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REMARKS ON EXTREMUM PROBLEMS IN H^1

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1. Let U^n be the unit polydisc in the n complex variables C^n and T^n be its distinguished boundary, and let m_n denote the normalized Haar measure on T^n. $H^1(U^n)$ will denote the set of all holomorphic functions f on U^n such that $\|f\|_1 = \sup_{0 \leqq r<1} \int_{T^n} |f(rw)| \, dm_n(w) < \infty$. Every $f \in H^1(U^n)$ has radial limits $f^*(w) = \lim_{r\to 1} f(rw)$ for almost all $w \in T^n$. An $f \in H^1(U^n)$ is said to be outer if $\log|f(0)| = \int_{T^n} \log|f^*(w)| \, dm_n(w)$. The following result is shown in [12];

THEOREM A. *Let $f \in H^1(U^n)$ and f be outer and $1/f^* \in L^1(T^n)$. Then if $g \in H^1(U^n)$ and $\|g\|_1 = \|f\|_1$, and* arg $g^*(w) =$arg $f^*(w)$ *a.e. on T^n, it follows that $g = f$, where* arg f *denotes the argument of f.*

This is essentially a uniqueness theorem in extremum problems in $H^1(U^n)$. In the next section we shall extend this theorem to bounded symmetric domains. On the other hand in section 3 we shall discuss about the necessity of the assumptions posed on the above theorem.

2. Let D be a bounded symmetric domain in C^n, and $0 \in D$. D is circular and star-shaped with restect to the origin, that is, $tz \in D$ when $z \in D$ and $t \in C$ with $|t| \leqq 1$ [3]. It has the Bergman-Shilov boundary b, and b has a unique normalized measure μ invariant under the holomorphic automorphisms γ satisfying $\gamma(0)=0$. This μ is given by $d\mu(w) = V^{-1} \, ds(w)$, V the euclidean volume of b and $ds(w)$ the volume element at $w \in b$. A holomorphic function f in D is said to be in $N(D)$ if $\sup_{0 \leqq r<1} \int_b \log^+ |f(rw)| \, ds(w) < \infty$. In the same way as in U^n, it can be shown that every $f \in N(D)$ has radial limits $f^*(w) = \lim_{r\to 1} f(rw)$ for almost all $w \in b$. A holomorphic function f on D is said to be in $N_*(D)$ if $\{\log^+ |f(rw)|; 0<r<1\}$ forms a uniformly integrable family in $L^1(b)$. A holomorphic function f on D is said to be in the Hardy class $H^p(D)$ if $\|f\|_p = \sup_{0 \leqq r<1} \left(V^{-1} \int_b |f(rw)|^p\right.$

$ds(w)\Big)^{1/p} < \infty\ (p>0)$. A function f is said to be outer if both f and $1/f$ belong to $N_*(D)$. This definition coincides with the classical one in the case U^n. We state a characterization of $H^p(D)$.

THEOREM 1. *A function f on D is in $H^p(D)$ if and only if f is in $N_*(D)$ and $|f^*(w)|^p \in L^1(b)$.*

Another characterization is given in [3], but we do not use that here. Now in the same way as in [12], we have

THEOREM 2. *Let $f(z) \in H^1(D), f(z)$ be outer and $1/f^*(w) \in L^1(b)$. Then if $g(z) \in H^1(D)$ and $\|g\|_1 = \|f\|_1$ and if* $\arg g^*(w) = \arg f^*(w)$ *a.e. on b* (mod. 2π), *it follows that $g(z) = f(z)$.*

To prove these theorems, we use the following results. The first is due to A. Korányi [5] and the second will be found in [3].

LEMMA 1. *There exists the Poisson kernel $P(w, z)$ defined on (b, D) and satisfying*

(i) $P(w, z) \geqslant 0$ *for* $w \in b, z \in D$

(ii) $V^{-1} \int P(w, z)\, ds(w) = 1$ *for* $z \in D$

(iii) *For any fixed w_0 and a neighbourhood $N \subset b$ of w_0,* $\lim_{z \to w_0} \int_{w \notin N} P(w, z)\, ds(w) = 0$ *for* $z \in D$

(iv) *$P(w, z)$ is harmonic on D for every fixed $w \in b$*

(v) $f(z) = V^{-1} \int P(w\ z) f^*(w)\, ds(w)$ *for* $f \in H^p(D)$ $(p \geqslant 1)$

(vi) *$P(w, z)$ is continuous in w for every fixed $z \in D$*

(vii) $P(u, rw) = P(w, ru)$ *for $u, w \in b$ and* $0 < r < 1$.

