_l = \oplus_{\gamma \in \mathbf{Z}^I_{\geq 0}, Z(\gamma) \in l} H^\bullet_{\mathsf{G}_\gamma}(\mathsf{M}^{ss}_\gamma).$$

Let us denote by $S := S_{\gamma_1,\gamma_2,\gamma_3,(d_i)_{i\in I}}$ the scheme of short exact sequences

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

of such representations of the extended quiver $\widehat{Q}$ that $dim(E_i) = \gamma_i \in \mathbf{Z}^I_{\geq 0}, i = 1,2,3$, representation $E_1$ is framed, $E_2, E_3$ stable framed, and the morphism $E_2 \to E_3$ is equal to the identity at the vertex $i_0$.

There is a projection $\pi_{13} : S \to \mathsf{M}_{\gamma_1} \times \mathsf{M}^{st}_{\gamma_3,(d_i)_{i\in I}}$ which sends the short exact sequence $0 \to E_1 \to E_2 \to E_3 \to 0$ to the pair $(E_1, E_3)$, where we treat $E_1$ as a representation of $Q$. Similarly we have a projection $\pi_2$ to $E_2$. Notice that the latter is a proper morphism of $S$ to $\mathsf{M}^{st}_{\gamma_2,(d_i)_{i\in I}}$. Since the automorphism group of the moduli space of stable framed objects is trivial, we see that the morphism $\pi_{2*}\pi^*_{13}$ gives rise to a map of cohomology groups

$$H^\bullet_{\mathsf{G}_{\gamma_1}}(\mathsf{M}_{\gamma_1}) \otimes H^\bullet(\mathsf{M}^{st}_{\gamma_3,(d_i)_{i\in I}}) \to H^\bullet(\mathsf{M}^{st}_{\gamma_2,(d_i)_{i\in I}}).$$

**Proposition 4.1.1.** *The above map gives rise to a (left) $\mathcal{H}_l$-modules structure on the vector space $V^l := V^l_{(d_i)_{i\in I}} = \oplus_\gamma V^l_{\gamma,(d_i)_{i\in I}}$.*

*Proof.* Similar to the proof of associativity of the product on COHA given in [KoSo5]. ■

**Remark 4.1.2.** *The above considerations can be generalized to the case of quivers with potential.*

**Example 4.1.3.** *In the case of the quiver $Q_2$ (one vertex and two loops) and $d_1 = 1$ the moduli space $\mathsf{M}^{st}_{\gamma,d_1}, \gamma \in \mathbf{Z}_{\geq 0}$ is the same as the moduli space of representations of the free algebra $\mathbf{k}\langle x_1, x_2\rangle$ of dimension $\gamma$ which are cyclic. In other words, it is the moduli space of codimension $\gamma$ ideals in the free algebra with two generators, i.e. it is the non-commutative Hilbert scheme. The above Proposition claims that it carries a structure of module over the COHA for $Q_2$ (which is the infinite Grassmann algebra).*

Consider as an example COHA $\mathcal{H}$ of a quiver which has at least one vertex $i_0$ without loops. Then $\mathcal{H}$ is a module over the infinite Grassmann algebra (a.k.a free fermion algebra) $\Lambda^\bullet$. Indeed, consider $i_0$ as a quiver $Q_0$ (one vertex, no loops). We know that COHA of $Q_0$ is $\Lambda^\bullet$. Since it is a subalgebra of $\mathcal{H}$, it acts on $\mathcal{H}$ by left multiplication.

Let $Q$ be a quiver with the set of vertices $I$. Let us fix a set of non-negative integers $d = (d^i)_{i\in I}$ (not all equal to zero)and the dimension vector $\gamma = (\gamma^i)_{i\in I}$. Then we have an extended quiver $\widehat{Q}$ with the set of vertices $I \sqcup i_0$ and $d^i$ arrows from $i_0$ to $i \in I$. For a fixed central charge $Z : \mathbf{Z}^I \to \mathbf{C}$ the moduli space $\mathsf{M}^{st,l}_{\gamma,d}$ of stable framed representations of $Q$ of dimension $\gamma$ such that $Z(\gamma) \in l$ is a non-empty smooth variety of pure dimension $\sum_{i\in I} d^i\gamma^i - \chi(d,d)$, where $\chi(\alpha,\beta)$ is the Euler-Ringel bilinear form of $Q$ (see [EnRe], Prop. 3.6). Moreover it admits a projective morphism to the moduli space of polystables with the fixed slope.

## 4.2 Representations of COHA in general case

Let $\mathcal{A}$ be "good' abelian subcategory in the $3CY$ category $\mathcal{C}$ (see Introduction and Section 2.1). We assume the conditions on the potential $W$ which guarantee existence o COHA of $\mathcal{A}$ as well as moduli spaces of stable framed objects. Then considerations from the previous subsection can be generalized to this situation provided $\mathcal{A}$ satisfies some extra conditions.

First we assume as before that we are given a class map $cl : K_0(\mathcal{A}) \to \Gamma$, where $\Gamma \simeq \mathbf{Z}^n$ is a free abelian group. We assume that classes $cl(E)$ of objects of $\mathcal{A}$ belong to an additive monoid $\Gamma_+$ which is mapped to $\mathbf{Z}^n_{\geq 0}$ under the above identification $\Gamma \simeq \mathbf{Z}^n$.

