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Remarks on n-normal Operators

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Abstract. Let T be a bounded linear operator on a complex Hilbert space and $n, m \in \mathbb{N}$. Then T is said to be *n-normal* if $T^*T^n = T^nT^*$ and *(n,m)-normal* if $T^{*m}T^n = T^nT^{*m}$. In this paper, we study several properties of n-normal, (n,m)-normal operators. In particular, we prove that if T is 2-normal with $\sigma(T) \bigcap (-\sigma(T)) \subset \{0\}$, then T is polaroid. Moreover, we study subscalarity of n-normal operators. Also, we prove that if T is (n,m)-normal, then T is decomposable and Weyl's theorem holds for $f(T)$, where f is an analytic function on $\sigma(T)$ which is not constant on each of the components of its domain.

1. Introduction and Motivation

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle\ ,\ \rangle$ and $\mathcal{L}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *subnormal* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subset \mathcal{H}$ and $T = N|_{\mathcal{H}}$, *hyponormal* if $T^*T - TT^* \geq 0$. An operator T is said to be *scalar of order m* if it admits a spectral distribution of order m, i.e., if there is a continuous unital morphism $\Phi : C_0^m(\mathbb{C}) \longrightarrow \mathcal{L}(\mathcal{H})$ such that $\Phi(z) = T$, where z stands for the identity function on $\mathbb{C}$ and $C_0^m(\mathbb{C})$ for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m $(0 \leq m \leq \infty)$. An operator T is said to be *subscalar of order m* if it is similar to the restriction of a scalar operator of order m to an invariant subspace. It is known that subnormal operators are hyponormal and hyponormal operators are subscalar ([8]).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T - \lambda)f(\lambda) \equiv 0$ on G, we have $f(\lambda) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for $x \in \mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \longrightarrow \mathcal{H}$ such that $(T - \lambda)f(\lambda) \equiv x$ on G. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspace* of an operator $T \in \mathcal{L}(\mathcal{H})$ by $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$.

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For an operator $T \in \mathcal{L}(\mathcal{H})$, the quasinilpotent part of $T - \lambda$ is defined as

$$H_0(T-\lambda) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\}.$$

In general, $\ker((T-\lambda)^m) \subset H_0(T-\lambda)$ and $H_0(T-\lambda)$ is not closed. However, if λ is an isolated point of $\sigma(T)$, then $E_T(\{\lambda\})\mathcal{H} = H_0(T-\lambda)$ where

$$E_T(\{\lambda\}) = \frac{1}{2\pi i}\int_{\partial D}(z-T)^{-1}\,dz$$

denotes the Riesz idempotent corresponding to λ with D is a closed disk centered at λ which contains no other points of $\sigma(T)$. Hence $H_0(T-\lambda)$ is closed in this case.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is well known that Property (β) $\Longrightarrow$ Dunford's property (C) $\Longrightarrow$ SVEP, and the converse implications do not hold ([7, Proposision 1.2.19]). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{H} = \mathcal{X} + \mathcal{Y}$, $\sigma(T|_{\mathcal{X}}) \subset \overline{U}$ and $\sigma(T|_{\mathcal{Y}}) \subset \overline{V}$. Remark that T is decomposable if and only if T and T^* have the property (β) ([7, Theorem 2.5.19]).

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *isoloid* if every isolated point of $\sigma(T)$ belongs to the point spectrum of T. Hence, hyponormal operators are isoloid ([6, Theorem 2]). Of course, there are many classes of operators weaker than hyponormal which are isoloid. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *polaroid* if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be quasinilpotent if $\sigma(T) = \{0\}$.

In [3], S. A. Alzuraiqi and A. B. Patel introduced n-normal operators.

Definition 1.1. *For $n \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be n-normal if*

$$T^*T^n = T^nT^* \quad . \tag{1}$$

This definition seems natural. S. A. Alzraiqi and A. B. Patel proved characterizations of 2-normal, 3-normal and n-normal operators on $\mathbb{C}^2$. Also, they made several examples of n-normal operators and proved that T is n-normal if and only if T^n is normal. Also, they proved that if T is 2-normal with the following condition

$$\sigma(T) \bigcap (-\sigma(T)) = \emptyset, \tag{2}$$

then T is subscalar. If an operator $T \in \mathcal{L}(\mathcal{H})$ satisfies (2), then T is invertible automatically. Recently, the authors in [4] have studied spectral properties of an n-normal operator T satisfying the following condition (3).

$$\sigma(T) \bigcap (-\sigma(T)) \subset \{0\}. \tag{3}$$

It is a little weaker assumption than this condition (2). We define (n, m)-normality as follows.

Definition 1.2. *For $n, m \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be (n, m)-normal if*

$$T^{*m}T^n = T^nT^{*m}.$$

In this paper, we study several properties of n-normal or (m, n)-normal operators. In particular, we prove that if T is 2-normal with (3), then T is polarloid. We study subscalarity of n-normal operators. Moreover, we show that if T is (n, m)-normal, then T is decomposable and Weyl's theorem holds for $f(T)$, where f is an analytic function on $\sigma(T)$ which is not constant on each of the components of its domain.

The following proposition is important in this paper.

Proposition 1.3. ([3, Proposition 2.2]) *Let $T \in \mathcal{L}(\mathcal{H})$ and $n \in \mathbb{N}$. Then T is n-normal if and only if T^n is normal.*

Therefore, we have the following result.

Theorem 1.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be n-normal. Then T has the single-valued extension property.*

Proof. Since T^n is normal, it follows that T^n has the single-valued extension property. Hence T has the single-valued extension property by [1, Theorem 2.40]. □

2. 2-normal Operators

In this section, we study some properties of 2-normal operators. Let M be a subspace of $\mathcal{H}$. Then M is said to be a *reducing subspace* for T if $T(M) \subset M$ and $T^*(M) \subset M$, that is, M is an invariant subspace for T and T^*.

