REMARKS ON FIRST AND SECOND ORDER PERIODIC BOUNDARY VALUE PROBLEMS*

by

V. Lakshmikantham
Department of Mathematics
The University of Texas at Arlington
Arlington, Texas 76019

and

S. Leela
State University of New York
Geneseo, New York 14454

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1. Introduction.

We consider the first and second order periodic boundary value problems (PBVP for brevity)

$$u' = f(t,u),\ u(0) = u(2\pi), \tag{1.1}$$

and

$$-u'' = f(t,u),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi), \tag{1.2}$$

and obtain the existence of extremal solutions as limits of monotone iterates. In [2] and [3], the monotone method was employed for the problems (1.1) and (1.2) when the corresponding lower and upper solutions $\alpha(t), \beta(t)$ satisfy

$$\alpha(0) \leq \alpha(2\pi),\ \beta(0) \geq \beta(2\pi), \tag{1.3}$$

$$\alpha'(0) \geq \alpha'(2\pi),\ \beta'(0) \leq \beta'(2\pi), \tag{1.4}$$

in addition to other conditions. Furthermore, when f is increasing, it is shown [2,3] that the conditions (1.3) and (1.4) are tantamount to assuming the existence of a periodic solution. Accordingly, the problem of proving the existence of periodic solutions when the conditions (1.3) and (1.4) are violated becomes important and this question has been open. Our main objective in this paper is to investigate this open problem.

2. First Order Periodic Boundary Value Problem.

Consider the PBVP (1.1), where $f \in C[[0,2\pi] \times R, R]$. We list the following assumptions (relative to the lower and upper solutions $\alpha(t)$ and $\beta(t)$) for convenience:

(A_0) $\alpha_1\beta \in C^1[[0,2\pi],r], \alpha(t) \leq \beta(t)$ on $[0,2\pi]$ and $f(t,u_1) - f(t,u_2) \geq -M(u_1-u_2)$, $t \in [0,2\pi]$, for any u_1, u_2 such that $\alpha(t) \leq u_2 \leq u_1 \leq \beta(t)$ and $M > 0$;

(A_1)(i) $\alpha' \leq f(t,\alpha)$, $t \in (0,2\pi]$ and $\alpha(0) \leq \alpha(2\pi)$;

(ii) $\beta' \geq f(t,\beta)$, $t \in (0,2\pi]$ and $\beta(0) \geq \beta(2\pi)$;

(A_2)(i) $\alpha' \leq f(t,\alpha) - Mr_\alpha$, $t \in (0,2\pi]$ and $\alpha(0) > \alpha(2\pi)$, where

$$r_\alpha = [\alpha(0) - \alpha(2\pi)] \frac{e^{2M\pi}}{e^{2M\pi}-1} ;$$

(ii) $\beta' \geq f(t,\beta) + Mr_\beta$, $t \in (0,2\pi]$ and $\beta(0) < \beta(2\pi)$, where

$$r_\beta = [\beta(2\pi) - \beta(0)] \frac{e^{2M\pi}}{e^{2M\pi}-1} ;$$

(A_3) (A_1)(i) and (A_2)(ii) hold;

(A_4) (A_1)(ii) and (A_2)(i) hold.

The following lemma is useful for our further discussion.

Lemma 2.1. Let $m \in C^1[[0,2\pi],R]$. Suppose that

(i) $m'(t) \leq -Mm(t)$, $t \in (0,2\pi]$ and $m(0) \leq m(2\pi)$,

or

(ii) $m'(t) \leq -Mm(t) - Mr$, $t \in (0,2\pi]$ and $m(0) > m(2\pi)$, where

$$r = [m(0) - m(2\pi)] \frac{e^{2M\pi}}{e^{2M\pi}-1} ,$$

holds. Then, in either case, $m(t) \leq 0$ on $[0,2\pi]$.