(i)—(v) are contained in [5] explicitely and (vi), (vii) are there implicitely. A complex-valued function h on D is said to be harmonic if $\Delta h = 0$ for each differential operator Δ of D with the Bergman metric, invariant under the holomorphic automorphisms of D.

LEMMA 2. *If f is plurisubharmonic in D, then we have for every* $0<r<1$

$$V^{-1}\int P(w,z)f(rw)\,ds(w)\geqslant f_r(z)=f(rz).$$

PROOF OF THEOREM 1. The necessity is well-known [1], and we shall show only the sufficiency. Since $\log^+|f(z)|$ is plurisubharmonic in D, we have by Lemma 2

$$\log^+|f(tz)|\leqslant V^{-1}\int P(w,z)\log^+|f(tw)|\,ds(w)\ (z\in D, 0<t<1). \tag{1}$$

As $\{\log^+|f(tw)|\}$ is a uniformly integrable family by the assumption, there is a sequence $t_j\to 1$ such that $\log^+|f(t_jw)|$ tends to an integrable function in the weak topology. The limiting function is clearly $\log^+|f^*(w)|$. As $P(w,z)$ is continuous in w if $z\in D$ is fixed, we have, letting $t_j\to 1$ in (1),

$$\log^+|f(rw)|\leqslant V^{-1}\int P(u,rw)\log^+|f^*(u)|\,ds(u)\quad (0<r<1, w\in b).$$

Consequently, since e^{pt} is a convex function, we have

$$\max(1,|f(rw)|^p)\leqslant V^{-1}\int P(u,rw)\max(1,|f^*(u)|^p)\,ds(u).$$

Therefore we have

$$|f(rw)|^p\leqslant 1+V^{-1}\int P(u,rw)|f^*(u)|^p\,ds(u). \tag{2}$$

Thus using $P(u,rw)=P(w,ru)$ $(u,w\in b)$, we have, after integrating (2) with respect to w,

$$\sup_{0\leqslant r<I}\int|f(rw)|^p\,ds(w)\leqslant V+\int|f^*(w)|^p\,ds(w).$$

This completes the proof.

PROOF OF THEOREM 2. By Theorem 1, we have $1/f\in H^1(D)$. Hence the assumption implies that $h(z)=g(z)/f(z)\in H^{1/2}(D)$ and $h^*(w)=g^*(w)/f^*(w)\geqslant 0$ a.e. on b. Since $s(w)$ is a measure invariant under multiplication by $e^{i\theta}$, we have

$$(2\pi V)^{-1}\int k(e^{i\theta}w)\,d\theta\,ds(w)=V^{-1}\int k(w)\,ds(w)$$

for every nonnegative or integrable function k on b. We have thus for almost all $w\in b$,

$$h_w^*(e^{i\theta}) = h^*(e^{i\theta}w) \geqslant 0, \text{ a. e. } \theta \in [0, 2\pi].$$

We have also for almost all $w \in b$ $h_w(z) = h(zw) \in H^{1/2}(U)$. Hence $h_w(z)$ is constant in virtue of Neuwirth-Newman's theorem, which asserts that every $H^{1/2}(U)$ function with nonnegative boundary values a. e. is constant. Now since discs$\{zw; z \in U\}_{w \in b}$ intersect at 0, we obtain that $h(zw) = h_w(z) = h(0)$ for amost all $w \in b$. Since $b/2 = \{w/2;\ w \in b\}$ is the Bergman-Shilov boundary of $D/2 = \{z/2;\ z \in D\}$ and $h(z)$ is holomorphic on the closure of $D/2$, we have $h(z) = h(0)$ in D. Hence we have $g(z) = f(z)$ in D, because $h(0) = g(0)/f(0)$ and $g(0) = V^{-1}\int g^*(w)\, ds(w) = V^{-1}\int f^*(w)\, ds(w) = f(0)$. This completes the proof.

