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REMARKS ON NONLINEAR SCHRÖDINGER EQUATIONS IN ONE SPACE DIMENSION

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Abstract. We consider the initial value problem for nonlinear Schödinger equations,

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial^2 u = F(u, \partial u, \overline{u}, \partial\overline{u}), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{†}$$

where $\partial = \partial_x = \partial/\partial x$ and $F : \mathbb{C}^4 \to \mathbb{C}$ is a polynomial having neither constant nor linear terms. Without a smallness condition on the data u_0, it is shown that (†) has a unique local solution in time if u_0 is in $H^{3,0} \cap H^{2,1}$, where $H^{m,s} = \{f \in \mathcal{S}' : \|f\|_{m,s} = \|(1+x^2)^{\frac{s}{2}}(1-\Delta)^{\frac{m}{2}} f\|_2 < \infty\}$, $m, s \in \mathbb{R}$.

1. Introduction. We consider the initial value problem for nonlinear Schrödinger equations,

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial^2 u = F(u, \partial u, \overline{u}, \partial\overline{u}), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $\partial = \partial_x = \partial/\partial x$, u is a complex valued function of $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, $\overline{u}$ is a complex conjugate of u, and F denotes a complex valued polynomial defined on $\mathbb{C}^4$ such that

$$F(z) = F(z_1, z_2, z_3, z_4) = \sum_{\substack{d \le |\alpha| \le \rho \\ \alpha \in \mathbb{Z}_+^4}} a_\alpha z^\alpha, \tag{1.2}$$

where we have used the standard notation for multi-indices. We assume that there exists $a_{\alpha_0} \neq 0$ for some $\alpha_0 \in \mathbb{Z}_+^4$ with $|\alpha_0| = d$, since, as we see below, the lowest degree d of the polynomial F, rather than the highest degree ρ, determines the character of the problem. The main results in this paper are the following. For notation, see below.

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Theorem 1.1. *Let F be as in (1.2) with $d \geq 3$. Then for any $u_0 \in H^{3,0}$, there exists a unique solution $u(\cdot)$ of (1.1) defined in the interval $[0,T]$, $T = T(\|u_0\|_{3,0}) > 0$ with $T(\theta) \to \infty$ as $\theta \to 0$, satisfying*

$$u \in C([0,T]; H^{2,0}) \cap C_w(0,T; H^{3,0}). \tag{1.3}$$

Theorem 1.2. *Let F be as in (1.2) with $d = 2$. Then for any $u_0 \in H^{3,0} \cap H^{2,1}$, there exists a unique solution $u(\cdot)$ of (1.1) defined in the interval $[0,T]$, $T = T(\|u_0\|_{3,0} + \|u_0\|_{2,1}) > 0$ with $T(\theta) \to \infty$ as $\theta \to 0$, satisfying*

$$u \in C([0,T]; H^{2,1}) \cap C_w(0,T; H^{3,0}). \tag{1.4}$$

In problem (1.1) the difficulty arises from the fact that the nonlinearity of F involves the first derivatives ∂u and $\partial \overline{u}$, which could cause the so-called loss of derivatives so long as we make direct use of the standard methods, such as the energy estimates, the space-time estimates, and so on. One way to avoid this difficulty is to impose some conditions on the form of the nonlinearity in order that the worst derivatives should be dropped after integration by parts [4, 8, 9]. Another way is to restrict the class of the initial data, or equivalently, the function space where we solve the initial value problem. This approach turned out to be successful in the spaces of analytic functions [1, 2, 5], where the loss of derivatives is absorbed by analyticity.

Recently, essential progress was made by Kenig, Ponce and Vega [7] by pushing forward the linear estimates associated with the Schrödinger group $\{\exp(\frac{i}{2}t\Delta)\}_{-\infty}^{\infty}$ and by introducing suitable function spaces where these estimates act naturally. In [7] the following theorems were proved.

