Remarks on partition functions of topological string theory on generalized conifolds

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Abstract

The notion of topological vertex and the construction of topological string partition functions on local toric Calabi-Yau 3-folds are reviewed. Implications of an explicit formula of partition functions for the generalized conifolds are considered. Generating functions of part of the partition functions are shown to be tau functions of the KP hierarchy. The associated Baker-Akhiezer functions play the role of wave functions, and satisfy q-difference equations. These q-difference equations represent the quantum mirror curves conjectured by Gukov and Sułkowski.

1 Introduction

Topological string theory is a simplified version of string theory in which topological properties of the target space are captured by dynamics of strings [1]. Some ten years ago, the method of "topological vertex" was introduced as a technique for calculating the "amplitudes" of topological string theory on toric (or, more precisely, local toric) Calabi-Yau 3-folds [2]. This method enables one to describe the amplitudes in the language of purely combinatorial notions such as partitions and (skew) Schur functions.

Local toric Calabi-Yau 3-folds are characterized by graphical data called "web diagrams" (which are dual to two-dimensional "toric diagrams"). The vertices of a web diagram represent copies of the simplest Calabi-Yau 3-fold $\mathbf{C}^3$. These $\mathbf{C}^3$'s

are glued together to form the 3-fold X in question. The web (or toric) diagram encodes the gluing data.

The topological vertex is the amplitude $C_{\alpha\beta\gamma}$ of topological string theory on $\mathbf{C}^3$ with boundary conditions imposed on string world sheets. The indices α, β, γ are integer partitions that specify the boundary conditions. These partitions are placed on the three edges emanating from the vertex. (Web diagrams are trivalent.) When the copies of $\mathbf{C}^3$ are glued together, their vertex weights are multiplied along with edges weights, and summed over all possible configurations of the partitions on the internal edges. The vertex weight $C_{\alpha\beta\gamma}$ itself is a somewhat complicated combination of special values of skew Schur functions. Thus the amplitude of topological string theory on X (also called the "partition function" from the point of view of statistical mechanics) is a sum of combinatorial quantities with respect to the partitions on the inner edges.

When the 3-fold X is the resolved conifold or its generalizations called "generalized conifolds", one can calculate the sum over partitions explicitly with the aid of the Cauchy identities for skew Schur functions. In this paper, we review this result and consider its implications in the context of "integrable hierarchies" [4, 5, 6] and "quantum mirror curves" [7, 8]. In particular, we present an explicit form of q-difference equations for "wave functions" [9] of the generalized conifold. This is a generalization of the known result on the resolved conifold. Although free fermions and vertex operators are very convenient tools [10, 11], we dare not use them and resort to the Cauchy identities.

2 Partitions and Schur functions

In this section, we recall some relevant notions from combinatorics of integer partitions and Schur functions [3].

2.1 Partitions

In this paper, partitions are understood to be decreasing sequence of non-negative integers λ_i, $k = 1, 2, \ldots$, in which only a finite number are non-zero:

$$\lambda = (\lambda_1, \lambda_2, \ldots), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad \exists N \; \forall i > N \; \lambda_i = 0.$$

The standard notations

$$l(\lambda) = \max\{i \mid \lambda_i \neq 0\}, \quad |\lambda| = \sum_{i \geq 1} \lambda_i, \quad \kappa(\lambda) = \sum_{i \geq 1} \lambda_i(\lambda_i - 2i + 1)$$

for the length, the weight and the second Casimir value are used throughout this paper. Partitions are in one-to-one correspondence with Young diagrams by identifying the parts λ_i, $i = 1, 2, \ldots$ with the lengths of rows. Let $^t\lambda$ denote the conjugate partition of λ. The parts $^t\lambda_j$, $j = 1, 2, \ldots$, of $^t\lambda$ are the lengths of columns of the

same Young diagram. The length $h(i,j)$ of the hook cornered at $(i,j) \in \lambda$ can be thereby expressed as

$$h(i,j) = \lambda_i - i + {}^t\lambda_j - j + 1.$$

Two partitions λ, μ are said to satisfy the inclusion relation $\lambda \supset \mu$ if the corresponding Young diagrams satisfy the inclusion relation (equivalently, if their parts λ_i, μ_i satisfy the inequalities $\lambda_i \geq \mu_i$ for $i \geq 1$). The difference of those Young diagram is denoted by λ/μ and called a skew Young diagram of shape λ/μ.

2.2 Schur and skew Schur functions

These partitions label the Schur functions $s_\lambda(x)$ and the skew Schur functions $s_{\lambda/\mu}(x)$ of a finite or infinite number of variables $x = (x_1, x_2, \cdots)$. In a combinatorial definition, $s_\lambda(x)$ is a sum of the form

$$s_\lambda(x) = \sum_{T \in \text{SSTab}(\lambda)} x^T, \tag{1}$$

where $\text{SSTab}(\lambda)$ denotes the set of all semi-standard tableaux of shape λ. By definition, a semi-standard tableau of shape λ is an array T of positive integers $T(i,j)$ that are put on the cells $(i,j) \in \lambda$ of the Young diagram and increasing in the rows and strictly increasing in the columns:

$$T(i+1,j) > T(i,j) \leq T(i,j+1).$$

The summand x^T is the monomial

$$x^T = \prod_{(i,j) \in \lambda} x_{T(i,j)}$$

determined by the entries of T. In the same sense, the skew Schur function $s_{\lambda/\mu}(x)$ is defined by the sum

$$s_{\lambda/\mu}(x) = \sum_{T \in \text{SSTab}(\lambda/\mu)} x^T \tag{2}$$

over the set $\text{SSTab}(\lambda/\mu)$ of all semi-standard tableaux of shape λ/μ. The summand is again a monomial of the form

$$x^T = \prod_{(i,j) \in \lambda/\mu} x_{T(i,j)}.$$

When $\lambda \not\supset \mu$, we define $s_{\lambda/\mu}(x) = 0$.