Proof. If the conclusion is false, there exists a $t_0 \in [0,2\pi]$ and an $\varepsilon > 0$ such that

$$m(t_0) = \varepsilon \quad \text{and} \quad m(t) \leq \varepsilon \quad \text{on} \quad [0,2\pi].$$

Let $t_0 \in (0,2\pi]$. Then $m'(t_0) \geq 0$. When either (i) or (ii) holds, it follows that

$$0 \leq m'(t_0) \leq -Mm(t_0) = -M\varepsilon < 0$$

which is a contradiction. If $t_0 = 0$ and (i) holds, then $m(2\pi) \geq m(0) = \varepsilon$ and hence $0 \leq m'(2\pi) \leq -Mm(2\pi) \leq -M\varepsilon < 0$ which is a contradiction. On the other hand, if $t_0 = 0$ and (ii) holds, the integration of the inequality

$$(m'+Mm)e^{Mt} \leq -Mre^{Mt}$$

from 0 to 2π yields the estimate

$$m(2\pi)e^{2M\pi} \leq m(0) + r(1-e^{2M\pi}).$$

Using the value of r, the above inequality reduces to $0 \leq m(0)(1-e^{2M\pi})$ i.e., $m(0) \leq 0$ and this contradicts $m(0) = \varepsilon > 0$. The lemma is therefore proved.

We are now in position to prove the following.

Theorem 2.1. Let (A_0) hold. Then, any one of the conditions (A_1)-(A_4) implies that there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0,2\pi]$ and that ρ, r are minimal and maximal solutions of PBVP (1.1) respectively.

Proof. Here, we indicate the proof for the case when (A_0) and (A_1) or (A_2) hold. The proofs of other two cases can be formulated on similar lines.

For any $\eta \in [\alpha,\beta] = \{\eta \in C[[0,2\pi],R]: \ \alpha(t) \leq \eta(t) \leq \beta(t),\ t \in [0,2\pi]\}$, consider the PBVP

$$u' = f(t,\eta(t)) - M(u-\eta(t)),\ u(0) = u(2\pi). \tag{2.2}$$

Rewriting (2.2) in the form $u' + Mu = \sigma(t)$, $u(0) = u(2\pi)$, where $\sigma(t) \equiv f(t,\eta(t)) + M\eta(t)$, it is easy to see that

$$u(t) = u(0)e^{-Mt} + \int_0^t \sigma(S)d^{-M(t-s)}ds$$

and

$$u(0) = u(2\pi) = \frac{1}{e^{2M\pi}-1}\int_0^{2\pi} \sigma(S)e^{MS}ds,$$

satisfies the PBVP (2.2). The uniqueness of solutions of (2.2) follows by Lemma 2.1. In fact, if u and v are two distinct solutions of (2.2), then setting $p = u - v$, we get

$$p' = -Mp, \; p(0) = p(2\pi),$$

which implies $p(t) \equiv 0$. Hence, for any $\eta \in [\alpha,\beta]$, we define a mapping A by $A\eta = u$, where u is the unique solution of (2.2). We shall show that

(i) $\alpha \leq A\alpha$, $\beta \geq A\beta$,

and (ii) A is monotone nondecreasing on $[\alpha,\beta]$.

To prove (i), we set $p = \alpha - \alpha_1$ where $\alpha_1 = A\alpha$. We then have, in case of (A_1),

$$p' \leq -Mp, \; p(0) \leq p(2\pi),$$

and, in case of (A_2),

$$p' \leq -Mp - Mr_\alpha, \; p(0) > p(2\pi).$$

Hence Lemma 2.1 implies $p(t) \leq 0$ on $[0,2\pi]$ proving $\alpha \leq A\alpha$. A similar proof holds for $\beta \geq A\beta$.

To prove (ii), let $\eta_1,\eta_2 \in [\alpha,\beta]$ such that $\eta_1 \leq \eta_2$. Let $A\eta_1 = u_1$ and $A\eta_2 = u_2$. Setting $p = u_1 - u_2$ and using (A_0), we obtain

$$p' \leq -Mp, \; p(0) = p(2\pi),$$

which implies by Lemma 2.1 that $p(t) \leq 0$. This means that A is monotone on $[\alpha,\beta]$.