Theorem 4.1 ([7]). *Let F be as in (1.2) with $d \geq 3$. Then for any $u_0 \in H^{\frac{7}{2},0}$ such that $\|u_0\|_{\frac{7}{2},0}$ is sufficiently small, there exists a unique solution $u(\cdot)$ of (1.1) defined in the interval $[0,T]$, $T = T(\|u_0\|_{\frac{7}{2},0}) > 0$ with $T(\theta) \to \infty$ as $\theta \to 0$, satisfying*

$$u \in C([0,T]; H^{\frac{7}{2},0}) \quad \textit{and} \quad \sup_{x \in \mathbb{R}} \left(\int_0^T |\partial^4 u(t,x)|^2 dt \right)^{\frac{1}{2}} < \infty.$$

Theorem 4.2 ([7]). *Let F be as in (1.2) with $d = 2$. Then for any $u_0 \in H^{\frac{11}{2},0} \cap H^{3,1}$ such that $\|u_0\|_{\frac{11}{2},0} + \|u_0\|_{3,1}$ is sufficiently small, there exists a unique solution $u(\cdot)$ of (1.1) defined in the interval $[0,T]$, $T = T(\|u_0\|_{\frac{11}{2},0} + \|u_0\|_{3,1}) > 0$ with $T(\theta) \to \infty$ as $\theta \to 0$, satisfying*

$$u \in C([0,T]; H^{\frac{11}{2},0} \cap H^{3,1}) \quad \textit{and} \quad \sup_{x \in \mathbb{R}} \left(\int_0^T |\partial^6 u(t,x)|^2 dt \right)^{\frac{1}{2}} < \infty.$$

Although no further restriction on the nonlinearity is made in these theorems, a smallness condition on the initial data is assumed. In fact, in [7] the following natural question is given.

For "large" data u_0 does (1.1) have a local solution?

In the present paper we have removed the smallness assumption and our results are positive answers to this question in the one dimensional case.

Our strategy of attacking the loss of derivatives is based on some observation of the structure of nonlinearity. By differentiating the equation with respect to x a few times, we easily observe the formation of the worst terms in the nonlinearity, which are classified into two categories: The terms which are handled by integration by parts and the terms which are not. Our conclusion in this paper is summarized as follows.

(1) The terms of second category can be absorbed by a gauge transformation.
(2) After the gauge transformation and a few times of differentiation of the equation (twice would suffice), the original equation is always transformed into a new system of equations where the usual energy estimates provide a sufficient method in order that the system proves to be closed with respect to differentiation.

As a result, we need no further restriction on the nonlinearity and on the data concerning the smallness and regularity conditions. A similar technique has been used on the derivative nonlinear Schrödinger equation [3, 5, 6], although no simple modification of the gauge transformation of [3, 5, 6] is fit for the general nonlinearity of the form (1.1).

We note finally that our method depends heavily on the fact that the space dimension is equal to one and the equation is single, while the method in [7] is applicable to higher dimensional cases as well as to the system.

We now give notation and function spaces.

Notation and function spaces. For simplicity

$$\partial_{z_j}\partial_{z_k}\partial_{z_l}F(z) = F_{jkl}, \quad \partial_{z_j}\partial_{z_k}F(z) = F_{jk}, \quad \partial_{z_j}F(z) = F_j.$$

We let

$$L^p = \{f : f \text{ is measurable on } \mathbb{R},\ \|f\|_p < \infty\},$$

where

$$\|f\|_{L^p}^p = \int_{\mathbb{R}} |f(x)|^p dx \quad \text{if } 1 \le p < \infty$$

and

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(x)| : x \in \mathbb{R}\} \quad \text{if } p = \infty,$$

and we let

$$H^{m,s} = \{f \in \mathcal{S}' : \|f\|_{m,s} = \|(1+x^2)^{\frac{s}{2}}(1-\Delta)^{\frac{m}{2}} f\|_2 < \infty\}, \ \ m, s \in \mathbb{R}.$$