In the case of N-variables, the entries of tableaux are restricted to $[N] = \{1, 2, \ldots, N\}$. Consequently, the Schur functions for partitions of length greater than N vanish,

$$s_\lambda(x_1, \ldots, x_N) = 0 \quad \text{if } l(\lambda) > N, \tag{3}$$

and the non-vanishing ones are given by the finite sum

$$s_\lambda(x_1, \ldots, x_N) = s_\lambda(x_1, \ldots, x_N, 0, 0, \ldots) = \sum_{T \in \mathrm{SSTab}(\lambda, [N])} x^T \tag{4}$$

over the set $\mathrm{SSTab}(\lambda, [N])$ of semi-standard tableaux of shape λ with entries in $[N]$.

The Schur and skew Schur functions are symmetric functions. This fact, far from being obvious in the foregoing definitions (1) and (2), becomes manifest in the Jacobi-Trudi formulae

$$s_{\lambda/\mu}(\boldsymbol{x}) = \det \left(h_{\lambda_i - \mu_j - i + j}(\boldsymbol{x}) \right)_{i,j=1}^{n} = \det \left(e_{{}^t\lambda_i - {}^t\mu_j - i + j}(\boldsymbol{x}) \right)_{i,j=1}^{m}, \tag{5}$$

where n and m are chosen to be such that $n \geq l(\lambda)$ and $m \geq l({}^t\lambda)$. The entries $h_k(\boldsymbol{x})$ and $e_k(\boldsymbol{x})$, $k = 0, 1, 2, \ldots$, of the determinants are the complete and elementary symmetric functions

$$h_k(\boldsymbol{x}) = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}, \quad e_k(\boldsymbol{x}) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k} \quad \text{for } k \geq 1,$$

$$h_0(\boldsymbol{x}) = e_0(\boldsymbol{x}) = 1.$$

2.3 Cauchy identities

The Schur functions satisfy the Cauchy identities

$$\sum_{\lambda \in \mathcal{P}} s_\lambda(\boldsymbol{x}) s_\lambda(\boldsymbol{y}) = \prod_{i,j \geq 1} (1 - x_i y_j)^{-1},$$

$$\sum_{\lambda \in \mathcal{P}} s_\lambda(\boldsymbol{x}) s_{{}^t\lambda}(\boldsymbol{y}) = \prod_{i,j \geq 1} (1 + x_i y_j), \tag{6}$$

where $\mathcal{P}$ denotes the set of all partitions. These identities are generalized to the skew Schur functions as

$$\sum_{\lambda \in \mathcal{P}} s_{\lambda/\mu}(\boldsymbol{x}) s_{\lambda/\nu}(\boldsymbol{y}) = \prod_{i,j \geq 1} (1 - x_i y_j)^{-1} \sum_{\lambda \in \mathcal{P}} s_{\mu/\lambda}(\boldsymbol{y}) s_{\nu/\lambda}(\boldsymbol{x}),$$

$$\sum_{\lambda \in \mathcal{P}} s_{\lambda/\mu}(\boldsymbol{x}) s_{{}^t\lambda/{}^t\nu}(\boldsymbol{y}) = \prod_{i,j \geq 1} (1 + x_i y_j) \sum_{\lambda \in \mathcal{P}} s_{{}^t\mu/{}^t\lambda}(\boldsymbol{y}) s_{\nu/\lambda}(\boldsymbol{x}). \tag{7}$$

When $\mu = \nu = \emptyset$, they reduce to (6). Moreover, since the Schur and skew Schur functions have the homogeneity

$$s_\lambda(Q\boldsymbol{x}) = Q^{|\lambda|} s_\lambda(\boldsymbol{x}), \quad s_{\lambda/\mu}(Q\boldsymbol{x}) = Q^{|\lambda| - |\mu|} s_{\lambda/\mu}(\boldsymbol{x}), \tag{8}$$

one can slightly generalize (6) and (7) as

$$\sum_{\lambda \in \mathcal{P}} Q^{|\lambda|} s_\lambda(x) s_\lambda(y) = \prod_{i,j \geq 1} (1 - Q x_i y_j)^{-1},$$

$$\sum_{\lambda \in \mathcal{P}} Q^{|\lambda|} s_\lambda(x) s_{{}^t\lambda}(y) = \prod_{i,j \geq 1} (1 + Q x_i y_j),$$