Next, let us fix a ray $l = \mathbf{R}_{\geq 0} \cdot e^{i\theta}$ in the upper half-plane, and a stability function $Z : \Gamma \to \mathbf{C}$ such that $Z(\Gamma_+)$ belongs to the upper half-plane. Then we have the category $\mathcal{A}_l$ of semistables with the central charge in $l$. Let us fix the framing functor $\Phi$. Then we can speak about framed and stable framed objects.

Recall that there is a notion of morphism of framed objects $(E_2, f_2) \to (E_3, f_3)$. An epimorphism $(E_2, f_2) \to (E_3, f_3)$ is a morphism in the category of framed objects which induces a homomorphism $H^0(\Phi(E_2)) \to H^0(\Phi(E_3))$ which sends $f_2$ to $f_3$ (see Section 3.1 for the notation).

Assume that $E_2$ and $E_3$ are semistable objects with central charges in the ray $l$. Then the kernel of the epimorphism $(E_2, f_2) \to (E_3, f_3)$ of framed objects does not have to be framed. Let us consider the stack $\mathcal{Z}_{\gamma_1,\gamma_2}$ of triples $(E_1, (E_2, f_2), (E_3, f_3))$ where:

a) $cl(E_1) = \gamma_1$, and $Z(\gamma_1) \in l$;

b) $(E_2, f_2)$ is stable framed, $cl(E_2) = \gamma_1 + \gamma_2$, $Z(cl(E_2) \in l$;

c) $(E_3, f_3)$ is stable framed, $cl(E_3) = \gamma_2$, $Z(cl(E_2) \in l$;

d) there is a epimorphism of framed objects $(E_2, f_2) \to (E_3, f_3)$ such that it induces (in the category of semistable objects with the central charge in $l$) a short exact sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0.$$

Recall that stable framed objects with fixed class $\gamma \in \Gamma_+$ form a scheme which we denote by $\mathcal{M}^{st,fr}_{\gamma}$. Then we have natural projections $p_2 : \mathcal{Z}_{\gamma_1,\gamma_2} \to \mathcal{M}^{st,fr}_{\gamma_1+\gamma_2}$ and $p_3 : \mathcal{Z}_{\gamma_1,\gamma_2} \to \mathcal{M}^{st,fr}_{\gamma_2}$ which are morphisms of stacks. Furthermore, let $\mathcal{M}_{\gamma}$ denotes the moduli stack of objects of $\mathcal{A}_l$. Then we have the natural projection $p_1 : \mathcal{Z}_{\gamma_1,\gamma_2} \to \mathcal{M}_{\gamma_1}$.

We will assume that:

i) if we consider the analog of the above situation with all $f_i = 0, i = 1, 2, 3$ (i.e. we work just in the abelian category $\mathcal{A}_l$) then the restriction of $p_2$ to $p_1^{-1}(E_1) \cap p_3^{-1}(E_3)$ is a morphism of smooth proper stacks;

ii) in general, for fixed $E_1$ and $(E_3, f_3)$ as above, the restriction of $p_2$ to $p_1^{-1}(E_1) \cap p_3^{-1}((E_3, f_3))$ is a morphism of smooth proper stacks.

By condition i) COHA $\mathcal{H}_l$ of the category $\mathcal{A}_l$ is well-defined as an associative algebra. For that we use the critical version of the cohomology from [Koso5] with trivial potential. Furthermore, repeating the construction from the previous subsection we obtain a structure of (left) $\mathcal{H}_l$-module on $V := V^l = \oplus_{\gamma \in \Gamma_+} H^{\bullet}(\mathcal{M}^{st,fr}_{\gamma})$.

**Remark 4.2.1.** *More generally, we can construct modules over COHA by considering the stack of objects whose Harder-Narasimhan filtration has consecutive factors with arguments of the central charge belonging to the interval* $[\theta, \pi]$. *If we have an exact short sequence*

$$0 \to E_1 \to E_2 \to E_3 \to 0,$$

*such that* $Arg\, Z(E_2) \in [\theta, \pi]$ *then* $Arg\, Z(E_3)$ *belongs to the same interval, while* $Arg\, Z(E_a) \in [0, \pi]$. *Then we get a representation of COHA in the cohomology of the stack of objects generated by semistables* $E$ *such that* $Arg\, Z(E) \in [\theta, \pi]$.

*Similarly, one can show that if* $V$ *is a strict sector in the plane then the graded "Cohomological Hall vector space"* $\mathcal{H}(V)$ *bounded from the left by a ray* $l$ *is a module over the COHA* $\mathcal{H}_l$ *associated with the ray.*

Furthermore, suppose that our abelian category $\mathcal{A}$ is a "good" subcategory of an ind-Artin $3CY$ category $\mathcal{C}$ endowed with orientation data. Let $W$ be the potential for $\mathcal{C}$. It gives rise to the sheaf of vanishing cycles $\phi_W$ on the stack of objects of $\mathcal{C}$. Then the pull-backs of $\phi_W$ to the stack of objects of $\mathcal{A}$ and subsequently to $\mathcal{M}_\gamma$ and $\mathcal{M}_\gamma^{st,fr}$ are well-defined. Then, similarly to [KoSo5] (and under the above assumption), the above construction (but this time with cohomolology groups with coefficients in $\phi_W$) gives rise to the module $V := V^l = \oplus_{\gamma \in \Gamma_+} H^\bullet(\mathcal{M}_\gamma^{st,fr}, \phi_W)$ over the COHA $\mathcal{H}_l$ of $\mathcal{A}_l$. The details will be explained elsewhere.