For any interval I of $\mathbb{R}$ and a Banach space B with the norm $\|\cdot\|_B$, we let $C(I; B)$ (resp. $C_w(I; B)$) be the space of continuous (resp. weakly continuous) functions from I to B, and we let $L^p(I; B)$ be the space consisting of strongly measurable B valued functions u defined on I such that $\|u(\cdot)\|_B \in L^p(I)$. Different positive constants will be denoted by the same letter C. If necessary, by $C_j(* \cdots *)$ we denote constants depending on the quantities appearing in parentheses.

2. Proof of the Theorems. Let

$$G_\pm(u) = \exp\Big(\int_{-\infty}^x \pm F_2 dy\Big) = \exp\Big(\int_{-\infty}^x \pm(F_2(u, \partial u, \overline{u}, \overline{\partial u}))(t, y) dy\Big).$$

F_2 is written as

$$F_2 = \alpha_1 u + \alpha_2 \overline{u} + \alpha_3 \partial u + \alpha_4 \partial \overline{u} + P(u, \partial u, \overline{u}, \partial \overline{u}),$$

where P is a polynomial having neither constant nor linear terms and α_j $(j = 1,2,3,4)$ are constants for $d = 2$ and $\alpha_j = 0$ for $d = 3$. If $\partial u, \partial \overline{u} \to 0$ as $x \to -\infty$, then

$$G_{\pm}(u) = \exp\Big(\int_{-\infty}^{x} \pm(\alpha_1 u + \alpha_2 \overline{u} + P)dy \pm \alpha_3 u \pm \alpha_4 \overline{u}\Big).$$

We let $u = u_1$, $\partial u = u_2$ and $u_3 = G_-\partial^2 u$. Then (1.1) is written as

$$i\partial_t u_1 + \frac{1}{2}\partial^2 u_1 = F(u_1, u_2, \overline{u_1}, \overline{u_2}). \tag{2.1}$$

Differentiating (2.1) with respect to x , we obtain

$$i\partial_t u_2 + \frac{1}{2}\partial^2 u_2 = F_1 \cdot u_2 + F_3 \cdot \overline{u_2} + F_2 \cdot G_+ u_3 + F_4 \cdot \overline{G_+ u_3}. \tag{2.2}$$

We again differentiate (2.2) with respect to x and then multiply the resulting equation by G_- to obtain

$$\begin{aligned} i\partial_t u_3 + \frac{1}{2}\partial^2 u_3 &= i\partial_t(G_-\partial^2 u) + \frac{1}{2}\partial^2(G_-\partial^2 u) \qquad (2.3)\\ &= (i\partial_t G_-)(\partial^2 u) + \frac{1}{2}(\partial^2 G_-)G_+ u_3 + G_-\partial(i\partial_t + \frac{1}{2}\partial^2)u_2 + \partial G_- \cdot \partial^3 u \\ &= -\int_{-\infty}^{x} i\partial_t F_2 dy \cdot u_3 + \frac{1}{2}\partial^2 G_- \cdot G_+ u_3 + G_-(F_{11}u_2^2 + 2F_{13}|u_2|^2 + F_{33}\overline{u_2}^2) \\ &+ 2F_{12}u_2 u_3 + 2F_{23}\overline{u_2}u_3 + 2F_{34}G_-\overline{G_+}\,\overline{u_2}\,\overline{u_3} + 2F_{14}G_-\overline{G_+}u_2\overline{u_3} + F_{22}G_+ u_3^2 \\ &+ 2F_{24}\overline{G_+}|u_3|^2 + F_{44}G_-\overline{G_+}^2\overline{u_3}^2 + F_1 u_3 + F_3 G_-\overline{G_+}\,\overline{u_3} \\ &+ F_4 G_-\overline{G_+}\partial\overline{u_3} + F_4 G_-\overline{\partial G_+}\,\overline{u_3}, \end{aligned}$$

