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REMARKS ON COMPLETE NON-COMPACT GRADIENT RICCI EXPANDING SOLITONS

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Abstract

In this paper, we study gradient Ricci expanding solitons (X, g) satisfying

$$Rc = cg + D^2 f,$$

where Rc is the Ricci curvature, $c < 0$ is a constant, and $D^2 f$ is the Hessian of the potential function f on X. We show that for a gradient expanding soliton (X, g) with non-negative Ricci curvature, the scalar curvature R has at most one maximum point on X, which is the only minimum point of the potential function f. Furthermore, $R > 0$ on X unless (X, g) is Ricci flat. We also show that there is exponentially decay for scalar curvature on a complete non-compact expanding soliton with its Ricci curvature being ε-pinched.

1. Introduction

In this paper, we continue our study on Ricci solitons [8], which are special solutions generated by one parameter family of diffeomorphisms to Ricci flow introduced by R. Hamilton in 1982 [7]. Ricci flow enjoys a remarkable property to improve Riemannian metrics on 3-manifolds (see [5] and [10]). It is an interesting and challenging subject to better understand the special solutions such as Ricci solitons to Ricci flow.

We assume in this paper that (X, g) is a gradient expanding soliton. Here is the definition of the gradient expanding soliton.

Definition 1. We call a Riemannian manifold (X, g) a gradient expanding soliton (in short, just call it an expanding soliton) if there is a smooth solution f on a Riemannian manifold (X, g) such that for some constant $c < 0$, it holds the equation

$$Rc = cg + D^2 f, \tag{1}$$

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on X, where $D^2 f$ is the Hessian matrix of the function f and Rc is the Ricci tensor of the metric g. We call the function f the potential function for the soliton (X, g). If $c > 0$ in (1), (X, g) is called a shrinking soliton; if $c = 0$, (X, g) is called a steady soliton.

In the study of Ricci flow, we often meet the following definition.

Definition 2. The Ricci curvature of a Riemannian manifold (X, g) is called ε-pinched if there is some $\varepsilon > 0$ such that the scalar curvature $R > 0$ on X and

$$Rc \geq \varepsilon R g$$

on X.

Throughout this paper, we shall assume that the Riemannian manifold (X, g) is a complete non-compact Riemannian manifold of dimension $n \geq 3$. We denote by R the scalar curvature of the metric g.

Our main result is the following

Main Theorem. *Assume that the Ricci curvature of the gradient expanding soliton (X, g) is non-negative. Then the scalar curvature R has at least one maximum point on X, which is the only assumed minimum point of the potential function f. Furthermore, $R > 0$ on X unless (X, g) is Ricci flat.*

The proof of this Theorem will be proved in section 3.

In section four, we will prove the following result

Theorem 3. *Assume that (X, g) is a gradient expanding soliton with its Ricci curvature being ε-pinched. Then its scalar curvature has the decay*

$$R(s) \leq R(o) e^{Cs - Cs^2}.$$

as the distance function s from a fixed point going to infinity, i.e., $s = d(x, o) \to +\infty$.

We remark that a similar but weaker decay result has been announced by L. Ni in Proposition 3.1 in [9]. We know the result for a while, and a reason for the delay of this present is that we try to prove non-existence of this kind of expanding solitons. However, it is still an open problem.

Throughout C will denote various uniform constants in different places.

2. Preliminary

We recall first some basic properties about Ricci solitons [7].

Taking the trace of both sides of (1), we have

$$R = nc + \Delta f. \tag{2}$$

Take a point $x \in X$. In local normal coordinates (x^i) of the Riemannian manifold (X, g) at a point x, we write the metric g as (g_{ij}). The corresponding Riemannian curvature tensor and Ricci tensor are denoted by $Rm = (R_{ijkl})$ and $Rc = (R_{ij})$ respectively. Hence,

$$R_{ij} = g^{kl} R_{ikjl}$$

and

$$R = g^{ij} R_{ij}.$$

We write the covariant derivative of a smooth function f by $Df = (f_i)$, and denote the Hessian matrix of the function f by $D^2 f = (f_{ij})$, where D the covariant derivative of g on X. The higher order covariant derivatives are denoted by f_{ijk}, etc. Similarly, we use the $T_{ij,k}$ to denote the covariant derivative of the tensor (T_{ij}). We write $T_j^i = g^{ik} T_{jk}$. Then the Ricci soliton equation is