Theorem 2.1. ([4]) *Let $T \in \mathcal{L}(\mathcal{H})$ be* 2*-normal and satisfy* (3). *Then the following statements hold.*
(i) *T is isoloid and $\sigma(T) = \sigma_a(T)$.*
(ii) *If z and w are distinct eigen-values of T and $x, y \in \mathcal{H}$ are corresponding eigen-vectors, respectively, then $\langle x, y\rangle = 0$.*
(iii) *If z, w are distinct values of $\sigma_a(T)$ and $\{x_n\}$, $\{y_n\}$ are the sequences of unit vectors in $\mathcal{H}$ such that $(T - z)x_n \to 0$ and $(T - w)y_n \to 0$ $(n \to \infty)$, then $\lim_{n\to\infty} \langle x_n, y_n\rangle = 0$.*
(iv) *If z and w are distinct eigen-values of T, then* $\ker(T - z) \perp \ker(T - w)$.
(v) *If z is a non-zero eigen-value of T, then* $\ker(T - z) = \ker(T^2 - z^2) = \ker(T^{*2} - \overline{z}^2) = \ker(T^* - \overline{z})$ *and hence* $\ker(T - z)$ *is a reducing subspace for T.*

In 2012, J. T. Yuan and G. X. Ji ([10, Lemma 5.2]) proved the following Lemma.

Lemma 2.2. *Let m be a positive integer, λ be an isolated point of $\sigma(T)$ and $E = E_T(\{\lambda\})$.*
(i) *Then the following assertions are equivalent.*
 (a) $E\mathcal{H} = \ker((T - \lambda)^m)$.
 (b) $\ker(E) = (T - \lambda)^m\mathcal{H}$.
 Hence λ is a pole of the resolvent of T and the order of λ is not greater than m.
(ii) *If λ is a pole of the resolvent of T and the order of λ is m, then the following assertions are equivalent:*
 (a) *E is self-adjoint.*
 (b) $\ker((T - \lambda)^m) \subset \ker((T - \lambda)^{*m})$.
 (c) $\ker((T - \lambda)^m) = \ker((T - \lambda)^{*m})$.

Next we show that if T is 2-normal and satisfies (2), then T is polaroid.

Theorem 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be* 2*-normal and satisfy* (3)*, and let λ be an isolated point of spectrum of T. Then λ is a pole of the resolvent, that is, T is polaroid and the following statements hold.*
(i) *If $\lambda = 0$, then* $H_0(T) = \ker(T^2) = \ker(T^{*2})$, *$E_T(\{0\})$ is self-adjoint and the order of* 0 *is not greater than* 2.
(ii) *If $\lambda \neq 0$, then* $H_0(T - \lambda) = \ker(T - \lambda) = \ker((T - \lambda)^*)$, *$E_T(\{\lambda\})$ is self-adjoint and the order of λ is* 1.

Proof. Let λ be an isolated point of spectrum of T.

(i) Assume that $\lambda = 0$. Since $\sigma(T^2) = \{z^2 : z \in \sigma(T)\}$, it follows that 0 is an isolated point of spectrum of T^2. We want to prove that $H_0(T) = H_0(T^2)$. Let $x \in H_0(T)$. Then $\|T^n x\|^{\frac{1}{n}} \longrightarrow 0$ and hence $\|T^{2n}x\|^{\frac{1}{2n}} = \left(\|T^{2n}x\|^{\frac{1}{n}}\right)^{\frac{1}{2}} \longrightarrow 0$ and $\|T^{2n}x\|^{\frac{1}{n}} \longrightarrow 0$. Hence $x \in H_0(T^2)$. Conversely, let $x \in H_0(T^2)$. Then $\|T^{2n}x\|^{\frac{1}{n}} \longrightarrow 0$ and so $\|T^{2n}x\|^{\frac{1}{2n}} = \left(\|T^{2n}x\|^{\frac{1}{n}}\right)^{\frac{1}{2}} \longrightarrow 0$. Since

$$\|T^{2n+1}x\|^{\frac{1}{2n+1}} \le \left(\|T\|\|T^{2n}x\|\right)^{\frac{1}{2n+1}} \le \|T\|^{\frac{1}{2n+1}} \left(\|T^{2n}x\|^{\frac{1}{2n}}\right)^{\frac{2n}{2n+1}} \longrightarrow 0 \ \ (n \to \infty),$$

it follows that $x \in H_0(T)$. Hence $H_0(T) = H_0(T^2)$.

Let $x \in E_T(\{0\}) = H_0(T) = H_0(T^2) = E_{T^2}(\{0\})$. Since T^2 is normal, it follows that $E_{T^2}(\{0\}) = \ker(T^2) = \ker(T^{*2})$. Hence $x \in \ker(T^2)$ and $E_T(\{0\}) \subset \ker(T^2)$. Therefore $E_T(\{0\}) = \ker(T^2) = \ker(T^{*2})$ and 0 is a pole of the resolvent of T and the order of 0 is not greater than 2 by Lemma 2.2.