where the terms of the right hand side of the last equality involving ∂u_3 cancel. Indeed, the third term of the right hand side of the second equality of (2.3) involves the term

$$G_-\partial(F_2 G_+ u_3) = (\partial F_2)u_3 + (F_2)^2 u_3 + F_2 \partial u_3,$$

while the fourth term is equal to

$$-F_2 G_-\partial^3 u = -F_2\partial u_3 - (F_2)^2 u_3$$

and therefore the contribution of those two terms is equal to $(\partial F_2)u_3$. The fact that ∂u_3 does not appear in (2.3) will enable us to derive a crucial energy estimate below, namely (2.14).

A direct calculation shows

$$\begin{aligned} i\partial_t F_2 &= F_{12} i\partial_t u + F_{22} i\partial_t \partial u + F_{23} i\partial_t \overline{u} + F_{24} i\partial_t \partial\overline{u} \qquad (2.4)\\ &= F_{12}(-\frac{1}{2}\partial^2 u + F) + F_{22}(-\frac{1}{2}\partial^3 u + \partial F) + F_{23}(\frac{1}{2}\partial^2\overline{u} - \overline{F}) + F_{24}(\frac{1}{2}\partial^3\overline{u} - \partial\overline{F}) \end{aligned}$$

$$= -\frac{1}{2}F_{12}G_+u_3 + \frac{1}{2}F_{23}\overline{G_+}\overline{u_3} + F_{12}F$$
$$- F_{23}\overline{F} + F_{22}(F_1u_2 + F_3\overline{u_2} + F_2G_+u_3 + F_4\overline{G_+}\overline{u_3})$$
$$- F_{24}(\overline{F_1}\overline{u_2} + \overline{F_3}u_2 + \overline{F_2G_+}\overline{u_3} + \overline{F_4}G_+u_3) - \frac{1}{2}\partial(F_{22}G_+u_3 - F_{24}\overline{G_+}\overline{u_3})$$
$$+ \frac{1}{2}(F_{122}G_+u_2u_3 + F_{222}G_+^2u_3^2 + F_{223}G_+\overline{u_2}u_3 - F_{124}\overline{G_+}u_2\overline{u_3}$$
$$- F_{234}\overline{G_+}\overline{u_2}\overline{u_3} - F_{244}\overline{G_+}^2\overline{u_3}^2),$$

$$\partial G_- = -F_2G_-, \tag{2.5}$$

$$\begin{aligned}\partial^2 G_- &= -\partial F_2 \cdot G_- + F_2^2G_- \\ &= -(F_{12}u_2 + F_{23}\overline{u_2} + F_{22}G_+u_3 + F_{24}\overline{G_+}\overline{u_3})G_- + F_2^2G_-,\end{aligned} \tag{2.6}$$

and

$$F_4G_-\overline{G_+}\partial\overline{u_3} = \Big(\int_{-\infty}^{x} \partial(F_4G_-\overline{G_+})dy\Big)\partial\overline{u_3} \tag{2.7}$$
$$= \Big(\int_{-\infty}^{x} ((F_{14}u_2 + F_{34}\overline{u_2} + F_{24}u_3G_+ + F_{44}\overline{u_3}\overline{G_+} + F_4(\overline{F_2} - F_2))G_-\overline{G_+})dy\Big)\partial\overline{u_3}$$

if $F_4G_-\overline{G_+} \to 0$ as $x \to -\infty$. We apply (2.4)–(2.7) to (2.3) to obtain, if $F_4G_-\overline{G_+} \to 0$ and $F_{22}G_+u_3 - F_{24}\overline{G_+}\overline{u_3} \to 0$ as $x \to -\infty$,