$$R_{ij} = f_{ij} + c g_{ij}.$$

Taking covariant derivative, we get

$$f_{ijk} = R_{ij,k}.$$

So we have

$$f_{ijk} - f_{ikj} = R_{ij,k} - R_{ik,j}.$$

By the Ricci formula we have that

$$f_{ijk} - f_{ikj} = R_{ijk}^l f_l.$$

Hence we obtain that

$$R_{ij,k} - R_{ik,j} = R_{ijk}^l f_l.$$

Recall that the contracted Bianchi identity is

$$R_{ij,j} = \frac{1}{2} R_i.$$

Upon taking the trace of the previous equation we get that

$$\frac{1}{2} R_i + R_i^k f_k = 0,$$

i.e.,

$$R_k = -2 R_k^j f_j. \tag{3}$$

Then at x,

$$D_k(|Df|^2 + R + 2cf) = 2f_j(f_{jk} - R_{jk} + 2c g_{jk}) = 0.$$

So,

$$|Df|^2 + R + 2cf = M, \tag{4}$$

where M is a constant.

In the remaining part of this section, we assume that there is for some constant $C > 0$ such that $0 \leq Rc \leq C$ on the expanding soliton (X, g). Then we have $|D^2 f| \leq C$ on X. Assume $f \geq 0$ and that o is a critical point of the potential function f. Then using the Taylor's expansion, we have

$$f(x) \leq Cd^2(x, o).$$

We now study the behavior of the potential function along a minimizing geodesic curve on the expanding soliton. A similar work has been done by G. Perelman [10] (see also [6]) where he tries to give some uniform bounds on potential function f on a shrinking soliton. Fix a point $o \in X$. Take any minimizing geodesic curve $\gamma(s)$ connecting x and the fixed point p, where s is the arc-length parameter. Write by $r = d(x, o)$ and $X = \gamma'(s)$. Assume that $r > 2$. Let $\{Y_i\}$ $(i = 1, \ldots n-1)$ be an orthonormal parallel vector fields along γ. Let Y be an orthogonal vector field along the curve γ vanishing at end points. Then the second variational formula [11] (see also [1]) tells us that

$$\int_0^r (|Y|^2 - \langle R(X, Y)Y, X\rangle)\, ds \geq 0.$$

Take Y to be sY_i on $[0,1], = Y_i$ on $[1, r - r_0]$ where $1 < r_0 < r$, and $\dfrac{r-s}{r_0} Y_i$. Adding over i gives that

$$\int_0^r Rc(X, X) \leq C_0(r_0) + \frac{n-1}{r_0} - \int_{r-r_0}^r \left(\frac{r - r_0}{r_0}\right)^2 Rc(X, X)\, ds,$$

which implies that for some constant $C > 0$,

$$\int_0^r Rc(X, X) \leq C. \tag{5}$$

Note that

$$\left(\int_0^r Rc(X, Y_1)\, ds\right)^2 \leq r \int_0^r |Rc(X, Y_1)|^2\, ds \leq s \sum_i \int_0^r |Rc(X, Y_i)|^2\, ds.$$

Thinking of Rc as self-adjoint linear operator on TX and taking a point-wise orthonormal frame $\{e_j\}$ as eigenvectors of $Rc = (\bigoplus \lambda_j)$, we have that

$$R = \sum_j \lambda_j$$

and for $X = \sum_j X_j e_j$,

$$\sum_i |Rc(X, Y_i)|^2 = \langle X, Rc^2 X\rangle = \sum_j \lambda_j^2 X_j^2 \leq RRc(X, X).$$

Then,

$$\left(\int_0^r Rc(X, Y_1)\, ds\right)^2 \leq Cs \int_0^r Rc(X, X) \leq C^2 s.$$