(9)

and

$$\sum_{\lambda \in \mathcal{P}} Q^{|\lambda|} s_{\lambda/\mu}(x) s_{\lambda/\nu}(y) = \prod_{i,j \geq 1} (1 - Q x_i y_j)^{-1} \sum_{\lambda \in \mathcal{P}} Q^{|\lambda|} s_{\mu/\lambda}(Qy) s_{\nu/\lambda}(Qx),$$

$$\sum_{\lambda \in \mathcal{P}} Q^{|\lambda|} s_{\lambda/\mu}(x) s_{{}^t\lambda/{}^t\nu}(y) = \prod_{i,j \geq 1} (1 + Q x_i y_j) \sum_{\lambda \in \mathcal{P}} Q^{|\lambda|} s_{{}^t\mu/{}^t\lambda}(Qy) s_{\nu/\lambda}(Qx).$$

(10)

3 Topological vertex and partition functions

In this section, we review the notion of topological vertex and the construction of the amplitudes (or partition functions) of topological string theory on local toric Calabi-Yau 3-folds. We refer details to Mariño's book [1] and references cited therein, in particular, the original paper [2] of Aganagic et al.

3.1 Topological vertex

The topological vertex depends on a common parameter q. We consider this parameter to be a complex number with $|q| > 1$. Relevant generating functions are expanded in negative powers of q.

The topological vertex has the combinatorial expression

$$C_{\alpha\beta\gamma} = s_\beta(q^\rho) q^{\kappa(\gamma)/2} \sum_{\nu \in \mathcal{P}} s_{\alpha/\nu}(q^{{}^t\beta+\rho}) s_{{}^t\gamma/\nu}(q^{\beta+\rho}),$$

(11)

where α, β, γ are partitions on the three (clockwise ordered) legs emanating from the vertex (Figure 1) and ρ is the infinite-dimensional vector

$$\rho = \left(-\frac{1}{2}, -\frac{3}{2}, \ldots, -i + \frac{1}{2}, \ldots \right).$$

The definition of the topological vertex thus contains special values of Schur and skew Schur function at

$$q^\rho = \left(q^{-i+1/2} \right)_{i=1}^\infty, \quad q^{\beta+\rho} = \left(q^{\beta_i - i + 1/2} \right)_{i=1}^\infty, \quad q^{{}^t\beta+\rho} = \left(q^{{}^t\beta_i - i + 1/2} \right)_{i=1}^\infty.$$

These special values, primarily being power series of q^{-1}, become rational functions of q. In particular, $s_\beta(q^\rho)$ has the hook formula

$$s_\beta(q^\rho) = \frac{q^{\kappa(\beta)/4}}{\prod_{(i,j) \in \beta} (q^{h(i,j)/2} - q^{-h(i,j)/2})}.$$

(12)
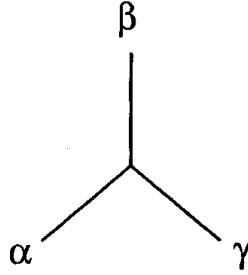
Figure 1: Partitions on legs of topological vertex

Unfortunately, no hook-like formula seems to be known for the special values $s_{\alpha/\nu}(q^{\,{}^t\beta+\rho})$ and $s_{\,{}^t\gamma/\nu}(q^{\beta+\rho})$.

An extremely nontrivial property of the topological vertex is the cyclic symmetry

$$C_{\alpha\beta\gamma} = C_{\beta\gamma\alpha} = C_{\gamma\alpha\beta}. \tag{13}$$

This property can be derived from a "crystal model" [10] of the topological vertex. It seems difficult to prove it directly from the conventional knowledge [3] on the Schur and skew Schur functions. In the special case where $\gamma = \emptyset$, the three quantities in (13) can be expressed as

$$C_{\alpha\beta\emptyset} = s_\beta(q^\rho)s_\alpha(q^{\,{}^t\beta+\rho}), \quad C_{\emptyset\alpha\beta} = s_\alpha(q^\rho)q^{\kappa(\beta)/2}s_{\,{}^t\beta}(q^{\alpha+\rho}),$$
$$C_{\beta\emptyset\alpha} = q^{\kappa(\alpha)/2}\sum_{\nu\in\mathcal{P}}s_{\beta/\nu}(q^\rho)s_{\,{}^t\alpha/\nu}(q^\rho). \tag{14}$$

Replacing $\beta \rightarrow \,{}^t\beta$ and using the hook formula (12), one can reduce the cyclic symmetry in this case to the identities

$$s_\alpha(q^\rho)s_\beta(q^{\alpha+\rho}) = q^{(\kappa(\alpha)+\kappa(\beta))/2}\sum_{\nu\in\mathcal{P}}s_{\,{}^t\alpha/\nu}(q^\rho)s_{\,{}^t\beta/\nu}(q^\rho) = s_\beta(q^\rho)s_\alpha(q^{\beta+\rho}). \tag{15}$$

A direct proof of (15) can be found in Zhou's paper [12].

3.2   Gluing topological vertices

The amplitude of topological string theory on a local toric Calabi-Yau 3-fold X is obtained by "gluing" the vertex weights along the internal edges of the web diagram. The vertex weight $C_{\alpha\beta\gamma}$ itself is the amplitude of the simplest 3-fold $\mathbf{C}^3$.