$$i\partial_t u_3 + \frac{1}{2}\partial^2 u_3 = P_1(u_1, u_2, u_3) \tag{2.8}$$
$$+ \Big(\int_{-\infty}^{x} ((F_{14}u_2 + F_{34}\overline{u_2} + F_{24}u_3G_+ + F_{44}\overline{u_3}\overline{G_+} + F_4(\overline{F_2} - F_2))G_-\overline{G_+})dy\Big)\partial\overline{u_3},$$

where

$$P_1(u_1, u_2, u_3) = -\Big\{\int_{-\infty}^{x} (-\frac{1}{2}F_{12}G_+u_3 + \frac{1}{2}F_{23}\overline{G_+}\overline{u_3} + F_{12}F - F_{23}\overline{F})dy\Big\}u_3 \tag{2.8'}$$
$$- \Big\{\int_{-\infty}^{x} F_{22}(F_1u_2 + F_3\overline{u_2} + F_2G_+u_3 + F_4\overline{G_+}\overline{u_3})dy\Big\}u_3$$
$$+ \Big\{\int_{-\infty}^{x} F_{24}(\overline{F_1}\overline{u_2} + \overline{F_3}u_2 + \overline{F_2G_+}\overline{u_3} + \overline{F_4}G_+u_3)dy\Big\}u_3$$
$$- \Big\{\int_{-\infty}^{x} \frac{1}{2}(F_{122}G_+u_2u_3 + F_{222}G_+^2u_3^2 + F_{223}G_+\overline{u_2}u_3$$
$$- F_{124}\overline{G_+}u_2u_3 - F_{234}\overline{u_2}\overline{u_3} - F_{244}\overline{G_+}^2\overline{u_3}^2)dy\Big\}u_3 + \frac{1}{2}(F_{22}G_+u_3 - F_{24}\overline{G_+}\overline{u_3})u_3$$
$$+ \frac{1}{2}\{-(F_{12}u_2 + F_{23}\overline{u_2} + F_{22}G_+u_3 + F_{24}\overline{G_+}\overline{u_3}) + F_2^2\}u_3$$
$$+ G_-(F_{11}u_2^2 + 2F_{13}|u_2|^2 + F_{33}\overline{u_2}^2) + 2F_{12}u_2u_3 + 2F_{23}\overline{u_2}u_3$$
$$+ 2F_{34}G_-\overline{G_+}\overline{u_2}\overline{u_3} + 2F_{14}G_-\overline{G_+}u_2\overline{u_3} + F_{22}G_+u_3^2 + 2F_{24}\overline{G_+}|u_3|^2$$
$$+ 2F_{44}G_-\overline{G_+}^2\overline{u_3}^2 + F_1u_3 + F_3G_-\overline{G_+}\overline{u_3} - F_4F_2G_-\overline{G_+}\overline{u_3}.$$

Proof of Theorem 1.1: It is now clear that the usual energy method is applicable to the system of equations (2.1), (2.2) and (2.8). Therefore we study the system of equations (2.1), (2.2) and (2.8) in the function space

$$X_T = \Big\{U = (u_1, u_2, u_3) : u_j \in C([0,T]; L^2) \cap L^\infty(0,T; H^{1,0}),\ j = 1,2,3,$$
$$|||U|||^2_{X_T} = \sum_{j=1}^{3} \sup_{0 \le t \le T} \|u_j(t)\|^2_{1,0} < \infty\Big\}.$$

We let $V = (v_1, v_2, v_3)$ in $X_T(R) = \{U \in X_T : |||U|||_{X_T} \le R\}$ and consider the linearized equations of (2.1), (2.2) and (2.8),

$$i\partial_t u_1 + \frac{1}{2}\partial^2 u_1 = F(v_1, v_2, \overline{v_1}, \overline{v_2}), \tag{2.9}$$

$$i\partial_t u_2 + \frac{1}{2}\partial^2 u_2 = F_1 \cdot v_2 + F_3 \cdot \overline{v_2} + F_2 \cdot G_+ v_3 + F_4 \cdot \overline{G_+ v_3}, \tag{2.10}$$