Hence, for any unit vector field Y along γ, orthogonal to X, we have

$$\int_0^r Rc(X, Y)\, ds \le C(\sqrt{s} + 1).$$

Using (1) we have

$$\frac{d^2 f(\gamma(s))}{ds^2} = Rc(X, X) - c \ge -c,$$

and

$$\frac{d(Yf)(\gamma(s))}{ds} = Rc(X, Y).$$

Then we have

$$\frac{df(\gamma(s))}{ds} \ge \frac{df(\gamma(s))}{ds}(0) - cs \ge -cs + C$$

and for $s > 2$,

$$|(Yf)(\gamma(s))| \le |(Yf)(\gamma(0))| + \int_0^s |Rc(X, Y)|\, ds \le C\sqrt{s}. \tag{6}$$

Therefore, we can conclude that at large distance from o the potential function f has its gradient making small angle with the gradient of the distance function from o.

3. Proof of Main Theorem

We now give the *proof of Main Theorem*: Assume that $Rc \ge 0$ on X. Then for any constant $c < 0$ we have $Rc - cg > 0$ on X. By (1) we know that

$$D^2 f = Rc - cg \ge -cg > 0, \quad on \ X.$$

Then the potential function f is locally strictly convex. Since (X, g) is a complete non-compact Riemannian manifold, we have that f has at most one critical point, i.e., the point where $\nabla f = 0$. Using $D^2 f > 0$, we know that if $p \in X$ is the critical point of f, then it is a non-degenerate minimum point of f.

Note that along any minimizing geodesic curve $\gamma(s)$ connecting x and the fixed point p, where s is the arc-length parameter, we have

$$\begin{aligned} \langle \nabla f, \gamma'(s) \rangle|_0^s &= \int_0^s f_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds}\, ds \\ &= \int_0^s (R_{ij} - cg_{ij}) \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds}\, ds \\ &= -cs + \int_0^s R_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds}\, ds \\ &\ge -cs > 0 \end{aligned} \tag{7}$$

This implies that $f(\gamma(s))$ is growing at infinity at least the quadratic rate $-c$ of the distance function. Then f has at least a minimum point in X.

Assume that o is the only critical point of f. Then by adding a constant, we can assume that and $f(o) = 0$ and $f > 0$ on $X - \{o\}$. Using (4), we know that

$$M = |Df|^2(o) + R(o) + 2cf(o) = R(o).$$

Using (3) we know that o is also the critical point of R.

Let $x \in X - \{o\}$. Taking a minimizing geodesic curve $\gamma(s)$ connecting x and the fixed point o, where s is the arc-length parameter, we again have by using (7)

$$\langle \nabla f, \gamma'(s) \rangle > -cs > 0.$$

This implies that the integral curves of ∇f in $X - \{o\}$ emanating from the point o to infinity. Take a integral curve $\sigma(t)$ ∇f in $X - \{o\}$. Then by (3) we have

$$\frac{d}{dt} R(\sigma(t)) = R_i f_i = -2Rc(\nabla f, \nabla f) \le 0. \tag{8}$$

Hence $R(x) \le R(o)$ for all $x \in X - \{o\}$. So, o is a maximum point of R.

By this we conclude that

Assertion 4. *Assume that the Ricci curvature of the gradient expanding soliton (X, g) is non-negative positive. Then the scalar curvature R has at most one maximum point of R, which is the only critical point of the potential function f.*

If $R(o) = 0$, then $R = 0$ on X. Hence $Rc = 0$ on X, that is to say that (X, g) is Ricci flat. So we have $R(o) > 0$. By the local strong maximum principle, we must have $R > 0$ on the whole space X.

This finishes the *proof of Main Theorem*.

In the remaining part of this section, we consider the behavior of f at infinity. Since

$$|Df|(x)^2 + 2cf(x) = R(o) - R(x) \ge 0,$$

we get that

$$|Df|^2 \ge -2cf = 2|c|f.$$

Then we have

$$|D\sqrt{f}| \ge \sqrt{\frac{|c|}{2}},$$

at where $f \ne 0$. Therefore, we have

$$\sqrt{f}(s) \ge \sqrt{\frac{|c|}{2}} s$$

and

$$f(s) \geq \frac{|c|}{2} s^2$$

along any minimizing geodesic curve $\gamma(s)$ connecting x and the fixed point o, where s is the arc-length parameter.