The internal lines, too, are also weighted. The n-th internal line of the web diagram is given the weight $(-Q_n)^{|\alpha_n|}$, where Q_n is the so called "Kähler parameter", and α_n is the partition assigned to one of the two vertices connected to this internal line. The other vertex have $\,{}^t\alpha_n$ on the same line, because its leg along this line is

given an opposite orientation therein. The negative sign in the weight is also related to this inversion of orientation.

Thus the n-th internal line and the two vertices on both ends altogether have the weight $C_{\alpha_n \beta_n \gamma_n}(-Q_n)^{|\alpha_n|} C^t{}_{\alpha_n \beta'_n \gamma'_n}$. This weight is further modified by the "framing factor" $(-1)^{r_n|\alpha_n|} q^{-r_n \kappa(\alpha_n)/2}$, where r_n is an integer determined by directional vectors v_n, v'_n of the legs carrying β_n, β'_n:

$$r_n = v_n \wedge v'_n = \det(v_n, v'_n).$$

The total amplitude Z, which is referred to as a "partition function" in the following, is obtained by summing the product of these weights over all possible configuration of partitions on the internal lines:

$$Z = \sum_{\alpha_1, \cdots, \alpha_N \in \mathcal{P}} \cdots C_{\alpha_n \beta_n \gamma_n}(-Q_n)^{|\alpha_n|}(-1)^{r_n|\alpha_n|} q^{-r_n \kappa(\alpha_n)/2} C^t{}_{\alpha_n \beta'_n \gamma'_n} \cdots \tag{16}$$

Let us illustrate the construction of the partition function in the case of the resolved conifold $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$. The web diagram has two vertices as shown in Figure 2. We assign the partitions $\alpha_0, \beta_1, \beta_2, \alpha_2$ to the external legs and the partition α_1 and the Kähler parameter Q to the internal line. The partition function $Z = Z^{\alpha_0 \alpha_2}_{\beta_1 \beta_2}$ is a sum of the form

$$Z^{\alpha_0 \alpha_2}_{\beta_1 \beta_2} = \sum_{\alpha_1 \in \mathcal{P}} C_{\alpha_1 \beta_1 \alpha_0}(-Q)^{|\alpha_1|} C^t{}_{\alpha_1 \beta_2 \alpha_2}. \tag{17}$$

(The framing factor vanishes in this case.) When $\alpha_0 = \alpha_2 = \emptyset$, the vertex weight are simplified to those shown in (14):

$$C_{\alpha_1 \beta_1 \emptyset} = s_{\beta_1}(q^\rho) s_{\alpha_1}(q^{{}^t\beta_1 + \rho}), \quad C^t{}_{\alpha_1 \beta_2 \emptyset} = s_{\beta_1}(q^\rho) s^t{}_{\alpha_1}(q^{{}^t\beta_2 + \rho}).$$

The partition function can be thereby written as

$$Z^{\emptyset\emptyset}_{\beta_1 \beta_2} = s_{\beta_1}(q^\rho) s_{\beta_2}(q^\rho) \sum_{\alpha_1 \in \mathcal{P}} (-Q)^{|\alpha_1|} s_{\alpha_1}(q^{{}^t\beta_1 + \rho}) s^t{}_{\alpha_1}(q^{{}^t\beta_2 + \rho}),$$

and, by the Cauchy identities (9), boils down to the well known product formula

$$Z^{\emptyset\emptyset}_{\beta_1 \beta_2} = s_{\beta_1}(q^\rho) s_{\beta_2}(q^\rho) \prod_{i,j=1}^{\infty} (1 - Q q^{{}^t\beta_{1,i} + {}^t\beta_{2,j} - i - j + 1}). \tag{18}$$

3.3   Implications of cyclic symmetry

Let us consider implications of the cyclic symmetry (13) of the vertex weights in the case of the resolved conifold.
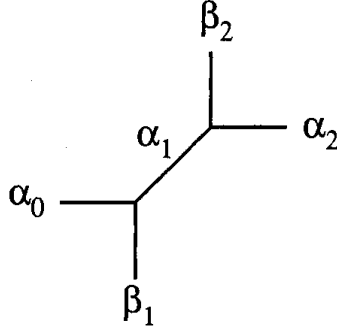
Figure 2: Partitions assigned to web diagram of resolved conifold

According to the consequences (14) of the cyclic symmetry, the vertex weights in the definition (17) of $Z^{\emptyset\emptyset}_{\beta_1\beta_2}$ have another expression:

$$C_{\alpha_1\beta_1\emptyset} = q^{\kappa(\alpha_1)/2} \sum_{\nu_1 \in \mathcal{P}} s_{\beta_1/\nu_1}(q^\rho) s_{{}^t\alpha_1/\nu_1}(q^\rho),$$

$$C_{{}^t\alpha_1\beta_2\emptyset} = q^{\kappa({}^t\alpha_1)/2} \sum_{\nu_2 \in \mathcal{P}} s_{\beta_2/\nu_2}(q^\rho) s_{\alpha_1/\nu_2}(q^\rho).$$

Substituting this expression in (17) and noting the general property

$$\kappa({}^t\lambda) = -\kappa(\lambda)$$

of the second Casimir value, one can rewrite $Z^{\emptyset\emptyset}_{\beta_1\beta_2}$ to the the following triple sum:

$$Z^{\emptyset\emptyset}_{\beta_1\beta_2} = \sum_{\alpha_1,\nu_1,\nu_2\in\mathcal{P}} (-Q)^{|\alpha_1|} s_{\beta_1/\nu_1}(q^\rho) s_{{}^t\alpha_1/\nu_1}(q^\rho) s_{\beta_2/\nu_2}(q^\rho) s_{\alpha_1/\nu_2}(q^\rho).$$