$$i\partial_t u_3 + \frac{1}{2}\partial^2 u_3 = P_1(v_1, v_2, v_3) \tag{2.11}$$
$$+ \Big(\int_{-\infty}^{x} ((F_{14} v_2 + F_{34}\overline{v_2} + F_{24} v_3 G_+ + F_{44}\overline{v_3}\overline{G_+} + F_4(\overline{F_2} - F_2))G_-\overline{G_+})dy\Big)\partial\overline{u_3},$$

with the initial data

$$u_1(0,x) = u_0, \quad u_2(0,x) = \partial u_0, \quad u_3(0,x) = G_-(u_0)\partial^2 u_0,$$

where we have used abbreviations such as

$$F_1 = (\partial_{v_1} F)(v_1, v_2, \overline{v_1}, \overline{v_2})$$

on the right hand sides of (2.10) and (2.11). By Sobolev's inequality, we see that $\|u_j(0)\|_{1,0}$, $j = 1,2,3$ being small is equivalent to $\|u_0\|_{3,0}$ being small. We define the mapping Φ by

$$U = \Phi V,$$

where $U = (u_1, u_2, u_3)$ is the solution of (2.9)–(2.11). We show that Φ has a fixed point in $X_T(R)$. By the usual energy estimates and Sobolev's inequality we have

$$\frac{d}{dt}(\|u_1(t)\|_2^2 + \|\partial u_1(t)\|_2^2)^{\frac{1}{2}} \le C \cdot (R^d + R^\rho), \tag{2.12}$$

$$\frac{d}{dt}(\|u_2(t)\|_2^2 + \|\partial u_1(t)\|_2^2)^{\frac{1}{2}} \le C \cdot (R^d + R^{2\rho-1})(1 + \exp(C \cdot (R^{d-1} + R^{\rho-1}))), \tag{2.13}$$

and

$$\frac{d}{dt}(\|u_3(t)\|_2^2 + \|\partial u_3(t)\|_2^2)^{\frac{1}{2}} \le C \cdot (R^d + R^{3\rho-2})(1 + \exp(C \cdot (R^{d-1} + R^{\rho-1})))$$
$$+ C \cdot (R^{d-1} + R^{2\rho-2})(1 + \exp(C \cdot (R^{d-1} + R^{\rho-1})))(\|u_3(t)\|_2^2 + \|\partial u_3(t)\|_2^2)^{\frac{1}{2}}. \tag{2.14}$$

To be specific, we derive the energy inequalities (2.12), (2.13), and (2.14) as follows. To derive (2.12), we multiply (2.13) by $(1-\partial^2)\overline{u_1}$, integrate over space, and take the imaginary part. The left hand side gives $\frac{d}{dt}(\|u_1(t)\|_2^2+\|\partial u_1(t)\|_2^2)$ and the right hand side is

$$\mathrm{Im}\langle F,(1-\partial^2)\overline{u_1}\rangle = \mathrm{Im}\langle F,\overline{u_1}\rangle + \mathrm{Im}\langle \partial F,\partial\overline{u_1}\rangle$$

by integration by parts. This leads to (2.12). By an analogous calculation with (2.10), we end up on the corresponding right hand side with terms such as

$$\mathrm{Im}\langle \partial(F_2G_+v_3),\partial\overline{u_2}\rangle = \mathrm{Im}\langle \partial F_2\cdot G_+v_3,\partial\overline{u_2}\rangle$$
$$+\mathrm{Im}\langle (F_2)^2G_+v_3),\partial\overline{u_2}\rangle + \mathrm{Im}\langle (F_2G_+\partial v_3,\partial\overline{u_2}\rangle,$$