## 4.3 Hecke operators associated with simple objects

Recall that in the classical Nakajima construction of the infinite Heisenberg algebra (see [Nak2]) one considers pairs of ideal sheaves $(J_2, J_3)$ on a surface $S$ such that $J_2 \subset J_3$ and $Supp(J_3/J_2) = \{x\}$, where $x$ is a fixed point. Then one has an epimorphism $\mathcal{O}_S/J_2 \to \mathcal{O}_S/J_3$. Let us compare this observation with the above construction of modules over COHA. We see that a fixation of $K$-theory classes $\gamma_i, i = 1, 2$ for the pair of objects $(E_2, E_3)$ along with an epimorphism $E_2 \to E_3$ corresponds in the Nakajima's construction to the fixation of $n_i, i = 2, 3$ such that $J_i \in Hilb_{n_i}(S)$ and to the above-mentioned epimorphism of the quotient sheaves.

In the construction of the module structure on the cohomology of stable framed objects we used the pushforward map associated with the projection to the middle term in the moduli space of short exact sequences

$$0 \to E_1 \to E_2 \to E_3 \to 0,$$

where $E_2, E_3$ are stable framed (we omit here $f_i, i = 2, 3$ from the notation). As a result, our construction gives rise to the "raising degree" operators $\mathcal{H}_{\gamma_1} \otimes V_{\gamma_2} \to V_{\gamma_1+\gamma_2}$ for the COHA action $\mathcal{H} \otimes V \to V$. There are no "lowering degree" operators, which would correspond to the projection to the term $E_3 = E_2/E_1$. The reason is similar to the one in the Nakajima's construction: such a projection is not proper.

Originally Nakajima solved the problem by considering points $x \in S$ which belong to a compact subset in $S$. We can use this idea and consider short exact sequences as above, where $E_1$ is a *simple* object which runs through a compact (in analytic topology) subset in the moduli scheme of simple objects of our abelian category $\mathcal{A}$.

Let us illustrate the construction in the case of quivers without potential and trivial stability condition. In that case stable framed objects are cyclic modules over the path algebra of the quiver. Then we should prove that there are sufficiently many cyclic modules with the fixed simple submodule and fixed cyclic quotient. This is guaranteed by the following result.

**Proposition 4.3.1.** *Let $A$ be an associative algebra, $(M_2, v_2), (M_3, v_3)$ be $A$-modules with marked elements $v_i \in M_i, i = 2, 3$ such that $v_3$ is a cyclic vector for $M_3$. Let $f : M_2 \to M_3$ be an epimorphism of $A$-modules such that $f(v_2) = v_3$ and such that $W = Ker(f)$ is a simple $A$-module. Suppose that the extension*

$$0 \to W \to M_2 \to M_3 \to 0$$

*is non-trivial. Then $v_2$ is a cyclic vector for $M_2$.*

*Proof.* Let $M_2' \subset M_2$ be the $A$-submodule generated by $v_2$. If $M_2' = M_2$ then we are done. Otherwise we have a non-trivial epimorphism $g : W \to M_2/M_2'$ of $A$-modules. Its kernel is a submodule of $W$. It must be trivial, since $W$ is simple. Hence $g$ is an isomorphism. Then the submodules $W$ and $M_2'$ determine the direct sum decomposition $M_2 = W \oplus M_2'$, where $M_2' \simeq M_3$. Hence the extension $0 \to W \to M_2 \to M_3 \to 0$ is trivial. This contradiction shows that $v_2$ is a cyclic vector. ■

**Corollary 4.3.2.** *For fixed $W, M_3$ the stack of cyclic modules $M_3$ which are middle terms in the above short exact sequence is a smooth projective scheme isomorphic to the projective space $\mathbf{P}(Ext^1(M_3, W))$.*

*Proof.* Follows from the Proposition. ■

**Remark 4.3.3.** *Similar result holds in case when $M_3$ is stable framed and $S$ is simple.*

Let now $\mathcal{M}^{simp} := \mathcal{M}^{simp}_{\mathcal{A}}$ be the moduli space of simple objects in the heart $\mathcal{A}$ of the "good" $t$-structure of an ind-Artin $3CY$ category $\mathcal{C}$ endowed with orientation data. Then $\mathcal{M}^{simp}_{\mathcal{A}}$ is a smooth separated scheme. Let $H^{\bullet}_{BM,c}(\mathcal{M}^{simp})$ denotes compactly supported Borel-Moore cohomology. As before we have two projections $\pi_1, \pi_3$ from the schemes of short exact sequences to its first and last term, i.e. to the moduli space $\mathcal{M}^{simp}$ of simple objects and to the moduli space $\mathcal{M}^{st}$ of stable framed objects correspondingly. Then the composition $\pi_{3,*} \circ \pi_1^*$ defines a collection of operations on $H^{\bullet}(\mathcal{M}^{st})$ parametrized by the elements of $H^{\bullet}_{BM,c}(\mathcal{M}^{simp})$. The above Proposition (or rather its analog for non-trivial stability condition) ensures that the operations are well-defined. Differently from the action of COHA defined in the previous subsection, these operations *decrease* the degree $\gamma \in \Gamma$.