REMARK. We have shown in the above proof implicitely that if $f \in H^{1/2}(D)$ and $f^* \geqslant 0$ a. e. on b, then f is constant.

3. Necessary conditions for Theorem A are the the followings,

(1) $\quad f/\|f\|_1$ is an extreme point of the unit ball of $H^1(U^n)$,

(2) $\quad f/(1-u)^2 \notin H^1(U)$ for every $u \in H^1(U^n)$ with $|u^*| = 1$ a. e.

In $H^1(U)$, (1) is equivalent to that f is an outer function and in particular has no zero in U. We have in [13] that there is an $f \in H^1(U^n)$ $(n \geqslant 2)$ such that it satisfies (1) but it is not an outer function. This suggests that to be outer (and in particular to be zero-free) is not necessary for the validity of Theorem A $(n \geqslant 2)$. We shall check it really in the following Theorem 5.

Next we suppose that f is outer. We have seen in [12] that $1/f \in L^1(T^n)$ is not superfluous in a sense, that is, for every $0 < p < 1$, there exists an $f \in H^1(U^n)$ such that f is outer and $1/f^* \in L^p(T^n)$ but it breaks the validity of Theorem A. We shall see in the following Theorem 3 that $1/f^* \in L^1(T)$ is not necessary for Theorem A even in the case $H^1(U)$.

Now we begin with an easy lemma.

LEMMA 3. *If $f(z)$ is a holomorphic function on the complex plane except* 1 *and it has at most a pole at* 1, *and if $f(e^{i\theta})$ is real for all real θ, then $f(z)$ has the following form*

$$f(z) = \sum_{j=0}^{k} a_j \left(i\frac{1+z}{1-z}\right)^j$$

for some nonnegative integer k and real a_j $(j = 0, \cdots, k)$. In particular, if $f(e^{i\theta}) \geqslant 0$, then $f(z)$ has the following from

$$f(z)=\sum_{j=0}^{2k} a_j\left(i\frac{1+z}{1-z}\right)^j,$$

where k is a nonnegative integer and $a_0, a_{2k}\geqslant 0$ and a_j $(j=1,\cdots,2k-1)$ are suitable reals.

PROOF. It is an easy matter to check that $i\dfrac{1+e^{i\theta}}{1-e^{i\theta}}$ is real for every real θ. Since $\dfrac{2}{1-z}=1+\dfrac{1+z}{1-z}$, $f(z)$ can be written as follows,

$$f(z)=\sum_{j=0}^{k} a_j\left(i\frac{1+z}{1-z}\right)^j$$

for some nonnegative integer k. Since $\left(i\dfrac{1+e^{i\theta}}{1-e^{i\theta}}\right)^j, j=0,\cdots,k$, are real valued functions are linearly independent, all coefficients a_j must be real. The second assertion is then easily verified.

Combining this lemma and an analytic continuation theorem for H^1, we have

THEOREM 3. *If $f(z)\in H^1(U)$ and if* $\arg f^*(e^{i\theta})=\arg(1-e^{i\theta})$ *a.e. on T* (mod. π), *then it follows that*

$$f(z)=a(1-z)+ib(1+z)$$

for some real numbers a and b. In particular if $\arg f^*(e^{i\theta})=\arg(1-e^{i\theta})$ *a.e. on T* (mod. 2π), *then we have*