where $\partial F_2\cdot v_3$ and $F_2\partial v_3$ are polynomials with powers d to ρ, and $(F_2)^2v_3$ is a polynomial with powers $2d-1$ to $2\rho-1$. This leads to (2.13). To derive (2.14), we multiply (2.11) by $(1-\partial^2)\overline{u_3}$, integrate over space and take the imaginary part. On the right hand side, there is a term involving $G_\pm$ and the derivatives of F multiplied by $\partial\overline{u_3}\cdot\partial^2\overline{u_3}$. Since $\partial\overline{u_3}\cdot\partial^2\overline{u_3}=\frac{1}{2}\partial(\partial(\overline{u_3})^2)$, we integrate by parts to obtain (2.14). We should emphasize here that if ∂u_3 had been present in (2.11) as well, no integration by parts would have been possible, and the estimate would not have gone through. From (2.12) and (2.13) it follows that

$$\sup_{0\le t\le T}\|u_1(t)\|_{1,0}+\sup_{0\le t\le T}\|u_2(t)\|_{1,0} \le \sum_{j=1}^{2}\|u_j(0)\|_{1,0}+C\cdot(R^d+R^\rho)(1+\exp(C\cdot(R^{d-1}+R^{\rho-1})))T. \tag{2.15}$$

Integrating the differential inequality (2.14), we have

$$\sup_{0\le t\le T}\|u_3(t)\|_{1,0}\le[\|u_3(0)\|_{1,0}+C_1(R)T]\exp(C_2(R)T), \tag{2.16}$$

where

$$C_1(R)=C\cdot(R^d+R^{3\rho-2})(1+\exp(C\cdot(R^{d-1}+R^{\rho-1}))),$$
$$C_2(R)=C\cdot(R^{d-1}+R^{2\rho-2})(1+\exp(C\cdot(R^{d-1}+R^{\rho-1}))).$$

From (2.15) and (2.16),

$$|||U|||_{X_T}\le\Big[\sum_{j=1}^{3}\|u_j(0)\|_{1,0}+C_1(R)T\Big]\exp(C_2(R)T). \tag{2.17}$$

We put $U^{(1)}=\Phi V^{(1)}$ and $U^{(2)}=\Phi V^{(2)}$, where $V^{(1)}$, $V^{(2)}\in X_T(R)$ and $U^{(1)}$, $U^{(2)}$ are the solutions of (2.9)–(2.11) with the same initial data. Then in the same way as in the proof of (2.17),

$$|||U^{(1)}-U^{(2)}|||_{Y_T}\le C_3(R)T|||V^{(1)}-V^{(2)}|||_{Y_T}, \tag{2.18}$$

where

$$C_3(R)=C\cdot(R^{d-2}+R^{2\rho-1})(1+\exp(C\cdot(R^{d-1}+R^{\rho-1})))$$

and

$$Y_T = \Big\{U = (u_1, u_2, u_3) : u_j \in C([0,T]; L^2),\ \ j = 1,2,3,$$
$$|||U|||_{Y_T}^2 = \sum_{j=1}^{3} \sup_{0\le t\le T} \|u_j(t)\|_2^2 < \infty\Big\}.$$

We note that the closed ball $X_T(R)$ is complete under the metric on Y_T. By (2.17) and (2.18) we see that the mapping Φ leaves $X_T(R)$ invariant and is a contraction in the metric on Y_T provided that T is chosen suitably according to the size of R and that R is chosen according to the size of $\|u_0\|_{3,0}$. By the contraction mapping principle, we see that there exists a unique local solution $U = (u_1, u_2, u_3)$ of (2.1), (2.2) and (2.3) defined in the interval $[0,T]$, $T = T(\|u_0\|_{3,0}) > 0$ with $T(\theta) \to \infty$ as $\theta \to 0$, satisfying

$$u_j \in C([0,T]; L^2) \cap L^\infty(0,T; H^{1,0}),\quad j = 1,2,3, \tag{2.19}$$

and

$$F_4 G_- \overline{G_+} \to 0,\quad F_{22} G_+ u_3 - F_{24} \overline{G_+} \overline{u_3} \to 0 \quad \text{as} \quad x \to -\infty.$$