(ii) Next we assume $\lambda \neq 0$. Then λ^2 is an isolated point of $\sigma(T^2)$ by [4, Lemma 2.1]. We will prove $H_0(T-\lambda) = H_0(T^2-\lambda^2)$. Let $x \in H_0(T-\lambda)$. Then $\|(T-\lambda)^n x\|^{\frac{1}{n}} \to 0$. Therefore we have

$$\|(T^2-\lambda^2)^n x\|^{\frac{1}{n}} \leq \|(T+\lambda)^n\|^{\frac{1}{n}} \|(T-\lambda)^n x\|^{\frac{1}{n}} \leq \|T+\lambda\| \|(T-\lambda)^n x\|^{\frac{1}{n}} \longrightarrow 0.$$

Hence $H_0(T-\lambda) \subset H_0(T^2-\lambda^2)$. Conversely, let $x \in H_0(T^2-\lambda^2)$. Since $T+\lambda$ is invertible by (3), we have

$$\begin{aligned} \|(T-\lambda)^n x\|^{\frac{1}{n}} &= \|(T+\lambda)^{-n}(T+\lambda)^n (T-\lambda)^n x\|^{\frac{1}{n}} \\ &\leq \| \left\{(T+\lambda)^{-1}\right\}^n \|^{\frac{1}{n}} \|(T^2-\lambda^2)^n x\|^{\frac{1}{n}} \leq \|(T+\lambda)^{-1}\| \|(T^2-\lambda^2)^n x\|^{\frac{1}{n}} \longrightarrow 0. \end{aligned}$$

Hence $H_0(T-\lambda) \supset H_0(T^2-\lambda^2)$ and $H_0(T-\lambda) = H_0(T^2-\lambda^2)$.

Let $x \in E_T(\{\lambda\})\mathcal{H}$. Since $E_T(\{\lambda\})\mathcal{H} = H_0(T-\lambda) = H_0(T^2-\lambda^2)$ and T^2 is normal, we have $H_0(T^2-\lambda^2) = E_{T^2}(\{\lambda^2\}) = \ker(T^2-\lambda^2)$. Hence $0 = (T^2-\lambda^2)x = (T+\lambda)(T-\lambda)x$. Since $T+\lambda$ is invertible by (3), we have $(T-\lambda)x = 0$. Hence $E_T(\{\lambda\})\mathcal{H} \subset \ker(T-\lambda)$ and $E_T(\{\lambda\})\mathcal{H} = \ker(T-\lambda)$. Hence λ is a pole of the resolvent of T and the order of λ is 1 by Lemma 2.2. Since $\ker(T-\lambda) = \ker((T-\lambda)^*)$ by [4, Theorem 2.6], it follows that $E_T(\{\lambda\})$ is self-adjoint by Lemma 2.2. □

Let D be a bounded open subset of $\mathbb{C}$ and $L^2(D,\mathcal{H})$ be the Hilbert space of measurable function $f : D \longrightarrow \mathcal{H}$ such that

$$\|f\|_{2,D} = \left(\int_D \|f(z)\|^2 \, d\mu(z) \right)^{\frac{1}{2}} < \infty,$$

where $d\mu$ is the planar Lebesgue measure. Let $W^2(D,\mathcal{H})$ be the Sobolev space with respect to $\overline{\partial}$ and of order 2 whose derivatives $\overline{\partial} f$ and $\overline{\partial}^2 f$ in the sense of distributions belong to $L^2(D,\mathcal{H})$. The norm $\|f\|_{W^2}$ is given by

$$\|f\|_{W^2} = \left(\|f\|_{2,D}^2 + \|\overline{\partial} f\|_{2,D}^2 + \|\overline{\partial}^2 f\|_{2,D}^2 \right)^{\frac{1}{2}} \quad \text{for } f \in W^2(D,\mathcal{H}).$$

Then in [3], S. A. Alzuraiqi and A. B. Patel proved the following result.

Proposition 2.4. ([3, Theorem 2.37]) *Let D be an arbitrary bounded disk in $\mathbb{C}$. If $T \in B(\mathcal{H})$ is 2-normal with* (1), *that is, $\sigma(T) \bigcap (-\sigma(T)) = \emptyset$, then the operator*

$$S = z - T : W^2(D,\mathcal{H}) \longrightarrow W^2(D,\mathcal{H})$$

is one to one.

We will revise this result as follows.

Theorem 2.5. *Let D be an arbitrary bounded open disk in $\mathbb{C}$. If $T \in B(\mathcal{H})$ is 2-normal and the planar Lebesgue measure of $\sigma(T) \bigcap (-\sigma(T))$ is 0, then the operator*

$$S = z - T : W^2(D,\mathcal{H}) \longrightarrow W^2(D,\mathcal{H})$$

is one to one.

Proof. Let $f \in W^2(D, \mathcal{H})$ and $Sf = 0$. We show $f = 0$. Then

$$\begin{aligned}\|f\|_{W^2}^2 &= \|f\|_{2,D}^2 + \|\overline{\partial} f\|_{2,D}^2 + \|\overline{\partial}^2 f\|_{2,D}^2 \\ &= \int_D \|f(z)\|^2 d\mu(z) + \int_D \|\overline{\partial} f(z)\|^2 d\mu(z) + \int_D \|\overline{\partial}^2 f(z)\|^2 d\mu(z) < \infty,\end{aligned}$$

and

$$\begin{aligned}\|Sf\|_{W^2}^2 = \|(z-T)f\|_{W^2}^2 &= \|(z-T)f\|_{2,D}^2 + \|\overline{\partial}((z-T)f)\|_{2,D}^2 + \|\overline{\partial}^2((z-T)f)\|_{2,D}^2 \\ &= \|(z-T)f\|_{2,D}^2 + \|(z-T)\overline{\partial} f\|_{2,D}^2 + \|(z-T)\overline{\partial}^2 f\|_{2,D}^2 = 0.\end{aligned}$$

Hence

$$\|(z-T)\,\overline{\partial}^i f\|_{2,D}^2 = \int_D \|(z-T)\,\overline{\partial}^i f(z)\|^2 d\mu(z) = 0 \ \ (i = 0, 1, 2).$$