$$f(z)=a(1-z) \quad \textit{for some } a>0\,.$$

PROOF. Let ε be an arbitrary positive number and let $U_\varepsilon=\{z\in U; |z-1|>\varepsilon\}$. Since $\dfrac{f(z)}{1-z}$ is in $H^1(U_\varepsilon)$ clearly and $\dfrac{f^*(e^{i\theta})}{1-e^{i\theta}}$ is real for almost all $\theta\in\{|e^{i\theta}-1|\geqslant\varepsilon\}$, $\dfrac{f(z)}{1-z}$ can be continued analytically across the open arc $\{|e^{i\theta}-1|>\varepsilon\}$ in the same way as in [7] p.59. Since ε is arbitrary, we can assert that $\dfrac{f(z)}{1-z}$ is continued analytically on the extended complex plane except 1, so that $f(z)=f(1/\bar{z})$. Next, since $i\dfrac{1-z}{1+z}$ has real values on T, $i\dfrac{f(z)}{1+z}$ has also real values a.e. on T. It follows therefore that $\frac{f(z)}{1+z}$ is holomorphic on the extended complex plane except -1. These two facts show that $\frac{f(z)}{1-z}$ has at most a pole of order one at 1. Applying the above lemma to this $\frac{f(z)}{1-z}$, we have the desired conclusion.

REMARK. That the hypothesis of $f(z) \in H^1$ is not superfluous is shown by the function

$$f(z) = 1 - z - \frac{(1+z)^2}{1-z}$$

which has the same arguments as $1-z$ on $|z| = 1$, except $z = 1$, while $f(z) \in H^p$ $(0<p<1)$.

With a slight modification of the proof of the above theorem, we have

THEOREM 4. *Suppose that $f(z)$ is in $H^1(U)$. Let $|a| = 1$ and $\alpha > -1$. Assume further that*

$$\arg f^*(e^{i\theta}) = \arg\ (a - e^{i\theta})^\alpha \quad a.e. \text{ on } T \ (\text{mod. } \pi),$$

where we take $1^\alpha = 1$. Then $f(z)$ has the following form,

$$f(z) = (a-z)^\alpha \sum_{j=0}^{k} a_j \left(i\frac{a+z}{a-z}\right)^j \tag{3}$$

where $k = [\alpha] + 1$ if $\alpha \neq$ integer, $= \alpha$ if $\alpha =$ integer and $a_j =$ real $(j = 0, \cdots, k)$. In particular if

$$\arg f^*(e^{i\theta}) = \arg\ (a - e^{i\theta})^\alpha \quad a.e. \text{ on } T \ (\text{mod. } 2\pi),$$

then in (3) $k = 2m$, where m is the largest integer such that $2m \leqslant \alpha$ if α is an integer and $2m \leqslant \alpha + 1$ if α is not an integer, a_0, $a_{2m} \geqslant 0$ and a_j $(j = 1, \cdots, 2m-1)$ are some suitable real numbers.

Using Theorem 3 we have its n-dimensional form.

THEOREM 5. *Let $f(z) \in H^1(U^n)$ $(n \geqslant 2)$ and let*

$$\arg f^*(e^{i\theta_1}, \cdots, e^{i\theta_n}) = \arg\,(e^{i\theta_1} + e^{i\theta_2}) \quad a.e.\ on\ T^n \ (\mathrm{mod.} 2\pi). \tag{4}$$

Then it follows that

$$f(z) = \frac{\pi}{4}\|f\|_1(z_1 + z_2)\,.$$

PROOF. We have $f(z_1, e^{i\theta_2}, \cdots, e^{i\theta_n}) \in H^1(U)$ *for almost all* $(e^{i\theta_2}, \cdots, e^{i\theta_n}) \in T^{n-1}$ ([14] p. 326). Thus by Theorem 3, the assumption (4) implies that

$$f(z_1, e^{i\theta_2}, \cdots, e^{i\theta_n}) = a(e^{i\theta_2}, \cdots, e^{i\theta_n})\,(z_1 + e^{i\theta_2}) \text{ a. e. on } T^{n-1}, \text{ where } a(e^{i\theta_2}, \cdots, e^{i\theta_n}) > 0.$$

