J. Math. Soc. Japan
Vol. 13, No. 4, 1961

Remarks on differentiable structures on spheres

By Itiro TAMURA

(Received June 23, 1961)

J. Milnor [2] defined the invariant λ' for compact unbounded oriented differentiable $(4k-1)$-manifolds which are homotopy spheres and boundaries of π-manifolds at the same time, and proved that the invariant λ' characterizes the J-equivalence classes of these $(4k-1)$-manifolds for $k>1$. Recently S. Smale [3] has shown that a compact unbounded (oriented) differentiable n-manifold ($n \geqq 5$) having the homotopy type of S^n is homeomorphic to S^n and that two such manifolds belonging to the same J-equivalence class are diffeomorphic to each other if $n \neq 6$. Hence it turns out that the invariant λ' characterizes differentiable structures on S^{4k-1} which bound π-manifolds for $k>1$.

In this note we shall compute the invariant λ' of $B^7_{m,1}$ (S^3 bundles over S^4, see [4]) and show that every differentiable structure on S^7 can be expressed as a connected sum of $B^7_{m,1}$. We shall obtain also a similar result on S^{15}. Furthermore we shall show that $\bar{B}^8_{m,1} \cup_i D^8$ such that $m(m+1) \equiv 0 \bmod 56$ are 3-connected compact unbounded differentiable 8-manifolds with the 4th Betti number 1 and differentiable 8-manifolds of this type are exhausted by them, where $B^8_{m,1}$ are 4-cell bundles over S^4 ([4]). This will reveal that Pontrjagin numbers are not homotopy type invariants.

Notations and terminologies of this note are the same as in the previous paper [4]. We shall use them without a special reference.

1. The invariant λ' of $B^7_{m,1}$.

In the following $M_1^{n-1} \# M_2^{n-1}$ will denote the connected sum of two compact connected unbounded oriented differentiable $(n-1)$-manifolds M_1^{n-1} and M_2^{n-1} (Milnor [2]). Let W_1^n and W_2^n be two compact connected oriented differentiable n-manifolds with non-vacuous boundaries; let $f_1: D^{n-1} \to \partial W_1^n$ be an orientation-preserving differentiable imbedding and $f_2: D^{n-1} \to \partial W_2^n$ be an orientation-reversing differentiable imbedding. Then $W_1^n + W_2^n$ denotes the compact connected oriented differentiable n-manifold with boundary obtained from the disjoint union of W_1^n and W_2^n by identifying $f_1(x)$ with $f_2(x)$ ($x \in D^{n-1}$), making use of the device of "straightening the angle".

We choose an orientation of $B^7_{m,1}$ (resp. $B^{15}_{m,1}$) and that of $\bar{B}^8_{m,1}$ (resp. $\bar{B}^{16}_{m,1}$)

in such a way that they are consistent and

$$(\alpha_4 \cup \alpha_4)[\bar{B}^8_{m,1}, B^7_{m,1}] = 1$$
$$(\text{resp. } (\alpha_8 \cup \alpha_8)[\bar{B}^{16}_{m,1}, B^{15}_{m,1}] = 1).$$

It is known that any differentiable structure on S^7 is the boundary of a π-manifold (Milnor [2, § 6]). Let M^7_0 be the compact connected unbounded oriented differentiable 7-manifold which is homeomorphic to S^7 such that $\lambda'(M^7_0) = 1$, and let W^8_0 be the compact connected parallelizable oriented differentiable 8-manifold with the boundary $\partial W^8_0 = M^7_0$ such that $I(W^8_0) = 8$ (Milnor [2, § 4]).