Let i be $i = 0, 1, 2$. Since $(z-T)\,\overline{\partial}^i f(z) = 0$ for $z \in D$, if $z \in D \setminus \sigma(T)$, then $\overline{\partial}^i f(z) = 0$ because $z - T$ is invertible. This implies

$$\|(z-T)^* \overline{\partial}^i f\|_{2,D\setminus\sigma(T)}^2 = \int_{D\setminus\sigma(T)} \|(z-T)^* \overline{\partial}^i f(z)\|^2 d\mu(z) = 0.$$

Since

$$\begin{aligned}\|(z^2 - T^2)\overline{\partial}^i f\|_{2,D}^2 &= \int_D \|(z^2 - T^2)\overline{\partial}^i f(z)\|^2 d\mu(z) \\ &\le \left(\sup_{z\in D} \|z + T\|\right)^2 \int_D \|(z-T)\overline{\partial}^i f(z)\|^2 d\mu(z) = \left(\sup_{z\in D} \|z + T\|\right)^2 \|(z-T)\overline{\partial}^i f\|_{2,D}^2 = 0,\end{aligned}$$

we have $(z^2 - T^2)\,\overline{\partial}^i f(z) = 0$ for $z \in D$. Moreover, since T^2 is normal, this implies

$$\|(z^2 - T^2)^* \overline{\partial}^i f\|_{2,D}^2 = \int_D \|(z^2 - T^2)^* \overline{\partial}^i f(z)\|^2 d\mu(z) = 0.$$

Hence

$$0 = (z^2 - T^2)^* \overline{\partial}^i f(z) = (z+T)^*(z-T)^* \overline{\partial}^i f(z) \ \text{ for } z \in D.$$

If $z \in D \cap (\sigma(T) \setminus (-\sigma(T)))$, then $z + T$ and $(z+T)^*$ are invertible. Hence we obtain $(z-T)^* \overline{\partial}^i f(z) = 0$ for $z \in D \cap (\sigma(T) \setminus (-\sigma(T)))$. Since D is bounded, $\|\overline{\partial}^i f\|_{2,D}^2 < \infty$ and the planar Lebesgue measure of $\sigma(T) \bigcap (-\sigma(T))$ is 0, we have

$$\begin{aligned}\|(z-T)^* \overline{\partial}^i f\|_{2,D}^2 &= \int_{D\setminus\sigma(T)} \|(z-T)^* \overline{\partial}^i f(z)\|^2 d\mu(z) \\ &\quad + \int_{D\cap(\sigma(T)\setminus(-\sigma(T)))} \|(z-T)^* \overline{\partial}^i f(z)\|^2 d\mu(z) + \int_{D\cap\sigma(T)\cap(-\sigma(T))} \|(z-T)^* \overline{\partial}^i f(z)\|^2 d\mu(z) \\ &\le 0 + 0 + \max_{z\in D} \|(z-T)^*\|^2 \int_{D\cap\sigma(T)\cap(-\sigma(T))} \|\overline{\partial}^i f(z)\|^2 d\mu(z) = 0.\end{aligned}$$

By [8, Proposition 2.1], we obtain $\|(I - P)f\|_{2,D} = 0$. Hence $f(z) = (Pf)(z)$ for $z \in D$. Since $Sf = 0$, we have $(Sf)(z) = (z-T)f(z) = (z-T)(Pf)(z) = 0$ for $z \in D$. Since T has the single-valued extension property by Theorem 1.4 and Pf is analytic, it follows that $0 = (Pf)(z) = f(z)$ for $z \in D$. Hence $f = 0$ and S is one to one. □

3. *n*-normal Operators

Theorem 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be n-normal. Let $\sigma(T)$ be contained in an angle $< 2\pi/n$ with vertex in the origin, i.e., there exists $\theta_1 \in [0, 2\pi)$ such that*

$$\sigma(T) \subset W = \left\{ re^{i\theta} : 0 < r, \theta_1 < \theta < \theta_1 + \frac{2\pi}{n} \right\}.$$

Then T is subscalar of order 2.

Proof. Let D be an open bounded disk such that $\sigma(T) \subset D$. Take an open set U such that $\sigma(T) \subset U \subset \overline{U} \subset D \cap W$. Let $M : W^2(D, \mathcal{H}) \to W^2(D, \mathcal{H})$ be a multiplication operator such that $(Mf)(z) = zf(z)$ for $f \in W^2(D, \mathcal{H})$ and $z \in D$. Then M is scalar of order 2 with a spectral distribution defined by $\Phi(\phi)f = \phi f$ for $\phi \in C_0^2(\mathbb{C})$ and $f \in W^2(D, \mathcal{H})$. Since $(z - T)W^2(D, \mathcal{H})$ is M-invariant, it follows that $S : \mathcal{H}(D) = W^2(D, \mathcal{H})/\overline{(z - T)W^2(D, \mathcal{H})} \to \mathcal{H}(D)$ as

$$S\left(f + \overline{(z - T)W^2(D, \mathcal{H})}\right) \to Mf + \overline{(z - T)W^2(D, \mathcal{H})}$$

for $f \in W^2(D, \mathcal{H})$ is well defined and S is still scalar of order 2 with a spectral distribution

$$\tilde{\Phi}(\phi)\left(f + \overline{(z - T)W^2(D, \mathcal{H})}\right) = \phi f + \overline{(z - T)W^2(D, \mathcal{H})}$$

for $\phi \in C_0^2(\mathbb{C})$ and $f + \overline{(z - T)W^2(D, \mathcal{H})} \in \mathcal{H}(D)$. Let $V : \mathcal{H} \to \mathcal{H}(D)$ be as

$$Vh = 1 \otimes h + \overline{(z - T)W^2(D, \mathcal{H})}$$

for $h \in \mathcal{H}$ where $(1 \otimes h)(z) = h$ for $z \in D$. Then

$$VT = SV.$$

We prove that V is one to one and has dense range. Then $V\mathcal{H}$ is an invariant subspace of S and $T = S|_{V\mathcal{H}}$. Hence T is subscalar of order 2.