By the Cauchy identities (10) for skew Schur functions, the partial sum over α_1 can be expressed as

$$\sum_{\alpha_1\in\mathcal{P}} (-Q)^{|\alpha_1|} s_{{}^t\alpha_1/\nu_1}(q^\rho) s_{\alpha_1/\nu_2}(q^\rho)$$

$$= \prod_{i,j=1}^{\infty}(1 - Qq^{-i-j+1}) \sum_{\mu\in\mathcal{P}}(-Q)^{|\mu|} s_{\nu_1/{}^t\mu}(-Qq^\rho) s_{\nu_2/\mu}(-Qq^\rho).$$

Thus, up to the pre-factor $\prod_{i,j=1}^{\infty}(1 - Qq^{-i^j+1})$, the partition function turns into another triple sum:

$$Z^{\emptyset\emptyset}_{\beta_1\beta_2} = \prod_{i,j=1}^{\infty}(1 - Qq^{-i-j+1})\times$$

$$\times \sum_{\mu,\nu_1,\nu_2\in\mathcal{P}} (-Q)^{|\alpha_1|} s_{\beta_1/\nu_1}(q^\rho) s_{\nu_1/{}^t\mu}(-Qq^\rho) s_{\beta_2/\nu_2}(q^\rho) s_{\nu_2/\mu}(-Qq^\rho).$$

Let us note that the pre-factor is essentially the inverse of the MacMahon function

$$M(Q,q) = \prod_{n=1}^{\infty}(1 - Qq^n)^{-n} = \prod_{i,j=1}^{\infty}(1 - Qq^{i+j-1})^{-1} \qquad (19)$$

with q replaced by q. It is well known that the MacMahon function is a generating function for weighted enumeration of 3D Young diagrams [10].

The last triple sum is also somewhat remarkable, because the partial sums over ν_1 and ν_2 are special values of the so called "supersymmetric" skew Schur functions

$$s_{\lambda/\mu}(\boldsymbol{x} \mid \boldsymbol{y}) = \sum_{\nu \in \mathcal{P}} s_{\lambda/\nu}(\boldsymbol{x}) s_{{}^t\nu/{}^t\mu}(\boldsymbol{y}). \qquad (20)$$

Thus we find another expression of $Z^{\emptyset\emptyset}_{\beta_1\beta_2}$:

$$Z^{\emptyset\emptyset}_{\beta_1\beta_2} = \prod_{i,j=1}^{\infty}(1 - Qq^{-i-j+1})\sum_{\mu \in \mathcal{P}}(-Q)^{|\mu|}s_{\beta_1/\mu}(q^{\rho} \mid -Qq^{\rho})s_{\beta_2/{}^t\mu}(q^{\rho} \mid -Qq^{\rho}). \qquad (21)$$

The existence of the two expressions (18) and (21) of the partition function $Z^{\emptyset\emptyset}_{\beta_1\beta_2}$ is a special feature of the resolved conifold. In particular, this implies the non-trivial identities

$$s_{\beta_1}(q^{\rho})s_{\beta_2}(q^{\rho}) \prod_{i,j=1}^{\infty} \frac{1 - Qq^{{}^t\beta_{1,i} + {}^t\beta_{2,j} - i - j + 1}}{1 - Qq^{-i-j+1}}$$

$$= \sum_{\mu \in \mathcal{P}}(-Q)^{|\mu|}s_{\beta_1/\mu}(q^{\rho} \mid -Qq^{\rho})s_{\beta_2/{}^t\mu}(q^{\rho} \mid -Qq^{\rho}). \qquad (22)$$

3.4 Generalized conifolds

The product formula (18) is extended by Iqbal and Kashani-Poor [13] to "generalized conifolds", namely, local toric Calabi-Yau 3-folds whose toric diagrams are triangulations of a "strip" (see Figure 3). The associated web diagram is acyclic, and each vertex has a vertical external leg. We assign the partitions $\beta_1, \ldots, \beta_N$ to these vertical legs, the partitions α_0, α_N to the non-vertical legs of the leftmost and rightmost vertices, and the partitions $\alpha_1, \ldots, \alpha_N$ and the Kähler parameters $Q_1, \ldots, Q_N$ to the internal lines. The partition function $Z^{\alpha_0\alpha_N}_{\beta_1\ldots\beta_N}$ is thus given by a sum with respect to $\alpha_1, \ldots, \alpha_N$.

Following the notations of Nagao [14] and Sułkowski [15], we now introduce the indices $\sigma_n = \pm 1$, $n = 1, \ldots, N$, that represent the "type" of the vertices:

- $\sigma_n = +1$ if the vertical leg of the n-th vertex is "down".