Suppose that $B^7_{m,1}$ is diffeomorphic to $M^7_0 \# M^7_0 \# \cdots \# M^7_0$ (s-fold connected sum of M^7_0). Let $M^8 = \bar{B}^8_{m,1} \cup ((-W^8_0)+(-W^8_0)+ \cdots +(-W^8_0))$ (s-fold sum of $-W^8_0$) be the compact connected unbounded oriented differentiable 8-manifold obtained from the disjoint union of $\bar{B}^8_{m,1}$ and $(-W^8_0)+(-W^8_0)+ \cdots +(-W^8_0)$ identifying $\partial \bar{B}^8_{m,1} = B^7_{m,1}$ with $-\partial((-W^8_0)+(-W^8_0)+ \cdots +(-W^8_0)) = M^7_0 \# M^7_0 \# \cdots \# M^7_0$ by the diffeomorphism.

Index theorem $I(M^8) = \frac{1}{45}(7p_2(M^8) - p_1^2(M^8))[M^8]$ yields

$$7p_2(M^8)[M^8] = 45(1-8s) + 4(2m+1)^2 . \qquad (*)$$

Integrality of $\hat{A}$-genus $\hat{A}(M^8) = \frac{1}{2^7 \cdot 45}(-4p_2(M^8) + 7p_1^2(M^8))[M^8]$ implies

$$p_2(M^8)[M^8] \equiv 7(2m+1)^2 \qquad \text{mod } 2^5 \cdot 45 . \qquad (**)$$

By (*) and (**), we have

$$m(m+1) \equiv -2s \qquad \text{mod } 8.$$

Furthermore (*) implies

$$m(m+1) \equiv -2s \qquad \text{mod } 7.$$

Since there exist precisely 28 distinct differentiable structures on S^7 which form an abelian group under the connected sum (Smale [3]), we obtain therefore the following theorem.

THEOREM 1. *The invariant λ' of $B^7_{m,1}$ is equal to* $-\frac{m(m+1)}{2}$.

For example M^7_0 is diffeomorphic to $B^7_{10,1}$.

The following theorem is an immediate consequence of Theorem 1.

THEOREM 2. *$B^7_{m,1}$ and $B^7_{m',1}$ are diffeomorphic if and only if*

$$m(m+1) \equiv m'(m'+1) \qquad \text{mod } 56.$$

In particular $B^7_{m,1}$ is diffeomorphic to the standard S^7 if and only if

$$m(m+1) \equiv 0 \qquad \text{mod } 56.$$

Theorem 1 also implies

THEOREM 3. *Every differentiable structures on S^7 can be expressed by means of connected sums of $B^7_{m,1}$.*

The following theorem follows from Theorem 3.

THEOREM 4. *For any C^∞ differentiable structure on S^7, there exists a non-degenerate C^∞ function having one maximum, one minimum, and no other critical point.*

Now we consider differentiable structures on S^{15}. Since $\pi_{15+q}(S^q) \approx Z_2 + Z_{480}$ for large q, the order of the image of J-homomorphism $J_{15}: \pi_{15}(SO(q)) \to \pi_{15+q}(S^q)$ is equal to 480 and the greatest common divisor I_4 of $I(M)$ where M ranges over all almost parallelizable compact unbounded differentiable 16-manifolds is equal to 8×8128 (Milnor [**2**; Lemma 3.5]). Hence there exist precisely 8128 distinct differentiable structures on S^{15} which bound π-manifolds. Therefore by a similar argument as in the case of differentiable structures on S^7, we obtain the following theorems.

THEOREM 5. *If $B^{15}_{m,1}$ bounds a π-manifold, the invariant λ' of $B^{15}_{m,1}$ is equal to $-\frac{m(m+1)}{2}$.*

THEOREM 6. *Suppose that both $B^{15}_{m,1}$ and $B^{15}_{m',1}$ bound π-manifolds. Then they are diffeomorphic if and only if*

$$m(m+1) \equiv m'(m'+1) \qquad \text{mod } 16256.$$

In particular $B^{15}_{m,1}$ is diffeomorphic to the standard S^{15} if and only if it bounds a π-manifold and

$$m(m+1) \equiv 0 \qquad \text{mod } 16256.$$

Since cokernel of J_{15} is Z_2, $B^{15}_{m,1} \# B^{15}_{m,1}$ bounds a π-manifold (Milnor [**2**; Theorem 6.7]), and its invariant λ' is definable. We have

THEOREM 7. *The invarient λ' of $B^{15}_{m,1} \# B^{15}_{m,1}$ is equal to $-m(m+1)$.*

The proof is similar to that of Theorem 5.