It therefore follows that we can define the sequences $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that

$$\alpha_n = A\alpha_{n-1}, \; \beta_n = A\beta_{n-1}$$

and conclude that on $[0,2\pi]$,

$$\alpha = \alpha_0 \le \alpha_1 \le \alpha_2 \le \dots \le \alpha_n \le \beta_n \le \dots \le \beta_2 \le \beta_1 \le \beta_0 = \beta.$$

The rest of the proof of the theorem is exactly the same as in [2]. Hence the theorem is proved.

Remark. In [2], an existence theorem based on the Lyapunov-Schmidt method and modified function approach was proved under a variety of conditions, one of the assumptions being

$$D^+\alpha(0) \ge f(0,\alpha(0)) + 1 + |\alpha(0)| \quad \text{and} \quad \int_0^{2\pi} f(s,\alpha(s))ds \ge 0$$

(and a similar condition for β; see [1]) instead of $\alpha(0) \le \alpha(2\pi)$ and $\beta(0) \ge \beta(2\pi)$). We now recognize that this condition is impossible to realize in view of a lemma [see Lemma 1.2.2 in [1]; page 9].

3. Second Order Periodic Boundary Value Problem.

We now consider the PBVP (1.2) where $f \in C[[0,2\pi] \times R, R]$. Relative to the lower and upper solutions of (1.2), we list the following assumptions:

(B_0) $\alpha,\beta \in C^2[[0,2\pi],R]$, $\alpha(t) \le \beta(t)$ on $[0,2\pi]$ and for any u_1, u_2 such that $\alpha(t) \le u_2 \le u_1 \le \beta(t)$, $t \in [0,2\pi]$, $f(t,u_1) - f(t,u_2) \ge -M^2(u_1-u_2)$;

(B_1)(i) $-\alpha'' \le f(t,\alpha)$, $t \in [0,2\pi]$, $\alpha(0) = \alpha(2\pi)$ and $\alpha'(0) \ge \alpha'(2\pi)$;

(ii) $-\beta'' \ge f(t,\beta)$, $t \in [0,2\pi]$, $\beta(0) = \beta(2\pi)$ and $\beta'(0) \le \beta'(2\pi)$;

(B_2)(i) $\alpha(0) = \alpha(2\pi)$, $\alpha'(0) < \alpha'(2\pi)$ and for $t \in [0,2\pi]$,

$$-\alpha'' \le f(t,\alpha) - M^2 r_\alpha, \quad \text{where} \quad r_\alpha = \frac{[\alpha'(2\pi)-\alpha'(0)](e^{2M\pi}+1)}{2M(e^{2M\pi}-1)};$$

(ii) $\beta(0) = \beta(2\pi)$, $\beta'(0) > \beta'(2\pi)$ and for $t \in [0,2\pi]$,

$$-\beta'' \ge f(t,\beta) + M^2 r_\beta, \quad \text{where} \quad r_\beta = \frac{[\beta'(0)-\beta'(2\pi)](e^{2M\pi}+1)}{2M(e^{2M\pi}-1)};$$

(B_3) (B_1)(i) and (B_2)(ii) hold;

(B_4) (B_1)(ii) and (B_2)(i) hold.

A useful comparison lemma in this context is the following:

Lemma 3.1. Let $m \in C^2[[0,2\pi],R]$. Suppose that $m(0) = m(2\pi)$ and one of the following conditions is satisfied:

(i) $-m''(t) \le -M^2 m(t)$, $t \in [0,2\pi]$ and $m'(0) \ge m'(2\pi)$:

or (ii) $-m''(t) \le -M^2 m(t) - M^2 r$, $t \in [0,2\pi]$ and $m'(0) < m'(2\pi)$, where

$$r = \frac{[m'(2\pi)-m'(0)](e^{2M\pi}+1)}{2M(e^{2M\pi}-1)} .$$

Then $m(t) \le 0$ on $[0,2\pi]$.