**Remark 4.3.4.** *Let us recall that for any $i \in \pm\mathbf{Z}_{>0}$ Nakajima defines an operator $P[i]$ which corresponds to the $i$-th generator of the infinite Heisenberg algebra. In the above discussion the operator $P[i]$ corresponds to the direct sum $iS := S \oplus S \oplus .... \oplus S$ of $\pm i > 0$ of copies of the simple object $S$.*

Using the above construction one can extend a representation of COHA to a representation of a bigger algebra, which we loosely call "full COHA" (or double of COHA). We do not know how to define this algebra intrinsically. This is similar to the Nakajima's construction of the infinite Heisenberg algebra from two representations of the symmetric algebra: one is given by creation operators and another one is given by annihilation operators. Commuting creation and annihilation representations *in the representation space* Nakajima recovers the infinite Heisenberg algebra. One can also compare the above construction with the one in [Re1].

**Remark 4.3.5.** *Notice that differently from the the case of conventional Hall algebras, we do not know a compatible comultiplication on COHA. Hence we cannot apply directly the "Drinfeld double" construction of the double of Hopf algebra.*

We are going to consider a motivating example in the next subsection.

## 4.4 "Full" COHA-an example

It is well-known that one can construct finite-dimensional representations of quantized enveloping algebra of finite-dimensional semisimple Lie algebras, using framed stable representations of quivers. Let us recall the construction in the case of $U_q(sl(2))$.

In general, if take the stability function $\Theta = 0$ then every finite-dimensional representation of a quiver $Q$ is semistable. Then the moduli space of stable framed representations admit a simple description in terms of Grassmannians (see e.g. [Re1], Prop. 3.9). Let us take quiver $Q = A_1$. This quiver has one vertex $i_1$ and no arrows. Framing consists of adding a new vertex $i_0$ and $d$ arrows $i_0 \to i_1$. The stability function is trivial automatically, and one can easily see that for each dimension vector $\gamma \in \mathbf{Z}_{\geq 0}$ the moduli space $\mathcal{M}_{\gamma,d} = \mathsf{M}_\gamma^{\theta=0,st}$ of framed stable representations of dimension $\gamma$ is isomorphic to the Grassmannian $Gr(d-\gamma, d) \simeq Gr(\gamma, d)$. Then it is non-empty only for $\gamma \leq d$. Let us denote by $Gr(d)$ the "full Grassmannian" consisting of vector subspaces of $\mathbf{C}^d$ of all dimensions (it is disconnected). Then the moduli space of $d$-framed semistable representations of $Q_1$ is $Gr(d)$. Since for the trivial stability function the COHA associated with a ray $\theta = 0$ coincides with whole COHA, we obtain a representation of the infinite Grassmann algebra $\Lambda^\bullet$ in the finite-dimensional vector space $V := H^\bullet(Gr(d)) = \oplus_{0 \leq k \leq d} H^\bullet(Gr(k,d))$. One can write down explicitly the action of the natural generators of $\Lambda^\bullet$ on the cohomology classes of Schubert cells.

The space of $GL(d)$-invariant functions with finite support $Fun(Gr(d))^{GL(d)}$ is a module over the constructible Hall algebra of $A_1$. The constructible Hall algebra for the quiver $A_1$ is the polynomial algebra with one generator $z := \mathbf{1}_{\mathbf{C}}$, where the generator $z$ corresponds to the characteristic function $\mathbf{1}_1$ of $\mathsf{M}_1$ in the stack $\mathcal{M} = \sqcup_{\gamma\geq 0}\mathsf{M}_\gamma$. Indeed the Hall product gives an isomorphism of the constructible Hall algebra with the polynomial ring $\mathbf{C}[z]$. In each $Gr(k,d)$ we have only one $GL(d)$-orbit of the standard coordinate vector subspace $\mathbf{C}^k \subset \mathbf{C}^d$. Let us denote by $v_k, 0 \leq k \leq d$ the characteristic function of the corresponding $GL(d)$-orbit.

Let us consider the "minus" Hecke correspondence given by pairs $(V_{k-1} \subset V_k)$ with 1-dimensional factor $V_1$ and project to $V_{k-1}$:

$$0 \to V_{k-1} \to V_k \to V_1 \to 0.$$

Then by direct computation we obtain a representation of $\mathbf{C}[z]$ given by $\rho_-(z)v_k = \frac{q^k - q^{-k}}{q-q^{-1}}v_{k-1}, 1 \leq k \leq d$, and $\rho_-(z)v_0 = 0$, where the factor comes from the normalization of the cocycle $c(M,N)$ above as $q^{\chi(M,N)}$. The Euler-Ringel form $\chi$ on the pair of representations $E$ of dimension $a$ and $F$ of dimension $b$ is given by $\chi(E,F) = ab$. Similarly, consider the "plus" Hecke correspondence $(V_k \subset V_{k+1})$ and project to $V_{k+1}$. Then we get a representation of $\mathbf{C}[z]$ in $\mathcal{F}_n$ given by

$$\rho_+(z)v_k = \frac{q^{k+1} - q^{-k-1}}{q - q^{-1}}v_{k+1}, 0 \leq k \leq d-1, \rho_+(z)v_d = 0.$$

