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Remarks on Log Calabi-Yau Structure of Varieties Admitting Polarized Endomorphisms

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Abstract. We discuss the Calabi-Yau type structure of normal projective surfaces and Mori dream spaces admitting non-trivial polarized endomorphisms.

1. Introduction

Throughout this article, we work over the complex number field $\mathbb{C}$ except where otherwise stated. Recently Yuchen Zhang studied the following question in his paper [37]:

Question 1.1. Let $(X, 0)$ be a normal isolated singularity. If $\phi\colon (X, 0) \to (X, 0)$ is a noninvertible finite endomorphism, does there exist a boundary Δ such that (X, Δ) is log canonical?

The case of surfaces was studied by Wahl [33] and Favre [7]. Defining the volume of a normal surface singularity, Wahl was able to give a positive answer to the question in the surface case. This invariant was generalized by Boucksom, de Fernex and Favre [3] to isolated singularities of higher dimensional varieties. As a consequence, they obtained a positive answer to the question in the $\mathbb{Q}$-Gorenstein case. Unfortunately, as shown by Yuchen Zhang in [36], their generalization cannot be used to treat the non-$\mathbb{Q}$-Gorenstein case. Introducing a new volume refining the one introduced in [3], it is shown in [37] that Question 1.1 holds in the non-$\mathbb{Q}$-Gorenstein case for étale in codimension one endomorphisms.

In this article we study the following conjecture, which is a global version of Question 1.1:

Conjecture 1.2. *Let X be a normal projective varieties admitting a non-trivial polarized endomorphism. Then X is of Calabi-Yau type.*

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Compared with the study of Question 1.1, the study of Conjecture 1.2 is closer to the classification theory. Indeed Fujimoto and Nakayama classify smooth compact complex surfaces admitting non-trivial endomorphisms (which are not necessary polarized) [9]. For singular surfaces, however, the classification is more complicated and not complete. For higher dimension, Nakayama and Zhang study the global structure of varieties admitting non-trivial polarized endomorphisms from the viewpoint of the maximal rationally connected fibrations [24]. See also the recent paper [20]. And Zhang studies such a variety which is uniruled in more detail in [34]. For a more arithmetic point of view, see [35]. For positive characteristics, see [23]. For higher dimensional varieties, see [8, 20, 24, 30, 34].

In this article, we obtain the following partial result toward Conjecture 1.2:

Theorem 1.3. (Theorems 3.8 and 5.1) *Let X be a Mori dream space or a normal projective surface. If there exists a non-trivial polarized endomorphism on X, then there is an effective $\mathbb{Q}$-divisor Δ such that $K_X + \Delta \sim_{\mathbb{Q}} 0$ and (X, Δ) is log canonical, i.e., X is of Calabi-Yau type (see Definition 2.3).*

We combine the techniques of [15, Theorem 1.4] and [4, Theorem 1.1] for our proof of Theorem 1.3 for Mori dream spaces.

In the case of surfaces, we can check that Theorem 1.3 holds for smooth surfaces by Fujimoto-Nakayama's classification [8] and [22, Theorem 3]. But in general, since endomorphisms can not be lifted on the minimal resolution, it does not seem easy to reduce the singular case to the smooth one. Thus we need to argue more directly and discuss separated cases according to the anti-Kodaira dimension, see Section 5. We also treat Conjecture 1.2 in the étale in codimension one case, which is the global version of [37], in Section 4.

2. Preliminaries

In this section, we collect definitions and lemmas needed for the proof of Theorem 1.3.

Definition 2.1. Let X be a normal projective variety and $f\colon X \to X$ a surjective morphism. We call f *polarized* if there exists an ample Cartier divisor H and a positive $q \in \mathbb{Z}$ such that $f^*H \sim_{\mathbb{Z}} qH$. We call it *non-trivial* if f is not an isomorphism.

In this paper, we use the following standard terminologies:

Definition 2.2. (cf. [19, Definition 2.34], [31, Remark 4.2]) Let X be a normal variety and Δ be a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi\colon \widetilde{X} \to X$ be a birational morphism from a normal variety $\widetilde{X}$. Then we can write

$$K_{\widetilde{X}} = \pi^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E,$$

where E runs through all the distinct prime divisors on $\widetilde{X}$ and the $a(E,X,\Delta)$ are rational numbers. We say that the pair (X,Δ) is *sub log canonical* (for short *sub lc*) if $a(E,X,\Delta) \geq -1$ for every prime divisor E over X. In particular, we just call *log canonical* (for short *lc*) if Δ is effective and (X,Δ) is sub lc. We say that (X,Δ) is plt if $a(E,X,\Delta) > -1$ for every *exceptional* prime divisor E over X and Δ is effective. If $\Delta = 0$, we simply say that X has log canonical singularities.