Proof. Suppose the conclusion is not true. Then there exists a $t_0 \in [0,2\pi]$ and an $\varepsilon > 0$ such that

$$m(t_0) = \varepsilon \text{ and } m(t) \le \varepsilon, \ t \in [0,2\pi].$$

Let $t_0 \in (0,2\pi)$. Then $m'(t_0) = 0$ and $m''(t_0) \le 0$. It then follows that, when either (i) or (ii) holds,

$$0 \le -m''(t_0) \le -M^2 m(t_0) = -M^2\varepsilon < 0$$

which is a contradiction. Let $t_0 = 0$ or 2π and (i) hold. Then, $m(2\pi) = m(0) = \varepsilon$ and $m(t) \le \varepsilon$ on $[0,2\pi]$. This implies that $m'(0) \le 0$ and $m'(2\pi) \ge 0$. Using the boundary condition $m'(0) \ge m'(2\pi)$, we arrive at $m'(0) = m'(2\pi) = 0$. Hence, for $i = 0,2\pi$,

$$-m''(i) \le -M^2 m(i) = -M^2\varepsilon < 0.$$

This contradicts $m(0) = m(2\pi) = \varepsilon$.

Now, let $t_0 = 0$ or 2π and (ii) be satisfied. As before, we get $m(0) = m(2\pi) = \varepsilon$ and $m(t) \le \varepsilon$ on $[0,2\pi]$. Consider the problem

$$-m'' + M^2 m = \sigma(t),\ m(0) = m(2\pi),\ m'(0) + \lambda = m'(2\pi), \qquad (3.1)$$

where $\lambda = \frac{2rM(e^{2M\pi}-1)}{(e^{2M\pi}+1)}$, $\sigma = -M^2 r + \eta$, $\eta = \eta(t) \leq 0$. A solution of (3.1) is given by

$$m(t) = C_1 e^{Mt} + C_2 e^{-Mt} - \frac{e^{Mt}}{2M}\int_0^t \sigma(s)e^{-Ms}ds + \frac{e^{Mt}}{2M}\int_0^t \sigma(s)e^{Ms}ds,$$

with

$$C_1 = \frac{1}{2M(e^{2M\pi}-1)}\left[\int_0^{2\pi} \sigma(s)e^{+M(2\pi-s)}ds + \lambda\right],$$

and

$$C_2 = \frac{1}{2M(e^{2M\pi}-1)}\left[\int_0^{2\pi} \sigma(s)e^{Ms}ds + \lambda e^{2M\pi}\right].$$

Consequently, noting that $\eta(t) \leq 0$ and $m(0) = C_1 + C_2$, we obtain

$$m(0) \leq \frac{-M^2 r}{2M(e^{2M\pi}-1)} \int_0^{2\pi} [e^{Ms} + e^{M(2\pi-s)}]ds + r = 0.$$

Thus $m(0) = m(2\pi) \leq 0$ which is a contradiction. Hence the lemma is proved.

We shall now state a theorem analogous to Theorem 2.1 for the PBVP (1.2).

<u>Theorem 3.1</u>. Let (B_0) and any one of the assumptions (B_1) - (B_4) be satisfied. Then there exist sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0,2\pi]$, where $\rho(t)$ and $r(t)$ are the minimal and the maximal solution of PBVP (1.2) respectively.

<u>Proof</u>. We shall indicate the proof only for the cases when (B_1) or (B_2) holds. The proof in other cases can be constructed in a similar way.

For any $\eta \in [\alpha,\beta] = \{\eta \in C[[0,2\pi],R]:\ \alpha(t) \leq \eta(t) \leq \beta(t),\ t \in [0,2\pi]\}$, consider the linear PBVP

$$-u'' = f(t,\eta) - M^2(u-\eta),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi). \tag{3.2}$$