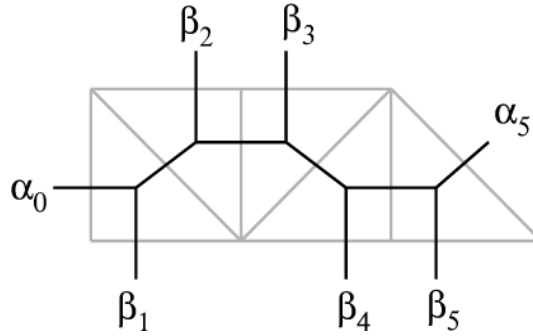- $\sigma_n = -1$ if the vertical leg of the n-th vertex is "up".

Figure 3: Web diagram on triangulated strip

For example, for the web diagram of Figure 3,

$$\sigma_1 = +1, \quad \sigma_2 = -1, \quad \sigma_3 = -1, \quad \sigma_4 = +1, \quad \sigma_5 = +1.$$

With these notations, one can summarize the result of Iqbal and Kashani-Poor [13] into the following beautiful formula:

$$Z^{\emptyset\emptyset}_{\beta_1\cdots\beta_N}$$

$$= s_{\beta_1}(q^\rho)\cdots s_{\beta_N}(q^\rho) \prod_{1\leq m<n\leq N} \prod_{i,j=1}^{\infty} \left(1 - Q_{m,n-1}q^{\,{}^t\beta_i^{(m)}+\beta_j^{(n)}-i-j+1}\right)^{-\sigma_m\sigma_n}. \tag{23}$$

Here we have introduced the abbreviation

$$Q_{m,n} = Q_m Q_{m+1}\cdots Q_n$$

and define $\beta^{(n)}$ as

$$\beta^{(n)} = \begin{cases} \beta_n & \text{if } \sigma_n = +1, \\ {}^t\beta_n & \text{if } \sigma_n = -1. \end{cases}$$

The method of Iqbal and Kashani-Poor is based on a nested use of the Cauchy identities. On the other hand, Nagao [14] and Sułkowski [15], generalizing the work of Eguchi and Kanno [16], presented a formula of $Z^{\alpha_0\alpha_N}_{\beta_1\cdots\beta_N}$ in terms of free fermions and vertex operators [10, 11]. (23) can be derived from the fermionic formula as well. This is also a place where integrable hierarchies come into the game, because the same fermions and vertex operators are also fundamental tools for integrable hierarchies [17].

4 Generating functions of partition functions

The partition functions $Z^{\alpha_0\alpha_N}_{\beta_1\cdots\beta_N}$ are amplitudes of "open" topological string theory. The partitions $\alpha_0, \alpha_N, \beta_1, \ldots, \beta_N$ represent "boundary conditions" to string world

sheets. One can construct amplitudes of "closed" topological string theory by multiplying these open string amplitudes with auxiliary Schur functions and summing over all possible configurations of partitions on the external legs. These generating functions are closely related to integrable hierarchies [4, 5, 6] and quantum mirror curves [7, 8].

4.1 Various generating functions

The partition functions $Z^{\emptyset\emptyset}_{\beta_1,\ldots,\beta_N}$ can be packed into the the generating function

$$Z^{\emptyset\emptyset}(x_1,\ldots,x_N) = \sum_{\beta_1,\ldots,\beta_N \in \mathcal{P}} Z^{\emptyset\emptyset}_{\beta_1,\ldots,\beta_N} s_{\beta_1}(x^{(1)}) \cdots s_{\beta_N}(x^{(N)}) \tag{24}$$

of the N-tuple of variables $x^{(n)} = (x_1^{(1)}, x_2^{(2)}, \ldots)$, $n = 1, \ldots, N$. This function can be specialized to the generating functions

$$Z_n(x) = Z^{\emptyset\emptyset}(\ldots, 0, \underset{(n)}{x}, 0, \ldots) = \sum_{\beta_n \in \mathcal{P}} Z^{\emptyset\emptyset}_{\ldots,\emptyset,\beta_n,\emptyset,\ldots} s_{\beta_n}(x) \tag{25}$$

of a single set of variables $x = (x_1, x_2, \ldots)$. As we shall see below, $Z_n(x)$'s are tau functions of the KP hierarchy [17] with respect to the "time variables"

$$t_k = \frac{1}{k} \sum_{i \geq 1} x_i{}^k, \quad k = 1, 2, \ldots, \tag{26}$$

which are nothing but the so called "power sums" divided by the degree k. Presumably, $Z^{\emptyset\emptyset}(t^{(1)}, \ldots, x^{(N)})$ will be a tau function of the "N-component" KP hierarchy with respect to the N-tuple of time variables

$$t_k^{(n)} = \frac{1}{k} \sum_{i \geq 1} x_i^{(n)k}, \quad k = 1, 2, \ldots, \quad n = 1, \ldots, N,$$

though we do not have a proof.

These are generating functions with the leftmost and rightmost partitions α_0, α_N being suppressed to $\emptyset$. If these partitions are turned on, a new generating function can be obtained:

$$Z_{\beta_1\cdots\beta_N}(y, z) = \sum_{\alpha_0, \alpha_N \in \mathcal{P}} Z^{\alpha_0\alpha_N}_{\beta_1\cdots\beta_N} s_{\alpha_0}(y) s_{\alpha_N}(z). \tag{27}$$

From the fermionic representation of Nagao [14] and Sułkowski [15], one can deduce that this is a tau function of the 2-component KP hierarchy or, rather, the Toda hierarchy (with the lattice coordinate s fixed to $s = 0$). Let us mention that this generating function is similar to the tau function in the melting crystal model of supersymmetric 5D $U(1)$ gauge theory [18, 19].