Definition 2.3. (cf. [29, Lemma-Definition 2.6]) Let X be a normal projective variety and Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that (X,Δ) is *log Calabi-Yau* if $K_X + \Delta \sim_{\mathbb{Q}} 0$ and (X,Δ) is log canonical. We say that X is of *Calabi-Yau type* if there exists an effective $\mathbb{Q}$-divisor Δ on X such that (X,Δ) is log Calabi-Yau.

The following two results are used in the proof of Theorem 1.3.

Lemma 2.4. [24, Lemma 2.1] *Let X be an n-dimensional normal projective variety admitting an endomorphism f such that there exists a nef and big Cartier divisor H and positive real number q such that $f^*H \equiv qH$, i.e., f is quasi-polarized. Then* $\deg f = q^n$ *and $|\lambda| = q$ for any eigenvalue λ of $f^*\colon N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$.*

The above lemma is obtained by a nice application of the Perron-Frobenius Theorem, see the proof of [24, Lemma 2.1].

Theorem 2.5. [4, Theorem 1.1] *Let X be a normal projective varieties such that there exists a non-trivial polarized endomorphism. If X is $\mathbb{Q}$-Gorenstein, then it has log canonical singularities.*

The above theorem is proved by using Odaka-Xu's lc modifications for pairs [26, Theorem 1.1] and a careful study of lc centers in [4].

And we have the same theorem for normal surfaces.

Theorem 2.6. *Let X be a normal projective surface such that there exists a non-trivial polarized endomorphism on X. Then it has log canonical singularities. In particular, X is $\mathbb{Q}$-Gorenstein.*

Proof. Apply [4, Proposition 3.1] and [33, Theorem (a) (b) (c), p. 626]. □

Lemma 2.7. *Let S be a normal projective rational surface. If $H^i(S,\mathcal{O}_S) = 0$ for $i = 1,2$, then S has rational singularities.*

Proof. Let $f\colon W \to S$ be a resolution of singularities. By the Leray spectral sequence, the vanishing of $H^i(S,\mathcal{O}_S)$ for $i = 1,2$ implies that $H^i(S, f_*\mathcal{O}_W) = 0$ for $i = 1,2$. Then we have $H^1(W,\mathcal{O}_W) \simeq H^0(S, R^1f_*\mathcal{O}_W)$ by the following exact sequence:

$$0 \to H^1(S,\mathcal{O}_S) \to H^1(W,\mathcal{O}_W) \to H^0(S,R^1f_*\mathcal{O}_W) \to H^2(S,f_*\mathcal{O}_W) \to \cdots .$$

Thus $H^0(S, R^1 f_* \mathcal{O}_W) = 0$. Now the support of $R^1 f_* \mathcal{O}_W$ is a finite set. Thus we have $R^1 f_* \mathcal{O}_W = 0$. □

Remark 2.8. Let C be a normal projective curve admitting a separated non-trivial endomorphism. Then $(C, \frac{1}{q-1}R)$ is a Calabi-Yau pair, where R is the ramification divisor and q is degree of the endomorphism.

3. Mori dream space case

We prove in this section Conjecture 1.3 for Mori dream spaces.

Mori dream spaces were first introduced by Hu and Keel [17].

Definition 3.1. A normal projective variety X is called a $\mathbb{Q}$-*factorial Mori dream space* (or *Mori dream space* for short) if X satisfies the following three conditions:

(i) X is $\mathbb{Q}$-factorial, $\mathrm{Pic}(X)$ is finitely generated, and $\mathrm{Pic}(X)_{\mathbb{Q}} \simeq \mathrm{N}^1(X)_{\mathbb{Q}}$,

(ii) $\mathrm{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles,

(iii) there exists a finite collection of small birational maps $f_i \colon X \dashrightarrow X_i$ such that each X_i satisfies (i) and (ii), and that $\mathrm{Mov}(X)$ is the union of the $f_i^*(\mathrm{Nef}(X_i))$.

Remark 3.2. Over the complex number field, the finite generation of $\mathrm{Pic}(X)$ is equivalent to the condition $\mathrm{Pic}(X)_{\mathbb{Q}} \simeq \mathrm{N}^1(X)_{\mathbb{Q}}$.