Combining $\rho_-$ and $\rho_+$ together we obtain the standard $d$-dimensional representation of the quantized enveloping algebra $U_q(sl(2))$ where the "positive" generator $E$ is represented by $\rho_-(z)$ while the "negative" generator $F$ is represented by $\rho_+(z)$. Then the commutator $[E,F]$ maps $v_k$ to $\frac{q^{2k}-q^{2k}}{q-q^{-1}}v_k$. From this formula one can recover the action of the Cartan generators $K, K^{-1}$.

Let us now consider COHA of the quiver $A_1$. Recall, it is isomorphic to the algebra $\Lambda^\bullet = \Lambda^\bullet(\xi_1, \xi_2, ....), deg\, \xi_{2i+1} = 2i+1, i \geq 0$.

The general construction gives us a representation of COHA for the quiver $A_1$ in the finite-dimensional vector space $V := H^\bullet(Gr(d)) = \oplus_{0\leq k\leq d}H^\bullet(Gr(k,d))$.

Let us choose a subspace in each $H^\bullet(Fun(Gr(k,d))), 0 \leq k \leq d$ spanned by the cohomology classes corresponding to $(\mathbf{C}^*)^d$-fixed points. We denote this basis by $e_j := \mathbf{1}_{\mathbf{C}_{j_1,\dots j_k}}$ (recall that the fixed points correspond to coordinate subspaces $\mathbf{C}_{j_1,\dots j_k} \subset \mathbf{C}^d$ spanned by the standard basis vectors $f_{j_1}, ..., f_{j_k}, j_1 < j_2 < ... < j_k$). We can identify the graded vector space $V$ with the quotient $\Lambda^\bullet(\xi_1, ..., \xi_d)/I_d$, where $I_d$ is the graded subspace (in fact ideal) spanned by monomials $\xi_{i_1} \wedge ... \wedge \xi_{i_l}, l \geq d+1$. Then we have standard representation of $\Lambda^\bullet$ in $V$ by creation operators: $a_n^* : e_j \mapsto \xi_n \wedge e_j$.

We can consider another action of COHA on $V$ by looking at the short exact sequences

$$0 \to E_1 \to E_2 \to E_3 \to 0,$$

where $E_1$ and $E_2$ are stable framed of the same slope, and $E_3$ is just a representation without framing. This gives a representation of $\Lambda^\bullet$ on $V$ by annihilation operators $a_n : e_j \mapsto i_{\xi_n}(e_j)$, where $i_{\xi_n}$ is the contraction operator which delete the variable $\xi_n$ from the monomial $e_j$.

Then, as we discussed above in the case of constructible Hall algebra, one can combine both actions of COHA into a single representation. In this way one sees representations of the Lie algebras $so(2d)$.[5] One can speculate, that the "full" COHA for the quiver $A_1$ should be the infinite Clifford algebra $Cl_c$ with generators $\xi_n^{\pm}, n \in 2\mathbf{Z}+1$ and the central element $c$, subject to the standard anticommuting relations between $\xi_n^+$ (resp. $\xi_n^-$) as well as the relation $\xi_n^+\xi_m^- + \xi_m^-\xi_n^+ = \delta_{nm}c$. In the case of finite-dimensional representations $c \mapsto 0$ and we see two representations of the infinite Grassmann algebra, which are combined in the representations of the orthogonal Lie algebra. It is also tempting to speculate, that $Cl_c$ is the Clifford algebra associated with the positive part of the affine Lie algebra $sl(2)$ (hence the relation to the quiver $A_1$).

The general construction of "full" COHA for arbitrary quiver with potential is not known.

# 5 Representations of COHA motivated by physics and geometry

In this section we are going describe some interesting classes of representations of COHA. Details of the constructions will appear elsewhere. The reader can consider this section as a collection of speculations. The details will appear elsewhere.

## 5.1 Fukaya categories of conic bundles and gauge theories from class $S$

We illustrate here general considerations in the case of $SL(2)$ Hitchin integrable systems. In that case the general conjecture (F.1) from the Introduction of [ChDiManMoSo] admits a very precise interpretation. Namely, with a point of the universal cover of the base of Hitchin system on a curve, say, $C$ one can associate a compact Fukaya category ("compact" means that it is generated by local systems supported on compact Lagrangian submanifolds) of the local Calabi-Yau 3-fold. The latter is uniquely determined by the corresponding spectral curve. The compact Fukaya category is endowed with the natural t-structure generated by SLAGs which are 3-dimensional Lagrangian spheres. The central charge of the corresponding stability condition is given by the period map of the Liouville form restricted to the spectral curve. According to the general theory developed in Section 8 of [KoSo1] categories generated by spherical collections are in one-to-one correspondence with pairs $(Q, W)$, i.e. quivers with potential. Hence we can speak about corresponding COHA, in particular, about representations in the cohomology of stable framed objects of the category $Crit(W)$.