Writing $f(t,\eta) + M^2\eta \equiv \sigma(t)$, we can find a solution $u(t)$ of (3.2) given by

$$u(t) = C_1e^{Mt} + C_2e^{-Mt} - \frac{e^{Mt}}{2M}\int_0^t \sigma(s)e^{-Ms}ds + \frac{e^{-Mt}}{2M}\int_0^t \sigma(s)e^{Ms}ds, \tag{3.3}$$

where $C_1 = \dfrac{e^{2M\pi}}{2M(e^{2M\pi}-1)} \displaystyle\int_0^{2\pi} \sigma(s)e^{-Ms}ds$,

and

$$C_2 = \frac{1}{2M(e^{2M\pi}-1)} \int_0^{2\pi} \sigma(s)e^{Ms}ds.$$

We claim that this solution $u(t)$ is unique. If not, let $v(t)$ be another solution of (3.2). Then setting $p(t) = v(t) - u(t)$, we see that

$$-p'' = -M^2p,\ p(0) = p(2\pi),\ p'(0) = p'(2\pi).$$

Hence, by Lemma 3.1, it follows that $p(t) \equiv 0$, which shows $v(t) \equiv u(t)$.

We can now define a mapping A by $A\eta = u$, for any $\eta \in [\alpha,\beta]$, where u is the unique solution of PBVP (3.2). We show that (i) $\alpha \leq A\alpha$, $\beta \geq A\beta$ and (ii) A is monotone nondecreasing on $[\alpha,\beta]$.

To prove (i), let $p = \alpha - \alpha_1$ where $\alpha_1 = A\alpha$. Then, we have in case of (B_1),

$$-p'' \leq -M^2p,\ p(0) = p(2\pi),\ p'(0) \geq p'(2\pi),$$

and in case of (B_2),

$$-p'' \leq -M^2p - M^2r,\ p(0) = p(2\pi),\ p'(0) < p'(2\pi),$$

where $r = [p'(2\pi) - p'(0)]\,\dfrac{(e^{2M\pi}+1)}{2M(e^{2M\pi}-1)}$.

Hence, by Lemma 3.1, we get $p(t) \leq 0$, implying $\alpha \leq A\alpha$. Similar arguments hold for $\beta \geq A\beta$.

To prove (ii), let $\eta_1, \eta_2 \in [\alpha, \beta]$ such that $\eta_1 \leq \eta_2$. Let $A\eta_1 = u_1$ and $A\eta_2 = u_2$. Setting $p = u_1 - u_2$ and using (B_0), we get

$$-p'' \leq -M^2 p, \; p(0) = p(2\pi), \; p'(0) = p'(2\pi).$$

Hence, by Lemma 3.1, we have $p(t) \leq 0$ and this implies that A is monotone on $[\alpha, \beta]$.

It therefore follows that we can define sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that

$$\alpha_n = A\alpha_{n-1}, \; \beta_n = A\beta_{n-1} \quad \text{with} \quad \alpha_0 = \alpha, \; \beta_0 = \beta,$$

and on $[0, 2\pi]$,

$$\alpha = \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0 = \beta.$$

Furthermore, it is easily seen that there exists a $N > 0$ which depends only on α, β such that $|\alpha_n'|$, $|\beta_n'| \leq N$ on $[0, 2\pi]$. It then follows by employing standard arguments that $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$, $\lim_{n\to\infty} \beta_n(t) = r(t)$ uniformly and monotonically on $[0, 2\pi]$. It is easy to show that $\rho(t)$ and $r(t)$ are solutions of PBVP (1.2) in view of the fact that α_n, β_n satisfy

$$-\alpha_n'' = f(t, \alpha_{n-1}) - M^2(\alpha_n - \alpha_{n-1}), \alpha_n(0) = \alpha_n(2\pi), \alpha_n'(0) = \alpha_n'(2\pi),$$
$$-\beta_n'' = f(t, \beta_{n-1}) - M^2(\beta_n - \beta_{n-1}), \beta_n(0) = \beta_n(2\pi), \beta_n'(0) = \beta_n'(2\pi)$$

and possess the integral representation similar to (3.3). It can be proved by induction argument that for any solution u of (1.2) with $\alpha \leq u \leq \beta$, we have $\alpha \leq \alpha_n \leq u \leq \beta_n \leq \beta$ on $[0, 2\pi]$ and this proves that ρ, r are extremal solutions of PBVP (1.2). The proof is therefore complete.

Remark. Unfortunately, our present method does not seem to work if f in (1.2) depends on u'. At this time, that problem remains open.

References.

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