On a Mori dream space, as its name suggests, we can run an MMP for any divisor.

Proposition 3.3. [17, Proposition 1.11] *Let X be a $\mathbb{Q}$-factorial Mori dream space. Then for any divisor D on X, a D-MMP can be run and terminates.*

Moreover the finiteness of nef cone implies the following lemmas:

Lemma 3.4. *Let X be a Mori dream space such that there exists a non-trivial polarized endomorphism f. Then there exists $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ such that $f^{m*} = \alpha \cdot \mathrm{Id}$ on $\mathrm{NS}_{\mathbb{R}}(X)$.*

Proof. Note that an endomorphism acts on the nef and pseudo effective cones. From the property of Mori dream spaces, the nef cone is rational polyhedral. Thus f^m trivially acts on the set of extremal ray of the nef cone of X for a sufficiently large and divisible m. By Lemma 2.4, $f^{m*} = \alpha \cdot \mathrm{Id}$ on the nef cone, where α is q in Lemma 2.4. The nef cone contains some basis of $\mathrm{NS}_{\mathbb{Q}}(X)$. Thus Lemma 3.4 follows. □

Lemma 3.5. *Let X be a projective scheme and H a semi-ample $\mathbb{Q}$-divisor. Suppose that there exists an endomorphism $f \colon X \to X$ such that $f^*H \sim_{\mathbb{Q}} qH$ for some $q \in \mathbb{Q}$. Let $g \colon X \to Y$ be the algebraic fibre space induced by H. Then f induces a polarized endomorphism f_Y of Y.*

Proof. Take the Stein factorization $g'\colon X \to Y'$ of $g \circ f\colon X \to Y$. Then we have an isomorphism $g \simeq g'$ since g and g' are algebraic fibre spaces and we see that the same curves are contracted by looking at the intersection numbers. Let $h\colon Y \simeq Y' \to Y$ be the induced map. We see that h gives a polarization. Indeed by definition we have a $\mathbb{Q}$-ample divisor H_Y such that $g^*H_Y \sim_{\mathbb{Q}} H$. Thus we have also $h^*H_Y \sim_{\mathbb{Q}} qH_Y$ by the projection formula. $\square$

Lemma 3.6. *Let $\pi\colon X \to S$ be an algebraic fiber space of normal projective varieties and H a Cartier divisor such that the section ring $R_\pi(X,H)$ is a finitely generated $\mathcal{O}_S$-algebra, where*

$$R_\pi(X,H) := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \pi_*\mathcal{O}_X(mH).$$

Suppose that there exists endomorphisms $f\colon X \to X$ and $f_S\colon S \to S$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ S \arrow[r, "f_S"] & S \end{tikzcd}$$

commutes, $f^*H \sim_{\mathbb{Z},\pi} qH$ for some $q \in \mathbb{Z}$, and $f_S^*H_S \sim_{\mathbb{Z},\pi} qH_S$ for some ample Cartier divisor H_S on S (i.e., f_S is polarized). Let $g\colon X \dashrightarrow Y$ be a dominant rational map to $Y = \operatorname{Proj}_S R_\pi(X,H)$. Then f induces a polarized endomorphism f_Y of Y over S such that the diagram

$$\begin{tikzcd} Y \arrow[r, "f_Y"] \arrow[d, "\pi_Y"'] & Y \arrow[d, "\pi_Y"] \\ S \arrow[r, "f_S"] & S \end{tikzcd}$$

commutes.

Proof. We have

$$f^*\colon R_\pi(X,H) \to R_\pi(X,qH).$$

Then we obtain a morphism $f_Y\colon Y \to Y$ over S satisfying the diagram

$$\begin{tikzcd} Y \arrow[r, "f_Y"] \arrow[d] & Y \arrow[d] \\ S \arrow[r, "f_S"] & S \end{tikzcd}$$

commutes. We show that f_Y is polarized. Indeed let H' be the tautological bundle of $\operatorname{Proj}_S R_\pi(X,mH)$. Then we see that $f_Y^*H' \sim_{\mathbb{Z}} qH'$. Thus $H_Y := H' + l\pi_Y^*H_S$ is ample on Y for $l \gg 0$. Since $f_Y^*H_Y \sim_{\mathbb{Z}} qH_Y$, f_Y is also polarized. $\square$