Recall that surface operators correspond to points of the curve $C$. In terms of the corresponding local Calabi-Yau 3-fold they are complex 2-dimensional manifolds. Hence one can look for the the moduli space of SLAGs with the

[5] I thank to Xinli Xiao for making explicit computations.

boundary which belongs to a complex surface inside of our local Calabi-Yau 3-fold. This can be thought of as the Fukaya-Seidel category of thimbles (see [Se]).

Furthermore, the operation of connected Lagrangian sum plays a role of an extension in the compact Fukaya category. This operation underlies the product structure on the COHA $\mathcal{H}^{Q,W}$. Let us observe that one can form a connected Lagrangian sum of a Lagrangian submanifold without boundary and the one with the boundary on a fixed complex surface. Mimicking the definition of the product on COHA with the "moduli space of Lagrangian connected sums" instead of the subvariety $\mathsf{M}_{\gamma_1,\gamma_2} \subset \mathsf{M}_{\gamma_1+\gamma_2}$ (see [KoSo2], Section 2), one obtains the $\mathcal{H}^{Q,W}$-module structure on the cohomology of the moduli space of SLAGs with the boundary on the fixed complex surface in our local Calabi-Yau 3-fold. Alternatively, following Paul Seidel, one can consider the double cover of the Calabi-Yau 3-fold branched along the divisor given by the complex surface. Then Lagrangian submanifolds with boundary lift to closed ones in the branched cover. One can form an equivariant Lagrangian connected sum, and then interpret that as an operation on the original Lagrangian submanifolds with boundary.

## 5.2 Resolved conifold and quivers

Let $X = tot(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$ be the resolved conifold. We denote the zero section of the corresponding vector bundle by $C_0 \simeq \mathbf{P}^1$. Let us fix a point $p_0 := 0 \in C_0$.

Let $\mathcal{A}$ be the abelian category of *perverse coherent sheaves* on $X$ topologically supported on $C_0$ (see e.g. [NagNak], [Tod] for descriptions convenient for our purposes; in [NagNak] our category $\mathcal{A}$ was denoted by $Per_c(X/Y)$, where $X \to Y$ is the crepant resolution of the conifold singularity $Y = \{xy - zw = 0\}$).

It is known (see e.g. [NagNak]) that $\mathcal{A}$ is equivalent to the abelian category $Crit(W)$ associated with the pair $(Q, W)$, where $Q$ is a quiver with two vertices $i_1, i_2$ two arrows $a_1, a_2 : i_1 \to i_2$, two arrows $b_1, b_2 : i_2 \to i_1$ and "Klebanov-Witten potential" $W = a_1b_1a_2b_2 - a_1b_2a_2b_1$. In particular, for any $\gamma = (\gamma^1, \gamma^2) \in \mathbf{Z}^2_{\geq 0}$ the stack of objects of $Crit(W)$ of dimension $\gamma$ is equivalent to the stack of such representations of $Q$ of dimension $\gamma$ in coordinate vector spaces, which belong to the critical locus of the function $Tr(W)$.

We recall that the category of perverse coherent sheaves carries a family of geometrically defined *weak stability conditions* (see e.g. [Tod]). In the case of the category $Crit(W)$ there is a class of (slope) stability conditions. The equivalence of two categories leads to the "chamber" structure of the space of stability conditions on $\mathcal{A}$ described in [NagNak]: some of the (infinitely many) chambers correspond to the stability conditions of the quiver origin, while "at infinity" we have chambers corresponding to different choices of the weak stability.

Similar story is with framed perverse coherent sheaves and framed representations of $(Q, W)$ (i.e. critical points of $Tr(W)$ considered as a function on the space of representations of the extended quiver $\widehat{Q}$ obtained from $Q$ by adding an extra vertex $i_0$ and an arrow $i_0 \to i_1$.)

In particular, when we are in the "quiver chamber", we can (after a choice of a stability condition on $\mathcal{A}$ which belongs to the above class) speak about the moduli space of stable framed objects of the fixed slope. For a given $(l,n) \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}$ and certain choice of stability condition on $Rep(Q)$, and certain slope $\theta$ which depends on $(l,n)$, the space of stable framed representations of $(Q,W)$ with the slope $\theta$ is isomorphic to the moduli space of PT stable pairs $P(l,n)$ (see [NagNak]). There is no unique $\theta$ which serves all $(l,n)$.

It follows from the previous section that:

**Proposition 5.2.1.** *For a choice of stability conditions in the "quiver chamber", COHA $\mathcal{H}^{(Q,W)}$ acts on the cohomology of the moduli space of stable framed representations of $(Q,W)$ having fixed slope.*

Let us observe that if we have a morphism $f : E_2 \to E_3$ of PT stable pairs which is surjective in degree zero (i.e. on the sheaves supported on $C_0$) then $Ker(f)$ is a coherent sheaf scheme-theoretically supported on $C_0$. Passing to the cohomology groups we reformulate the above Proposition by saying that COHA of the corresponding category acts on the cohomology of the moduli space of stable PT pairs. We expect the same result to hold for "geometric chambers" were one uses weak stability conditions.