Note that using (2) we have

$$|Df|^2(s) = -2cf(x) + R(o) - R(x) \leq -2cf(x) + R(o) \leq Cs^2 + R(o).$$

Hence, for $s \gg 1$,

$$C_4 s \leq |Df|(s) \leq C_5 s. \tag{9}$$

4. ε pinched solitons

We give a proof of Theorem 3 below. We try to make the proof more transparent and self-contained.

Proof of Theorem 3. Recall that the Ricci curvature of the non-shrinking soliton (X, g) is ε-pinched, i.e., for some $\varepsilon > 0$ we have that $R > 0$ on X and

$$Rc \geq \varepsilon R g$$

on X. Then using the maximum principle, we know that either $R = 0$ on X or $R > 0$. If $R = 0$ on X, then by the pinching condition we know that (X, g) is Ricci flat.

Assume that $R > 0$ on X. Then as before, the potential function f is locally strictly convex. Since (X, g) is a complete non-compact Riemannian manifold, we have that f has at most one critical point, i.e., the point where $\nabla f = 0$. Assume that we have a critical point for f, saying that it is $o \in X$. Then using (3), we know it is also a critical point of R. Using (8), we know that is the maximum point for R. In particular, we know that R is a bounded function on X, saying that $D > 0$ is the upper bound.

Using (3) and the ε-pinched condition, we have that

$$-R|\nabla f|^2 \leq \langle \nabla R, \nabla f \rangle = -2Rc(\nabla f, \nabla f) \leq -\varepsilon R |\nabla f|^2.$$

Taking a minimizing geodesic curve $\gamma(s)$ connecting x and a fixed point o, where s is the arc-length parameter, we have

$$\begin{aligned} \langle \nabla f, \gamma'(s) \rangle|_0^s &= \int_0^s f_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds}\, ds \\ &= \int_0^s (R_{ij} - c g_{ij}) \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds}\, ds \\ &= -cs + \int_0^s R_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds}\, ds. \end{aligned} \tag{10}$$

This implies that there is a constant C_2 such that

$$\langle \nabla f, \gamma'(s) \rangle \geq -cs + \int_0^s \varepsilon R\, ds \geq -cs + \int_0^1 R\, ds \geq -cs + C_2 \geq C_2$$

for $s \gg 1$.

Using (5) and the pinching condition, we have that

$$\int_0^s R\, ds \leq C_6.$$

Using the pinching condition again, (10) also implies that

$$\langle \nabla f, \gamma'(s) \rangle \leq -cs + \int_0^s R\, ds \leq -cs + D.$$

Therefore, the angle between ∇f and the gradient of the distance function from o is almost fixed.

Then, using (3) and the ε-pinched condition, we have for some constant $C_3 > 0$,

$$(R^{-1})_s = -R^{-2} \langle \nabla R, \gamma'(s) \rangle = 2R^{-2} Rc(\nabla f, \gamma'(s)).$$

Using (6) and (9), we obtain that

$$\begin{aligned} Rc(\nabla f, \gamma'(s)) &= |\nabla f| Rc(\gamma', \gamma') + 0(\sqrt{s}) \\ &\geq \varepsilon R |\nabla f| + 0(\sqrt{s}) \geq R(Cs - C), \end{aligned}$$

we have

$$(R^{-1})_s \geq 2R^{-1}(Cs - C).$$

This implies that

$$(\log R)_s \leq C - Cs$$

and

$$R(s) \leq R(o) e^{Cs - Cs^2}.$$

This implies that $R \to 0$ exponentially as $s \to +\infty$. *This completes the proof of Theorem* 3.

Theorem 3 tells us that for such (X, g) we have

$$A = \limsup_{s \to \infty} Rs^2 = 0.$$

Added in proof. Some of our results has been cited in Prop. 7.3 in the recent paper of Brendle and Schoen: *Sphere theorems in geometry*, arxiv:0904.2604v2. We refer to this paper for recent deep results of S. Brendle and R. Schoen.

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