Lemma 3.7. *Let X be a Mori dream space admitting a non-trivial polarized endomorphism. Then $\kappa(-K_X) \geq 0$ if K_X is $\mathbb{Q}$-Cartier.*

Proof. By Lemma 3.4, we can see that $f^{m*}K_X \sim_{\mathbb{Q}} lK_X$ for some integer $l \geq 2$ and $m \geq 1$. On the other hand, by the ramification formula, we have

$$K_X = f^{m*}K_X + R,$$

where R is the ramification divisor, in particular, it is effective. Thus we conclude that $R \sim_{\mathbb{Q}} (1-l)K_X$. □

Finally we prove that Conjecture 1.3 holds for Mori dream spaces.

Theorem 3.8. *Let X be a Mori dream space such that there exists a non-trivial polarized endomorphism. Then X is of Calabi-Yau type.*

Proof. Running a $(-K_X)$-MMP, we have an anti-minimal model Y of X. By Lemmas 3.4, 3.5 and 3.6, Y has a non-trivial polarized endomorphism. Thus Y has log canonical singularities by Theorem 2.5 and $-K_Y$ is semi-ample by Lemma 3.7. By the arguments of [15, Theorem 1.5], X is of Calabi-Yau type. □

4. Étale in codimension one case

We see that Conjecture 1.2 is true when f is étale in codimension one and X has $\mathbb{Q}$-Gorenstein singularities. Note that [37, Theorem 1.1] implies that the following proposition holds for smooth projective varieties:

Proposition 4.1. *Let X be a normal $\mathbb{Q}$-Gorenstein projective variety such that there exists a non-trivial endomorphism with étale in codimension one such that $f^*H = qH$ for some nef and big $\mathbb{R}$-Cartier divisor and some $q \in \mathbb{R}$. Then X is log canonical and $K_X \sim_{\mathbb{Q}} 0$.*

Proof. Since f is étale in codimension one, we have

$$K_X = f^*K_X.$$

Thus we have $K_X \equiv 0$ by Lemma 2.4 and that $q > 1$. On the other hand, log-canonicity follows from Theorem 2.5. Moreover we have

$$K_X \sim_{\mathbb{Q}} 0$$

by [14, Theorem 1.2], [18, Theorem 7], or [5, Theorem 1]. □

5. Surfaces

In this section, we give a proof of Conjecture 1.2 for surfaces:

Theorem 5.1. *Let X be a normal projective surface such that there exists a non-trivial polarized endomorphism. Then X is of Calabi-Yau type.*

Before starting a proof, we discuss a classification of surfaces admitting non-trivial endomorphisms after Fujimoto and Nakayama. Actually those surfaces are classified up to regular equivalence when they are smooth.

We begin by two results about non-rational smooth surfaces.

Proposition 5.2. [35, Proposition 2.3.1] *Let $\mathbb{P}_C(E)$ be a $\mathbb{P}^1$-bundle over an elliptic curve C. Then $\mathbb{P}_C(E)$ has a non-trivial polarized endomorphism if and only if $E = \mathcal{O}_C \oplus M$ with M semi-ample or E is indecomposable of odd degree.*

Proof. See [35, Proposition 2.3.1]. □

In particular, in the above two cases, the log anti-canonical divisor with some boundary is semi-ample:

Proposition 5.3. (cf. [2, Proposition 1.6]) *Let $S = \mathbb{P}_C(E)$ be a $\mathbb{P}^1$-bundle over an elliptic curve C. If S has a non-trivial polarized endomorphism, then $-(K_S + Z)$ is semi-ample for some (possibly zero) divisor Z such that (S, Z) is plt.*

Proof. By Proposition 5.2, we see $E = \mathcal{O}_C \oplus M$ with M semi-ample or E is indecomposable of odd degree. First we show Proposition 5.3 for the indecomposable E. Then E is a semi-stable vector bundle (cf. [1], [16, V. Theorem 2.15 and Exercise 2.8]). Thus $-K_S$ is nef by [25] or [21, Theorem 3.1] and never big nor numerically trivial. Thus $-K_S$ generates the extremal ray of the nef cone which is deferent from the fibre of $S \to C$. On the other hand, the linear system $|-2K_S|$ is a pencil by the Riemann-Roch theorem. Thus S has an elliptic fibration and $-K_S$ defines this fibration. Thus $-K_S$ is semi-ample. In the case of $E = \mathcal{O}_C \oplus M$ with M ample, let Z be the negative section $Z^2 = -\deg M$. Then we can see that $-(K_S + Z)$ is nef and big, and (S, Z) is plt. Moreover $-(K_S + Z)|_Z = -K_Z$ is trivial. Thus $-(K_S + Z)$ is semi-ample by the base point free theorem [13, Theorem 1.1]. In the last part of $E = \mathcal{O}_C \oplus M$ with M torsion, take the étale morphism $C' \to C$ corresponding to M. Then by the base change of S by $C' \to C$ is isomorphic to $\mathbb{P}^1 \times C'$. Moreover $\mathbb{P}^1 \times C' \to S$ is also étale. Thus $-K_S$ is semi-ample since so is $-K_{\mathbb{P}^1 \times C'}$. □