## 5.3 Vertically framed sheaves

There is a potential application of the representation theory of COHA of the resolved conifold to knots. It is motivated by the conjecture from [ORS], its reformulation in [DiHuSo] and the proof in [Mau1] of the "unrefined" version.

If we would like to incorporate algebraic knots in the story, we should add coherent sheaves on $X$ supported on a singular algebraic curve $C_K$ which defines the knot via the intersection with the $S^3$-boundary of a small ball around the singularity. More precisely we consider coherent sheaves which are "vertically framed" along $C_K$ (see details [DiHuSo]). Stable vertically framed coherent sheaves provide a natural generalization to stable pairs from [PT1] (this is justified at the level of physics in [DiShVa]). In the language of gauge theory inclusion the curve $C_K$ to the story corresponds to a choice of surface operator.

Let us recall some details following [DiHuSo].

Let $X = tot(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$ be the resolved conifold, $C$ a planar complex algebraic curve with the only singular point $p$. In [DiHuSo] the authors used the abelian category of $C$-framed perverse coherent sheaves which is a full subcategory $\mathcal{A}^C \subset D^b(Coh(X))$. Roughly speaking, $\mathcal{A}^C$ consists of complexes $E$ of coherent sheaves on $X$ such that the cohomology sheaves $H^i(E)$ are nontrivial for $i \in \{0,-1\}$ only, and those cohomology sheaves are topologically supported one the union $C \cup \mathbf{P}^1$ (see loc.cit. Section 2.2 and below for more precise description). The category $\mathcal{A}^C$ is closed under extensions. Since it is a full subcategory of the category of perverse coherent sheaves $\mathcal{A} \subset D^b(X)$, it was used in the loc. cit. for developing the theory of $C$-framed stable pairs analogous to the one of stable pairs of Pandharipande and Thomas (the latter can be interpreted in terms of $\mathcal{A}$, see e.g. [Tod]). After fixing Kähler class $\omega$ on the compactification $\overline{X}$ defined in [DiHuSo] one defines a family of weak stability conditions on $\mathcal{A}^C$ associated with an explicitly given family of slope functions $\mu_b := \mu_{(\omega,b\omega)}$. Then one can speak about $C$-framed (semi)stable sheaves, meaning weakly (semi)stable objects of $\mathcal{A}^C$ with respect to the slope function $\mu_b$. [6] For "very negative" value of $b$ the moduli space $\mathcal{P}_b^s(X,C,r,n)$ of $C$-framed $\mu_b$-stable objects $E$ with $ch(E) = (-1,0,[C]+r[\mathbf{P}^1],n)$ is isomorphic to the moduli space of stable framed pairs on $X$ in the sense of Pandharipande and Thomas which are $C$-framed. If we move the value of $b$ from $b = -\infty$ to a small positive number (which depend on $r$) the above moduli space of $\mu_b$-stable objects experiences finitely many wall-crossings. One of the main results of [DiHuSo] is a theorem which relates the moduli space of $\mu_b$-stable objects of $\mathcal{A}^C$ for small $b > 0$ with the punctual Hilbert schemes from [ORS], thus making a link to HOMFLY polynomials of algebraic knots. The moduli space $\mathcal{P}_b^{ss}(X,C,r,n)$ of $\mu_b$ semistable objects is a $\mathbf{C}^*$-gerbe over $\mathcal{P}_b^s(X,C,r,n)$.

Let us fix $(r,n) \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}$ and consider the full subcategory $\mathcal{A}^C_{r,n} \subset \mathcal{A}^C$ consisting of objects $E$ such that $ch(E) = (-1,0,[C]+r[\mathbf{P}^1],n)$. Let $E_1$ be a pure dimension one sheaf on $X$ supported on $\mathbf{P}^1$ (hence it belongs to $\mathcal{A}^C$ as well), and let $E_3 \in Ob(\mathcal{A}^C_{r_3,n_3})$. Then we see that the middle term $E_2$ of an extension in $\mathcal{A}^C$

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

belongs to $\mathcal{A}^C_{r_2,n_2}$ for some $r_2, n_2$.

Let $\mathcal{M}^C := \cup_{r,n}\mathcal{M}^C_{r,n}$ be the moduli space (stack) of the objects $E$ which belong the category $\mathcal{A}^C_{r,n}$ for some $r,n$. Let $\mathcal{M}^{\mathbf{P}^1}$ be the moduli space (stack) of objects of the category $Coh_{\mathbf{P}^1}(X)$ of coherent sheaves on $X$ supported on $\mathbf{P}^1$. Let $\mathcal{N}$ be the moduli space (stack) of short exact sequences as above. We have the following projections: $\pi_{13} : \mathcal{N} \to \mathcal{M}^C \times \mathcal{M}^{\mathbf{P}^1}, (E_1,E_2,E_3) \mapsto (E_1,E_3)$ and $\pi_2 : \mathcal{N} \to \mathcal{M}^C, (E_1,E_2,E_3) \mapsto E_2$.

Then we can apply the same procedure as for framed representations of quivers using the composition $\pi_{2*}\pi_{13}^*$. Then e.g. in the case of COHA it gives us the module structure $H^\bullet(\mathcal{M}^{\mathbf{P}^1}) \otimes H^\bullet(\mathcal{M}^C) \to H^\bullet(\mathcal{M}^C)$ over the COHA of the category $Coh_{\mathbf{P}^1}(X)$, where by $H^\bullet$ we denote an appropriate stack cohomology.