The most general generating function is, of course, obtained by turning on all partitions:

$$Z(\boldsymbol{y}, \boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(N)}, \boldsymbol{z})$$

$$= \sum_{\alpha_0, \beta_1, \dots, \beta_N, \alpha_N \in \mathcal{P}} Z^{\alpha_0 \alpha_N}_{\beta_1, \dots, \beta_N} s_{\alpha_0}(\boldsymbol{y}) s_{\beta_1}(\boldsymbol{x}^{(1)}) \cdots s_{\beta_N}(\boldsymbol{x}^{(N)}) s_{\alpha_N}(\boldsymbol{z}). \qquad (28)$$

It is natural to expect that this function, too, becomes a tau function of the multi-component KP hierarchy.

4.2 Generating functions as tau functions

Let us explain why $Z_n(\boldsymbol{x})$'s may be thought of as tau functions of the KP hierarchy. This is based on the fact that the coefficients of $Z_n(\boldsymbol{x})$ take the factorized form

$$Z^{\emptyset\emptyset}_{\dots,\emptyset,\beta_n,\emptyset,\dots} = s_{\beta_n}(q^\rho) \prod_{i=1}^{\infty} f_{\beta_{n,i}-i+1} \prod_{i=1}^{\infty} g^{\iota\beta_{n,i}-i+1}, \qquad (29)$$

where

$$f_k = \prod_{m=1}^{n-1} \prod_{j=1}^{\infty} (1 - Q_{m,n-1}q^{k-j})^{-\sigma_m}, \qquad g_k = \prod_{m=n+1}^{N} \prod_{j=1}^{\infty} (1 - Q_{n,m-1}q^{k-j})^{-\sigma_m}$$

if $\sigma_n = +1$ and

$$f_k = \prod_{m=n+1}^{N} \prod_{j=1}^{\infty} (1 - Q_{m,n-1}q^{k-j})^{\sigma_m}, \qquad g_k = \prod_{m=1}^{n-1} \prod_{j=1}^{\infty} (1 - Q_{n,m-1}q^{k-j})^{\sigma_m}$$

if $\sigma_n = -1$.

We first consider the simplified generating function

$$Z(\boldsymbol{x}) = \sum_{\beta \in \mathcal{P}} s_\beta(q^\rho) s_\beta(\boldsymbol{x}). \qquad (30)$$

This amounts to letting $Q_m = 0$ for $m = 1, \dots, N$, in $Z_n(\boldsymbol{x})$. Since $s_{\beta_n}(q^\rho) = C_{\emptyset\beta_n\emptyset}$, this is nothing but the generating function for $\mathbf{C}^3$. By the simplest Cauchy identities (6), one can rewrite $Z(\boldsymbol{x})$ to an infinite product of the form

$$Z(\boldsymbol{x}) = \prod_{i,j=1}^{\infty} (1 - x_i q^{-j+1/2})^{-1} = \prod_{i=1}^{\infty} \Phi_{q^{-1}}(x_i), \qquad (31)$$

where $\Phi_q(x)$ is the the quantum dilogarithmic function

$$\Phi_q(x) = \prod_{j=1}^{\infty} (1 - xq^{j-/2})^{-1} = \exp\left(\sum_{k=1}^{\infty} \frac{q^{k/2}x^k}{k(1-q^k)} \right). \qquad (32)$$

From the exponential form of the quantum dilogarithm, one can see that $Z(x)$ is an exponential function of a linear combination of the KP time variables:

$$Z(x) = \exp\left(\sum_{k=1}^{\infty} \frac{q^{-k/2}t_k}{k(1-q^{-k})}\right) = \exp\left(\sum_{k=1}^{\infty} \frac{t_k}{k[k]}\right).$$ (33)

Here we have introduced the notation

$$[k] = q^{k/2} - q^{-k/2}$$ (34)

that is commonly used in the literature on the topological vertex. Since any exponential function of a linear form of the time variables is a (trivial) tau function, $Z(x)$ is indeed a tau function.

We now return to $Z_n(x)$. The coefficients (29) of its Schur function expansion are obtained from those of $Z(x)$ by multiplying $\prod_{i=1}^{\infty} f_{\lambda_i-i+1} g^{\iota_{\lambda_i-i+1}}$. It is known in the theory of integrable hierarchies [17] that this is a very special type of transformations on the space of tau functions. The tau functions of the KP hierarchy in general have Schur function expansions

$$\tau(x) = \sum_{\lambda \in \mathcal{P}} a_\lambda s_\lambda(x)$$

in which the coefficients a_λ are Plücker coordinates of an infinite dimensional Grassmann manifold (the so called "Sato Grassmannian"). The action of $GL(\infty)$ on this manifold induces transformations of tau functions. In particular, diagonal transformations

$$a_\lambda \mapsto a_\lambda \prod_{i=1}^{\infty} f_{\lambda_i-i+1} \prod_{i=1}^{\infty} g^{\iota_{\lambda_i-i+1}}$$

of the Plücker coordinates are realized by the action of diagonal matrices. Being derived from the (trivial) tau function $Z(x)$ by these transformations, $Z_n(x)$, too, is a tau function.