Thus we show the following by using Fujimoto-Nakayama's classification.

Corollary 5.4. *Let S be a smooth projective surface admitting a non-trivial polarized endomorphism. Then S is of Calabi-Yau type.*

Proof. By the Bertini's theorem and [35, Proposition 2.3.1], Proposition 5.3 implies Corollary 5.4 for non-rational surfaces. On the other hand, a smooth rational projective surface admitting a non-trivial endomorphism is toric (see [22, Theorem 3]), thus a Mori dream space. □

However it seems difficult to have such a classification for singular surfaces. The main obstruction to this classification for singular surfaces seems to be the non-liftability of endomorphisms to minimal resolutions. Thus we cannot reduce the problem to smooth cases.

We start the proof of Theorem 5.1. First we show Theorem 5.1 for rational surfaces with rational singularities:

Theorem 5.5. *Let S be a normal projective rational surface with only rational singularities such that there exists a non-trivial polarized endomorphism. Then S is of Calabi-Yau type.*

Proof. Since S has rational singularities S, is $\mathbb{Q}$-factorial. Moreover we can use the following lemma due to Zhang:

Lemma 5.6. [34, Theorem 2.7] *Let S be a projective $\mathbb{Q}$-Gorenstein surface admitting an endomorphism f such that $f^*H \equiv qH$ for some $q > 1$ and a big line bundle H. If $K_S \not\equiv 0$, then there exist $m \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$ such that $f^{m*} = \alpha \cdot \mathrm{Id}$ on $\mathrm{NS}_{\mathbb{Q}}(S)$.*

Assuming $K_S \not\equiv 0$, and since the result does not depend on the multiple of f considered, we may assume that $f^* = q \cdot \mathrm{Id}$ on $\mathrm{NS}_{\mathbb{Q}}(S)$. By the same proof as in Lemma 3.7, we have $\kappa(-K_S) \geq 0$. We argue case by case according to the value of $\kappa(-K_S)$.

Case 1. When $\kappa(-K_S) = 2$.

In this case, $-K_S$ is big. Take the minimal resolution $\pi\colon T \to S$. We have

$$\pi^*K_S = K_T + E,$$

where E is a π-exceptional divisor. Thus $-K_T$ is big. By [17, 1.11 Proposition] (cf. [27] for a more general statement) and [32], S is a Mori dream space. Thus S is of Calabi-Yau type by Theorem 3.8.

Case 2. When $\kappa(-K_S) \leq 1$.

Take the Zariski decomposition $-K_S \sim_{\mathbb{Q}} P + N$. Now P is nef and $\kappa(P) = \nu(P) = 1$ or $P \sim_{\mathbb{Q}} 0$. In particular P is semi-ample. By Lemma 5.6 and $\kappa(N) = 0$, $f^*N = qN$. Thus after replacing f with a power of f, we may assume that every component of N is a totally invariant divisor of f. Note that $N \leq \frac{1}{q-1}R$. Let N_{red} be the reduced sum of prime divisors in N. Then (S, N_{red}) is log canonical by [4, Corollary 3.3]. Since $N \leq \frac{1}{q-1}R$ and the components are total invariant, we have $N \leq N_{\mathrm{red}}$.

Therefore we see that a pair (X, N) is log canonical. Moreover since P is semi-ample, we see that S is of Calabi-Yau type. □

Finally we treat the cases of singular non-rational surfaces and singularities and give a proof of Theorem 5.1.