Claim. If $Vh_n \to 0$, then $h_n \to 0$.

Let $Vh_n \to 0$. Then there exists $f_n \in W^2(D, \mathcal{H})$ such that

$$\|(z - T)f_n + 1 \otimes h_n\|_{W^2}^2 = \|(z - T)f_n + 1 \otimes h_n\|_{2,D}^2 + \|(z - T)\overline{\partial} f\|_{2,D}^2 + \|(z - T)\overline{\partial}^2 f\|_{2,D}^2 \to 0.$$

Let $\zeta = \exp(2\pi i/n)$. Then

$$\begin{aligned}\|(z^n - T^n)\overline{\partial}^i f\|_{2,D}^2 &= \int_D \|(z^n - T^n)\overline{\partial}^i f_n(z)\|^2 d\mu(z) = \int_D \left\| \prod_{k=1}^{n} (\zeta^k z - T)\overline{\partial}^i f_n(z) \right\|^2 d\mu(z) \\ &\le \sup_{z \in D} \left\| \prod_{k=1}^{n-1} (\zeta^k z - T) \right\|^2 \int_D \|(z - T)\overline{\partial}^i f_n(z)\|^2 d\mu(z) \to 0.\end{aligned}$$

Since T^n is normal, we have

$$\|(z^n - T^n)^* \overline{\partial}^i f\|_{2,D}^2 \to 0.$$

If $z \in \overline{U}$, then $\zeta^k z \notin \sigma(T)$ for $k = 1, 2, \cdots, n-1$ by the assumption. Hence

$$\begin{aligned}\|(z-T)^*\overline{\partial}^i f_n\|^2_{2,\overline{U}} &= \int_{\overline{U}} \|(z-T)^*\overline{\partial}^i f_n(z)\|^2 d\mu(z)\\ &= \int_{\overline{U}} \prod_{k=1}^{n-1}\left((\zeta^k z - T)^{-1}\right)^*\left(\prod_{k=1}^{n-1}(\zeta^k z - T)^*\right)(z-T)^*\overline{\partial}^i f_n(z) d\mu(z)\\ &\le \prod_{k=1}^{n-1}\sup_{z\in\overline{U}}\left\|\left((\zeta^k z - T)^{-1}\right)^*\right\|^2 \ \left\|(z^n - T^n)^*\overline{\partial}^i f_n\right\|^2_{2,D} \to 0.\end{aligned}$$

Since

$$\begin{aligned}\int_{D\setminus\overline{U}} \|\overline{\partial}^i f_n(z)\|^2 d\mu(z) &= \int_{D\setminus\overline{U}} \|(z-T)^{-1}(z-T)\overline{\partial}^i f_n(z)\|^2 d\mu(z)\\ &\le \sup_{z\in D\setminus\overline{U}} \|(z-T)^{-1}\|^2 \int_{D\setminus\overline{U}} \|(z-T)\overline{\partial}^i f_n(z)\|^2 d\mu(z)\\ &\le \sup_{z\in D\setminus\overline{U}} \|(z-T)^{-1}\|^2 \|(z-T)\overline{\partial}^i f_n\|^2_{2,D} \to 0,\end{aligned}$$

we have

$$\begin{aligned}\|(z-T)^*\overline{\partial}^i f_n\|^2_{2,D} &= \int_{D\setminus\overline{U}} \|(z-T)^*\overline{\partial}^i f_n(z)\|^2 d\mu(z) + \int_{\overline{U}} \|(z-T)^*\overline{\partial}^i f_n(z)\|^2 d\mu(z)\\ &\le \sup_{z\in D\setminus\overline{U}} \|(z-T)^*\|^2 \int_{D\setminus\overline{U}} \|\overline{\partial}^i f_n(z)\|^2 d\mu(z) + \|(z-T)^*\overline{\partial}^i f_n\|^2_{2,\overline{U}} \to 0.\end{aligned}$$

Let P be the orthogonal projection of $L^2(D, \mathcal{H})$ onto $A^2(D, \mathcal{H})$. Then there exists a constant $0 < C_D$ such that

$$\|(1-P)f_n\|_{2,D} \le C_D\left(\|(z-T)^*\overline{\partial} f_n\|_{2,D} + \|(z-T)^*\overline{\partial}^2 f_n\|_{2,D}\right) \to 0$$

by Proposition 2.1 of [8]. Hence

$$\begin{aligned}\|(z-T)Pf_n + 1\otimes h_n\|_{2,D} &\le \|(z-T)f_n + 1\otimes h_n\|_{2,D} + \|(z-T)(1-P)f_n\|_{2,D}\\ &\le \|(z-T)f_n + 1\otimes h_n\|_{2,D} + \sup_{z\in D}\|z-T\|\|(1-P)f_n\|_{2,D} \to 0.\end{aligned}$$