Now we can use the weak stability condition defined by the slope function $\mu_b$. More precisely, let us choose a stability parameter $b$ satisfying the condition (3.1) of Lemma 3.1 from [DiHuSo] and consider $\mu_b$-semistable objects $E$ of $\mathcal{A}^C$ such that $ch(E) = (-1,0,[C]+r[\mathbf{P}^1],n)$, where $(r,n) \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}$ is fixed, and $\mathbf{P}^1$, as before, denotes the zero section of the resolved conifold bundle. Then we can repeat the above definition but this time in the exact sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

we will assume that $E_2$ and $E_3$ are weakly semistable objects with respect to $\mu_b$, and $E_1$, as before, is an arbitrary coherent sheaf on $X$ supported on $\mathbf{P}^1$.

[6] In [DiHuSo] the authors considered stable vertically framed sheaves on the compactification $\overline{X}$. The corresponding moduli spaces were projective. Considerations with non-compact submanifold $X$ gives rise to quasi-projective moduli spaces. We ignore these technicalities in this paper.

There is an explicit description of $\mu_b$-semistable and $\mu_b$-stable objects of $\mathcal{A}^C_{r,n}$ for sufficiently small positive $b$ given in [DiHuSo], Section 3. For example a $\mu_b$-stable object $E$ fits into an exact short sequence

$$0 \to E_C \to E \to \mathcal{O}_{\mathbf{P}^1}(-1)^r \to 0,$$

where $E_C = (\mathcal{O}_X \to F_C)$ is a stable pair on $X$ in the sense of Pandharipande and Thomas, with the sheaf $F_C$ scheme theoretically supported on $C$ (and satisfying some non-degeneracy conditions, see [DiHuSo], Proposition 3.3 for the details). Similarly, any $\mu_b$-semistable object fits into an exact sequence where instead of $\mathcal{O}_{\mathbf{P}^1}(-1)^r$ one has a sheaf $G$ topologically supported on $\mathbf{P}^1$ (and $ch_2(G) = r[\mathbf{P}^1]$) which is a direct image (under the embedding $i : \mathbf{P}^1 \to X$) of the vector bundle $\oplus_{1\le j\le m}\mathcal{O}(a_j)^{r_j}$ with $a_1 > ... > a_m \geq -1$. The Harder-Narasimhan filtration of $G$ (with respect to the $\omega$-slope defined by $\chi(G)/r$) therefore have consecutive factors with slopes $a_j/r$.

The above considerations lead to the idea of using $C$-framed stable sheaves as " stable framed objects" which should give rise to representations of COHA of $X$ similarly to the case of quivers (and PT stable pairs). At this time it is not clear how far this idea can be developed. Indeed, computations made by E. Diaconescu, show that if in the short exact sequence $0 \to F \to E_1 \to E_2 \to 0$ the terms $E_1, E_2$ are $C$-framed stable then $F$ is isomorphic to $\mathcal{O}(-2)^n$. Probably,in order to obtain interesting representations, one should also include short exact sequences of the type $0 \to E_1 \to E_2 \to F \to 0$, where $E_1, E_2$ are $C$-framed stable. This should lead to the representation of "full" COHA (see below). We expect that it will be affine $sl(2)$.

**Remark 5.3.1.** *The above story with $C$-framed sheaves is related to algebraic knots. As for the general knots, we expect that the following is true.*

*Recall that the resolved conifold $X$ is a non-compact Calabi-Yau 3-fold. For any non-compact real analytic Lagrangian submanifold $L \subset X$ with "good behavior at infinity" we expect to have a well-defined stack $Coh_{\le 1}(X, L)$ of real analytic sheaves on $X$ (considered as real analytic manifold) with the following properties:*

*a) Every $F \in Coh_{\le 1}(X, L)$ has topological support, which is an immersed 2-dimensional real-analytic submanifold of $X$. Moreover, the support without boundary is an immersed non-compact complex analytic curve. The restriction of $F$ to the complement of the boundary is a a coherent sheaf on the corresponding complex manifold.*

*b) The boundary of the support of each $F \in Coh_{\le 1}(X, L)$ belongs to $L$.*

*c) The stack $Coh_{\le 1}(X, L)$ is a countable union of real-analytic stacks of finite type. It is naturally the stack of objects of the abelian category of real-analytic sheaves on $X$ satisfying conditions a) and b).*

*In particular, sheaves $F$ with pure support are those for which the support is an immersed "bordered Riemann surfaces" in the sense of [KatzLiu].*

*We expect that despite of the analytic nature of objects, there is a theory of stability structures for this category, as well as the notion of stable framed object.*

*Notice that we can consider extensions $0 \to F \to E_1 \to E_2 \to 0$, where $E_1, E_2$ are objects of $Coh_{\leq 1}(X, L)$, while $F$ is the usual coherent sheaf on $X$ with support on $C_0 = \mathbf{P}^1$. We expect that this operation leads to the action of COHA on the cohomology of framed stable objects in $Coh_{\leq 1}(X, L)$, similarly to the case of $C$-framed stable sheaves.*

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