Proof of Theorem 5.1. By Theorem 2.6, K_S is $\mathbb{Q}$-Cartier. We may assume that K_S is not pseudo-effective by Lemma 5.6. Indeed by the same proof as in Lemma 3.7, we have $\kappa(-K_S) \geq 0$. Run an MMP for S by [12, Theorem 1.1]. Thus we have

$$S \to S' \to C,$$

where $\varphi\colon S \to S'$ is a composition of divisorial contractions and $\pi\colon S' \to C$ is a Mori fiber space. By Lemma 5.6, we may assume that this is f-equivariant, i.e., there exist polarized endomorphisms $f'\colon S' \to S'$ and $g\colon C \to C$ commuting with φ and π. Let R be the ramification divisor of f. Then the strict transformation R' of R on S' is the ramification divisor of R'. And by Lemma 5.6, we see that

$$\varphi^*\left(K_{S'} + \frac{1}{q-1}R'\right) = K_S + \frac{1}{q-1}R \sim_{\mathbb{Q}} 0.$$

Case 3. When $\dim C = 1$.

In this case, C is a smooth rational curve or an elliptic curve. In the case of rational curves, S is a rational surface. And also we have $R^i\pi_*\mathcal{O}_{S'} = 0$ for $i > 0$ by relative Kodaira vanishing theorem for log canonical surfaces (cf. [11, Theorem 8.1]). Now since C is smooth and rational, we have $H^i(S', \mathcal{O}_{S'}) = 0$ for $i = 1, 2$ by the Leray spectral sequence. We have $H^i(S, \mathcal{O}_S) = 0$ for $i = 1, 2$ by again using the Leray spectral sequence. Then S has rational singularities by Lemma 2.7. On the other hand, in the case of elliptic curves, we apply the canonical bundle formulas for a pair $(S', \frac{1}{q-1}R')$, after replacing f by its positive power. Note that $(S', \frac{1}{q-1}R')$ is log canonical over the generic point of π by using [6, Theorem 5.1]. Indeed by [6, Theorem 5.1] we can find some general point $y \in C$ such that $F_y := \pi^*y$ is reduced and smooth, and f induces an endomorphism $f_y\colon F_y \to F_y$. We get by adjunction formula

$$K_{F_y} = f_y^*(K_{F_y}) + R'_{|F_y}$$

as the ramification formula for f_y. Thus the coefficients of $\frac{1}{q-1}R'_{|F_y}$ are less than or equal to one by Remark 2.8. Thus $(S', \frac{1}{q-1}R')$ is log canonical over the generic point of π.

Let

$$B = \sum_{P:\text{Weil divisor on } Y} (1 - t_P)P,$$

where

$$t_P := \sup\left\{t \in \mathbb{R} \mid (X, \Delta + t\pi^*P) \text{ is sub lc over } P\right\}.$$

Then by [10, Theorem 4.1.1 in the arXiv version] we have a pseudo-effective divisor M such that

$$K_{S'} + \frac{1}{q-1}R' = \pi^*(K_C + B + M).$$

Note that B is effective. Now since $K_{S'}+\frac{1}{q-1}R' \equiv 0$ and $K_C \sim 0$ we have $B = 0$ and $M \equiv 0$. So we see that $(S', \frac{1}{q-1}R')$ is log canonical by the definition of B and $K_S' + \frac{1}{q-1}R' \equiv 0$. Thus we also see $(S, \frac{1}{q-1}R)$ is log Calabi-Yau pair.

Case 4. When $\dim C = 0$.

When S is not rational, we see that S' is isomorphic to the cone of an elliptic curve E by [28, Corollary 5.4.4]. And also we may assume that $f^{-1}O = O$, where O is the vertex of this cone by [4, Lemma 2.10].

This induces a polarized endomorphism of the elliptic curve E by the fact that any dominant rational map of varieties induces a morphism between their Albanese varieties (cf. [20, Corollary 1.4]). If φ is not an isomorphism, then there exists a point $P \neq O \in S'$ such that $f^{-1}P = P$. Thus the endomorphism of E is ramified. But this is a contradiction. Thus φ is an isomorphism. Thus S is isomorphic to the cone of an elliptic curve. In particular, it is of Calabi-Yau type.

In the case of a rational surface S, the surface S' is lc and $-K_{S'}$ is ample. Then $H^i(S', \mathcal{O}_{S'}) = 0$ for $i = 1, 2$ (cf. [11, Theorem 8.1]). By the Leray spectral sequence, $H^i(S, \mathcal{O}_S) = 0$ for $i = 1, 2$. Then S has rational singularities by Lemma 2.7. We already have shown Theorem 5.1 in the case of rational surfaces with rational singularities in Theorem 5.5. $\square$

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