Hence

$$\|(z-T)Pf_n + 1\otimes h_n\|_{\infty,U} = \sup_{z\in U}\|(z-T)Pf_n(z) + h_n\| \to 0$$

by [8, Lemma 1.1]. Define $\Psi : A^2(U, \mathcal{H}) \to \mathcal{H}$ as

$$\Psi(g) = \frac{1}{2\pi i}\int_{\partial G}(z-T)^{-1}g(z)dz$$

for $g \in A^2(U, \mathcal{H})$ where G is an open set such that $\sigma(T) \subset G \subset \overline{G} \subset U$ and ∂G is a Jordan curve. Since

$$\|\Psi(g)\| \le \frac{1}{2\pi}\max_{z\in\partial G}\|(z-T)^{-1}\|\|g\|_{\infty,U}\ell(\partial G)$$

for $g \in A^2(U, \mathcal{H})$ where $\ell(\partial G)$ denotes the length of ∂G and

$$(z-T)Pf_n + 1\otimes h_n \in A^2(U, \mathcal{H}),$$

we have

$$\begin{aligned}\Psi((z-T)Pf_n + 1\otimes h_n) &= \frac{1}{2\pi i}\int_{\partial G}(z-T)^{-1}\left((z-T)Pf_n(z)+h_n\right)dz\\ &= \frac{1}{2\pi i}\int_{\partial G}\left(Pf_n(z)+(z-T)^{-1}h_n\right)dz = 0 + h_n \to 0.\end{aligned}$$

□

Corollary 3.2. *Under the same hypothesis as in Theorem 3.1, if $\sigma(T)$ has nonempty interior, then T has a nontrivial invariant subspace.*

Proof. By the hypothesis, T is subscalar of order 2 from Theorem 3.1. Since $\sigma(T)$ has nonempty interior, we get this result from [5, Theorem 2.1]. □

In [9], C.R. Putnam proved that if T is hyponormal, then

$$\pi\|T^*T - TT^*\| \le m(\sigma(T))$$

where m is the Lebesque measure in the complex plane. This is well known as Putnam's inequality.

Lemma 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be n-normal and let $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace for T. Then the following assertions hold.*

(i) $(T|_{\mathcal{M}})^n$ *is subnormal.*

(ii) *Let $\sigma(T|_{\mathcal{M}}) = \{\lambda\}$. Then $T|_{\mathcal{M}} = \lambda$ if $\lambda \neq 0$ and $(T|_{\mathcal{M}})^n = 0$ if $\lambda = 0$.*

Proof. (i) Since $(T|_{\mathcal{M}})^n = T^n|_{\mathcal{M}}$ and T^n is normal, $(T|_{\mathcal{M}})^n$ is subnormal.

(ii) Suppose $\lambda = 0$. Then $(T|_{\mathcal{M}})^n$ is subnormal by (1) and

$$\sigma\left((T|_{\mathcal{M}})^n\right) = \{z^n | z \in \sigma(T|_{\mathcal{M}})\} = \{0\}.$$

It follows that $(T|_{\mathcal{M}})^n = 0$ by Putnam's inequality.

Suppose $\lambda \neq 0$. Then $(T|_{\mathcal{M}})^n$ is subnormal and $\sigma\left((T|_{\mathcal{M}})^n\right) = \{\lambda^n\}$. It follows that $(T|_{\mathcal{M}})^n = \lambda^n$ by Putnam's inequality. Since $\sigma(T|_{\mathcal{M}}) = \{\lambda\}$ and

$$0 = (T|_{\mathcal{M}})^n - \lambda^n = \left(\prod_{k=1}^{n-1}(T|_{\mathcal{M}} - \lambda\zeta^k)\right)(T|_{\mathcal{M}} - \lambda),$$

we have

$$T|_{\mathcal{M}} - \lambda = \left(\prod_{k=1}^{n-1}(T|_{\mathcal{M}} - \lambda\zeta^k)\right)^{-1} \cdot 0 = 0,$$

where $\zeta = \exp(2\pi i/n)$. □

Definition 3.4. *Let $\lambda \in \sigma(T)$ be arbitrary, $n \in \mathbb{N}$ and $\zeta := \exp(2\pi i/n)$. We say that T has property (n) at λ if*

$$\lambda\zeta^k \notin \sigma(T) \;\; \textit{for } k = 1, \cdots, n-1.$$

Remark. We do not need the assumption that λ is an isolated point of $\sigma(T)$ in the following theorem.

Theorem 3.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be n-normal. Then the following assertions hold.*
(i) $H_0(T) = H_0(T^n) = \ker(T^n) = \ker(T^{*n})$.
(ii-1) *If* $\lambda \neq 0$, *then* $H_0(T-\lambda) = \ker(T-\lambda)$.
(ii-2) *If* $\lambda \neq 0$ *and T has property* (n) *at* λ, *then* $H_0(T-\lambda) = \ker(T-\lambda) = \ker((T-\lambda)^*)$.

Proof. (i) Since T^n is normal, we have $H_0(T) \subset H_0(T^n) = \ker(T^n) = \ker(T^{*n})$. It is known that $\ker(T^n) \subset H_0(T)$. Hence $H_0(T) = H_0(T^n) = \ker(T^n) = \ker(T^{*n})$.

(ii-1) We claim $H_0(T-\lambda) \subset H_0(T^n - \lambda^n)$.

Let $x \in H_0(T-\lambda)$ and $\zeta = \exp(2\pi i/n)$. Then

$$\begin{aligned}\|(T^n-\lambda^n)^m x\|^{\frac{1}{m}} &= \|(T-\lambda\zeta)^m(T-\lambda\zeta^2)^m\cdots(T-\lambda\zeta^{m-1})^m(T-\lambda)^m x\|^{\frac{1}{m}}\\ &\leq \|T-\lambda\zeta\|\|T-\lambda\zeta^2\|\cdots\|T-\lambda\zeta^{m-1}\|\|(T-\lambda)^m x\|^{\frac{1}{m}} \longrightarrow 0\ (m\to\infty).\end{aligned}$$

Hence $x \in H_0(T^n-\lambda^n)$.