Consequently we have

$$f^*(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n}) = a(e^{i\theta_2}, \cdots, e^{i\theta_n})\,(e^{i\theta_1} + e^{i\theta_2}) \tag{5}$$

a. e. on T^n. We can hence continue $a(e^{i\theta_2}, \cdots, e^{i\theta_n})$ analytically into U^{n-1} by

$$a(z_2, \cdots, z_n) = \frac{f(e^{i\theta_1}, z_2, \cdots, z_n)}{e^{i\theta_1} + z_2} \qquad (z_2, \cdots, z_n) \in U^{n-1}\,,$$

where $f(e^{i\theta_1}, z_2, \cdots, z_n) \in H^1(U^{n-1})$ for some fixed $\theta_1 \in [0, 2\pi)$ ([14] p.326). Since $e^{i\theta_1} + z_2$ is an outer function in U^{n-1}, it follows that $a(z_2, \cdots, z_n) \in N_*(U^{n-1})$. Now we have, integrating (5), $\left(\frac{1}{2\pi}\right)^{n-1} \int_{T^{n-1}} a(e^{i\theta_2}, \cdots, e^{i\theta_n})\, d\theta_2 \cdots d\theta_n = \frac{\pi}{4}\|f\|_1 < \infty$. Hence by Theorem 1, $a(z_2, \cdots, z_n)$ is in $H^1(U^{n-1})$. Since every $H^1(U^{n-1})$ function can be represented by the Poisson integral of its boundary function and since $a(e^{i\theta_2}, \cdots, e^{i\theta_n})$ is nonnegative, it follows that $a(z_2, \cdots, z_n)$ is a real-valued holomorphic function on U^{n-1}. Consequently $a(z_2, \cdots, z_n)$ must be a constant function. Then it follows immediately that $a = \frac{\pi}{4}\|f\|_1$, which completes the proof.

4. Similar results for tube domains over cones corresponding to those of section 2 will be given elsewhere.

5. Appendix. Recently R.P.Feinerman has shown the following fact in [15].

THEOREM B. *Let* λ *be a real number,* $p \geqslant 1$, *and* $\beta = \frac{2}{\pi}$ arc tan λ *(principal values). If* $f(z)$ *is in* $H^p(U)$, *is real on* $(-1, 1)$ *and satisfies*

$$\lambda \operatorname{Re} f(e^{i\theta}) = -\operatorname{Im} f(e^{i\theta}) \qquad a.e.\ in\ (0, \pi)\,,$$

then

$$f(z) = C\left(\frac{1-z}{1+z}\right)^{\beta}$$

where C is real and (C is 0 if $p|\beta|\geqslant 1$), and we take the branch of $\left(\frac{1-z}{1+z}\right)^{\beta}$ *as* $1^{\beta}=1$.

We notice that we can prove this theorem easily by using our theorem 4 as follows. Note first that the null function satisfies the hypotheses of the above theorem. We may thus assume $f(z)\not\equiv 0$. We assume further $\beta\geqslant 0$. By assumption $1>\beta>-1$. Now if g is in H^q $(q>0)$ and g is real on $(-1,1)$, then Re $g(e^{i\theta})$ $=$ Re $g(e^{-i\theta})$ and Im $g(e^{i\theta})=-$Im $g(e^{-i\theta})$ a. e. on $(0,\pi)$. This follows immediately from the fact $g(z)=\overline{g(\bar{z})}$, gained by Schwarz reflection principle. Therefore, since $\left(\frac{1-z}{1+z}\right)^{\beta}$ satisfies the hypotheses except that of H^p, we have that

$$\arg f(e^{i\theta}) = \arg\left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{\beta} \quad \text{a. e. on } T \ (\text{mod. } \pi)\,,$$

or equivalently

$$\arg\ (1+e^{i\theta})^{\beta}f(e^{i\theta})=\arg\ (1-e^{i\theta})^{\beta} \quad \text{a. e. on } T \ (\text{mod. } \pi)\,.$$

As $f(z)$ is in H^p, $(1+z)^{\beta}f(z)$ is in H^1. Hence by Theorem 4 we have

$$(1+z)^{\beta}f(z) = a(1-z)^{\beta}+ib(1-z)^{\beta}\frac{1+z}{1-z}\,,$$

or

$$f(z) = a\left(\frac{1-z}{1+z}\right)^{\beta}+ib\left(\frac{1-z}{1+z}\right)^{\beta-1}\,,$$

for some real numbers a,b. Since $f(z)$ is real on $(-1,1)$, the second term must vanish. As $f(z)$ is in H^p, $p\beta$ must be smaller than 1. The same is true in case $-1<\beta<0$. This proves Theorem B.

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