For example $M^{15}_0 \# M^{15}_0$ is diffeomorphic to $B^{15}_{1882,1} \# B^{15}_{1882,1}$.

Theorem 7 implies

THEOREM 8. *Every differentiable structure on S^{15} bounding a π-manifold for which the invariant λ' takes on even value can be expressed by a connected sum of $B^{15}_{m,1}$.*

2. 3-connected compact unbounded differentiable 8-manifolds with the 4 th Betti number 1.

Combining Theorem 2 and a result of the previous paper [**4**; Theorem 1], we have the following theorem.

THEOREM 9. *If $m(m+1) \equiv 0$ mod 56, then $\bar{B}^8_{m,1} \cup_i D^8$ is a 3-connected compact unbounded differentiable 8-manifold with the 4 th Betti number 1, and every* *such differentiable* 8-*manifold is diffeomorphic to* $\bar{B}^8_{m,1} \cup_i D^8$ *with* m *satisfying* $m(m+1) \equiv 0 \bmod 56$.

Since the Euler-Poincaré characteristic of $\bar{B}^8_{m,1} \cup_i D^8$ is 3, these manifolds cannot carry any (weak) almost complex structure (Hirzebruch [**1**]).

Theorem 9 yields

THEOREM 10. *Pontrjagin numbers are not homotopy type invariants.*

In fact, for example, $\bar{B}^8_{0,1} \cup_i D^8$ and $\bar{B}^8_{48,1} \cup_i D^8$ have the same homotopy type and their Pontrjagin numbers are given as follows ([**4**; Section 1]):

$$p_1^2(\bar{B}^8_{0,1} \cup_i D^8)[\bar{B}^8_{0,1} \cup_i D^8] = 4\,,$$

$$p_2(\bar{B}^8_{0,1} \cup_i D^8)[\bar{B}^8_{0,1} \cup_i D^8] = 7\,,$$

$$p_1^2(\bar{B}^8_{48,1} \cup_i D^8)[\bar{B}^8_{48,1} \cup_i D^8] = 37636\,,$$

$$p_2(\bar{B}^8_{48,1} \cup_i D^8)[\bar{B}^8_{48,1} \cup_i D^8] = 5383\,.$$

This shows that L-genus (index theorem) is essentially the unique linear combination of Pontrjagin numbers which has the homotopy type invariance property. For example $\hat{A}$-genus is not homotopy type invariant.

Since $\bar{B}^8_{0,1} \cup_i D$ is homeomorphic to the quaternion projective plane, also the following follows from Theorem 9.

THEOREM 11. *There exist infinitely many compact unbounded differentiable* 8-*manifolds having the same homotopy type as the quaternion projective plane which are not diffeomorphic to each other.*

Making use of this result we can construct compact unbounded differentiable 12-manifolds having the homotopy type of the quaternion projective space whose Pontrjagin numbers are different each other (Tamura [**5**]).

University of Tokyo

References

[1] F. Hirzebruch, Komplexe Mannigfaltigkeiten, Proc. of Int. Congress of Math., Edinburgh, 1958.

[2] J. Milnor, Differentiable manifolds which are homotopy spheres, (mimeographed), Princeton University, 1959.

[3] S. Smale, The generalized Poincaré conjecture in higher dimensions, Bull. Amer. Math. Soc., **66** (1960), 373-375.

[4] I. Tamura, 8-manifolds admitting no differentiable structure, J. Math. Soc. Japan, **13** (1961), 377-382.

[5] I. Tamura, On certain 12-manifolds, to appear.