Since T^n is normal, $H_0(T^n-\lambda^n) = \ker(T^n-\lambda^n)$. Put $\mathcal{M} = \ker(T^n-\lambda^n)$. Then $\mathcal{M}$ is an invariant subspace of T and $\sigma((T|_{\mathcal{M}})^n) = \sigma(T^n|_{\mathcal{M}}) = \{\lambda^n\}$. Hence $\sigma(T|_{\mathcal{M}}) \subset \{\lambda, \lambda\zeta, \cdots, \lambda\zeta^{n-1}\}$. Put $\sigma(T|_{\mathcal{M}}) = \{\mu_1, \cdots, \mu_r\}$ with $\mu_i \neq \mu_j$ $(i \neq j)$ and $\mu_i^n = \lambda^n$ for $i = 1, 2, \cdots, r$. Let F_i be the Riesz idempotent corresponding to $\mu_i \in \sigma(T|_{\mathcal{M}})$. Then $F_iF_j = 0$ $(i \neq j)$, $F_1 + \cdots + F_r = I_{\mathcal{M}}$, $\sigma((T|_{\mathcal{M}})|_{F_i\mathcal{M}}) = \sigma(T|_{F_i\mathcal{M}}) = \{\mu_i\}$ and $\sigma\left(T|_{(I_{\mathcal{M}}-F_i)\mathcal{M}}\right) = \sigma(T|_{\mathcal{M}}) \setminus \{\mu_i\}$ for $i = 1, 2, \cdots, r$. This shows that $T|_{F_i\mathcal{M}} = \mu_i$ for $i = 1, 2, \cdots, r$ by Lemma 3.3. Put $C = (\|F_1\| + \|F_2\| + \cdots + \|F_r\|)^{-1} > 0$. Since

$$\begin{aligned}\|x\| = \|F_1x + F_2x + \cdots + F_rx\| &\leq \|F_1x\| + \|F_2x\| + \cdots + \|F_rx\|\\ &\leq (\|F_1\| + \|F_2\| + \cdots + \|F_r\|)\,\|x\|,\end{aligned}$$

we have

$$\|x\| \geq C(\|F_1x\| + \|F_2x\| + \cdots + \|F_rx\|) \text{ for all } x \in \mathcal{M}.$$

Let $0 \neq x \in H_0(T-\lambda) \subset \mathcal{M}$. Then

$$\begin{aligned}\|(T-\lambda)^n x\|^{\frac{1}{n}} = \|(T|_{\mathcal{M}}-\lambda)^n x\|^{\frac{1}{n}} &\geq \left(C\sum_{k=1}^{r}\|F_k(T|_{\mathcal{M}}-\lambda)^n x\|\right)^{\frac{1}{n}}\\ &= \left(C\sum_{k=1}^{r}\|(T|_{\mathcal{M}}-\lambda)^n F_kx\|\right)^{\frac{1}{n}} = \left(C\sum_{k=1}^{r}\|(T|_{F_k\mathcal{M}}-\lambda)^n F_kx\|\right)^{\frac{1}{n}}\\ &= C^{\frac{1}{n}}\left(\sum_{k=1}^{r}|\mu_k-\lambda|^n\|F_kx\|\right)^{\frac{1}{n}} \geq |\mu_k-\lambda|C^{\frac{1}{n}}\|F_kx\|^{\frac{1}{n}}.\end{aligned}$$

By letting $n \to \infty$, it follows that $F_kx = 0$ for all k such as $\mu_k \neq \lambda$. Hence if there does not exist k such that $\mu_k = \lambda$, then $x = F_1x + F_2x + \cdots + F_rx = 0$ which is a contradiction. Hence there exists a unique number $k' \in \{1, \cdots, r\}$ such that $\mu_{k'} = \lambda$ and $F_{k'}x = x$. Hence $x \in F_{k'}\mathcal{M} = \ker(T|_{F_{k'}\mathcal{M}} - \lambda) \subset \ker(T-\lambda)$. Hence $H_0(T-\lambda) \subset \ker(T-\lambda)$. Since the converse inclusion is clear, we have $H_0(T-\lambda) = \ker(T-\lambda)$.

(ii-2) Let T have property (n) at λ. Since T^n is normal, we have

$$H_0(T-\lambda) = \ker(T-\lambda) \subset H_0(T^n-\lambda^n) = \ker(T^n-\lambda^n) = \ker((T^n-\lambda^n)^*).$$

Conversely, let $y \in H_0(T^n-\lambda^n) = \ker(T^n-\lambda^n) = \ker((T^n-\lambda^n)^*)$. Then $(T^n-\lambda^n)y = 0$ and $(T^n-\lambda^n)^*y = 0$. Since $\lambda\zeta^k \notin \sigma(T)$ for $k = 1, \cdots, n-1$, it follows that

$$(T-\lambda)y = \left(\prod_{k=1}^{n-1}(T|_{\mathcal{M}} - \lambda\zeta^k)\right)^{-1}(T^n-\lambda^n)y = 0$$

and

$$(T-\lambda)^* y = \left(\prod_{k=1}^{n-1} (T|_{\mathcal{M}} - \lambda\zeta^k)^* \right)^{-1} (T^n - \lambda^n)^* y = 0.$$

Hence $H_0(T-\lambda) = \ker(T-\lambda) = \ker(T^n-\lambda^n) = \ker((T^n-\lambda^n)^*) \subset \ker((T-\lambda)^*)$. Since $\ker((T-\lambda)^*) \subset \ker((T^n-\lambda^n)^*)$ is clear, we have

$$H_0(T-\lambda) = \ker(T-\lambda) = \ker((T-\lambda)^*) = \ker(T^n-\lambda^n) = \ker((T^n-\lambda^n)^*).$$

□

Theorem 3.6. *Let $T \in \mathcal{L}(\mathcal{H})$ be n-normal. Then T is isoloid and polaroid.*
Moreover, let λ be an isolated point of the spectrum of T. Then λ is a pole of the resolvent and following statements hold.
(i) *If $\lambda = 0$, then $E_T(\{0\})\mathcal{H} = H_0(T) = H_0(T^n) = \ker(T^n) = \ker(T^{*n})$, $E_T(\{0\})$ is self-adjoint and the order of* 0 *is not greater than n.*
(ii) *If $\lambda \neq 0$, then $E_T(\{\lambda\})\mathcal{H} = H_0(T-\lambda) = \ker(T-\lambda)$ and the order of λ is* 1.