By uniqueness of solutions we conclude that

$$u_2 = \partial u_1,\quad u_3 = G_-(u_1)\partial^2 u_1 \quad \text{on} \quad [0,T], \tag{2.20}$$

and therefore the unique solution of (1.1) is given by $u = u_1$. Since $\partial u_3 = -F_2 G_- \partial^2 u + G_- \partial^3 u$, we have

$$\partial^3 u = G_+ \partial u_3 + F_2 \partial^2 u.$$

Hence by Sobolev's inequality,

$$\|\partial^3 u\|_2 \le C_4(\|u\|_{2,0}) \cdot (1 + \|\partial u_3\|_2). \tag{2.21}$$

From (2.19)-(2.21) it follows that

$$u \in L^\infty(0,T; H^{3,0}).$$

This and the integral equation associated with (1.1) imply

$$u \in C([0,T]; H^{2,0}),$$

which when combined with the boundedness of u with values in $H^{3,0}$ implies the weak continuity of u with values in $H^{3,0}$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2: Multiplying both sides of (2.1), (2.2) and (2.8) by x, we have

$$i\partial_t x u_j + \frac{1}{2}\partial^2 x u_j = \partial u_j + x(i\partial_t + \frac{1}{2}\partial^2) u_j,\quad j = 1,2,3, \tag{2.22}$$

where $x(i\partial_t + \frac{1}{2}\partial^2)u_j$, $j = 1,2,3$, are understood to be the right hand sides of (2.1), (2.2), (2.8) multiplied by x, respectively. In the same way as in the proof of

Theorem 1.1, we consider the system of equations (2.1), (2.2), (2.8) and (2.22) in the function space

$$X_T = \{U = (u_1, u_2, u_3) : u_j \in C([0,T]; L^2) \cap L^\infty(0,T; H^{1,0} \cap H^{0,1}),\ \ j = 1,2,3,$$
$$|||U|||_{X_T} = \sum_{j=1}^{3} \sup_{0\le t\le T} (\|u_j(t)\|_{1,0} + \|u_j(t)\|_{0,1}) < \infty\Big\}.$$

If we take into account

$$\int_{-\infty}^{x} (F_{12}G_+u_3 - F_{23}\overline{G_+}\overline{u_3})dy = \int_{-\infty}^{x} (F_{12}\partial^2 u - F_{23}\partial^2\overline{u})dy$$
$$= F_{12}u_2 - F_{23}\overline{u_2} - \int_{-\infty}^{x} (\partial F_{12} \cdot u_2 - \partial F_{23} \cdot \overline{u_2})dy$$

in (2.8′) and

$$\|G_\pm\|_\infty \le \exp\Big(C\Big\{\|u_1\|_{0,1} + \sum_{j=1}^{2}(\|u_j\|_{1,0} + \|u_j\|_{1,0}^{\rho})\Big\}\Big),$$

we see that in the same manner to the proof of Theorem 1.1 there exists a unique local solution $U = (u_1, u_2, u_3)$ of (2.1), (2.2), (2.8) and (2.22) defined in the interval $[0,T]$, $T = T(\|u_0\|_{3,0} + \|u_0\|_{2,1}) > 0$ with $T(\theta) \to \infty$ as $\theta \to 0$, satisfying

$$u_j \in C([0,T]; L^2) \cap L^\infty(0,T; H^{1,0} \cap H^{0,1}), \quad j = 1,2,3, \tag{2.23}$$

and

$$F_4G_-\overline{G_+} \to 0, \quad F_{22}G_+u_3 - F_{24}\overline{G_+}\overline{u_3} \to 0 \quad \text{as} \quad x \to -\infty.$$

The rest of the proof of Theorem 2 proceeds in the same way as that of Theorem 1.1, and so we omit it. This completes the proof of Theorem 1.2.

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