Proof. (i) Assume that 0 is an isolated point of $\sigma(T)$. Since $H_0(T) = H_0(T^n) = \ker(T^n) = \ker(T^{*n})$ by Theorem 3.5, we have $E_T(\{0\})\mathcal{H} = H_0(T) = H_0(T^n) = \ker(T^n) = \ker(T^{*n})$. Hence 0 is a pole of the resolvent of T, $E_T(\{0\})$ is self-adjoint and the order of pole is not greater than n by Lemma 2.2.
(ii) Next we assume λ is a nonzero isolated point of $\sigma(T)$. Since $H_0(T-\lambda) = \ker(T-\lambda)$ by Theorem 3.5, we have $E_T(\{\lambda\})\mathcal{H} = H_0(T-\lambda) = \ker(T-\lambda)$. Hence λ is a pole of the resolvent of T and the order of pole is 1 by Lemma 2.2. □

4. (n, m)-normal Operators

Definition 4.1. *For $n, m \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be (n, m)-normal if*

$$T^{*m} T^n = T^n T^{*m}.$$

From the definition, it is clear that T is (n, m)-normal if and only if T^* is (m, n)-normal. Moreover, if T^n is normal, then T is (n, m)-normal for every m. Indeed, since T^n is normal and $T^m \cdot T^n = T^n \cdot T^m$, it follows from Fuglede theorem that $T^{*m} \cdot T^n = T^n \cdot T^{*m}$. Hence T is (n, m)-normal. From [4], we restate the properties of (m, n)-normal operators.

Lemma 4.2. Let $T \in \mathcal{L}(\mathcal{H})$ be (n, m)-normal. Then the following statements hold.
(i) T^* is (m, n)-normal.
(ii) If T^{-1} exists, then T^{-1} is (n, m)-normal.
(iii) If $S \in \mathcal{L}(\mathcal{H})$ is unitary equivalent to T, then S is (n, m)-normal.
(iv) If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ which reduces T, then $T|_{\mathcal{M}}$ is (n, m)-normal on $\mathcal{M}$.
(v) If T is (n, m)-normal, then T^k is normal where k is the least common multiple of n and m.
(vi) If T is quasi-nilpotent, then T is nilpotent.

Proof. The proofs of the statements of (i), (ii), (iii), and (iv) are clearly holds by the definition.
(v) Let $k := n \cdot j$ and $k := m \cdot \ell$. Since T is (n, m)-normal, it follows that

$$T^{*k} T^k = \overbrace{T^{*m} \cdots T^{*m}}^{\ell} \cdot \overbrace{T^n \cdots T^n}^{j} = T^n \cdots T^n \cdot T^{*m} \cdots T^{*m} = T^k T^{*k},$$

which means that T^k is normal.
(vi) If T is quasi-nilpotent, i.e., $\sigma(T) = \{0\}$, then $\sigma(T^k) = \{0\}$ for every $k \in \mathbb{N}$. Let k_0 be the least common multiple of n and m. Then T^{k_0} is normal by Lemma 4.2 (v). Hence $T^{k_0} = 0$. □

Corollary 4.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be (n, m)-normal. Then T is isoloid and polaroid.*
Moreover, let λ be an isolated point of the spectrum of T. Then λ is a pole of the resolvent and following statements hold.
(i) *If $\lambda = 0$, then $H_0(T) = E_T(\{0\})\mathcal{H} = \ker(T^{nm}) = \ker(T^{*nm})$, $E_T(\{0\})$ is self-adjoint and the order of 0 is not greater than n.*
(ii) *If $\lambda \neq 0$, then $H_0(T - \lambda) = E_T(\{\lambda\})\mathcal{H} = \ker(T - \lambda)$ and the order of λ is 1.*

Proof. Since T^{nm} is normal by Lemma 4.2, we have these results from Theorem 3.6. □

We say that *Weyl's theorem holds* for T if

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T), \text{ or equivalently, } \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$, $\pi_{00}(T) = \{\lambda \in \text{iso}\,(\sigma(T)) : 0 < \dim\ker(T - \lambda) < \infty\}$, and $\text{iso}\,(\sigma(T))$ denotes the set of all isolated points of $\sigma(T)$.

Theorem 4.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be (n, m)-normal. Then the following statements hold.*
(i) *T is decomposable.*
(ii) *If f is an analytic function on $\sigma(T)$ which is not constant on each of the components of its domain, then Weyl's theorem holds for $f(T)$.*

Proof. (i) Since T^{nm}, T^{*nm} are normal by Lemma 4.2, it follows T^{nm} is decomposable. Hence T is decomposable by [7, Theorem 3.3.9].

(ii) Since T is polaroid by Theorem 3.6 or Corollary 4.3 and T has the single-valued extension property by Theorem 1.4, it follows that Weyl's theorem holds for $f(T)$ by [2, Theorem 3.14]. □

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