As regards the more universal generating function $Z^{\emptyset\emptyset}(x^{(1)},\ldots,x^{(N)})$, we are still unble to prove that this is a tau function of the N-component KP hierarchy. In this respect, the case of the resolved conifold is rather special and resemble the generating function (30) for $\mathbf{C}^3$. Let us specify this case.

Recall that the partition function of the resolved conifold has another expression (21). By a nested use of the Cauchy identities (10) for skew Schur functions, one can derive from this expression the following product formula:

$$Z^{\emptyset\emptyset}(x^{(1)}, x^{(2)})$$
$$= \prod_{i,j=1}^{\infty} \frac{(1 - Qq^{-i-j+1})(1 - Qx_i^{(1)}q^{-j+1/2})(1 - Qx_j^{(2)}q^{-i+1/2})(1 - Qx_i^{(1)}x_j^{(2)})}{(1 - x_i^{(1)}q^{-j+1/2})(1 - x_j^{(2)}q^{-i+1/2})}.$$ (35)

This formula can be further converted to the exponential form

$$Z^{\emptyset\emptyset}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})$$
$$= \prod_{i,j=1}^{\infty} (1 - Qq^{-i-j+1}) \exp\left(\sum_{k=1}^{\infty} \frac{(1 - Q^k)(t_k^{(1)} + t_k^{(2)})}{k[k]} - \sum_{k=1}^{\infty} kQ^k t_k^{(1)} t_k^{(2)}\right). \quad (36)$$

This expression shows that $Z^{\emptyset\emptyset}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})$ is a tau function of the Toda hierarchy (hence, of the 2-component KP hierarchy) that is independent of the lattice coordinate s. Actually, this is an "almost trivial" tau function of the Toda hierarchy, just as (30) is a trivial tau function of the KP hierarchy. It is remarkable that such tau functions have a non-trivial structure in the $\boldsymbol{x}$ variables.

4.3 Wave functions as Baker-Akhiezer function

We now specialize the variables in $Z_n(\boldsymbol{x})$ to $\boldsymbol{x} = (x, 0, 0, \dots)$. Since

$$s_\lambda(x, 0, 0, \dots) = \begin{cases} x^k & \text{if } \lambda = (k), \ k = 0, 1, 2, \dots, \\ 0 & \text{otherwise,} \end{cases} \quad (37)$$

the partition β_n in the definition (25) of $Z_n(\boldsymbol{x})$ is restricted to

$$\beta_n = (k), \quad k = 0, 1, 2, \dots.$$

Thus $Z_n(\boldsymbol{x})$ reduces to

$$Z_n(x) = \sum_{k=0}^{\infty} Z^{\emptyset\emptyset}_{\dots,\emptyset,(k),\emptyset,\dots} x^k.$$

Let $\Phi_n(x)$ denote the normalized generating function $Z_n(x)/Z_n(0)$, namely,

$$\Phi_n(x) = 1 + \sum_{k=1}^{\infty} a_k x^k, \quad a_k = \frac{Z^{\emptyset\emptyset}_{\dots,\emptyset,(k),\emptyset,\dots}}{Z^{\emptyset\emptyset}_{\dots,\emptyset,\emptyset,\emptyset,\dots}}. \quad (38)$$

This function is studied by Kashani-Poor [9] as a "wave function" in topological string theory. Actually, this function amounts to the "dual" Baker-Akhiezer function of the KP hierarchy. The genuine Baker-Akhiezer function corresponds to the generating function

$$\Psi_n(x) = 1 + \sum_{k=1}^{\infty} b_k(-x)^k, \quad b_k = \frac{Z^{\emptyset\emptyset}_{\dots,\emptyset,(1^k),\emptyset,\dots}}{Z^{\emptyset\emptyset}_{\dots,\emptyset,\emptyset,\emptyset,\dots}}. \quad (39)$$

of the partition functions restricted to

$$\beta_n = (1^k) = (\underbrace{1, \dots, 1}_{k}).$$

The sign factor $(-1)^k$ is inserted for matching with the usual definition of the Baker-Akhiezer function [17]. In the language of free fermions, $\Phi_n(x)$ and $\Psi_n(x)$ correspond to the fermion fields $\psi^*(x)$ and $\psi(x)$. To be more precise, the usual Baker-Akhiezer functions depend on the time variables $t = (t_1, t_2, \ldots)$ of the KP hierarchy; these time variables are now specialized to $t = 0$.

4.4 q-difference equations for wave functions

Kashani-Poor [9] pointed out, in the case of the resolved conifold, that these "wave functions" satisfy linear q-difference equations. Gukov and Sułkowski [7] interpreted these equations as a realization of the "quantum mirror curve" of the resolved conifold.

One can derive those q-difference equations for the generalized conifolds as well. Since the cases of $\Phi_n(x)$ and $\Psi_n(x)$ are parallel, let us explain the derivation for $\Phi_n(x)$ in detail.

The first thing to do is to express the coefficients of this power series in an explicit form. To this end, apply the formula (23) to the case where

$$\beta_n = (k), \quad \beta_m = \emptyset \quad \text{for } m \neq n.$$

The hook formula (12) implies that $s_{\beta_n}(q^\rho) = s_{(k)}(q^\rho)$ can be expressed as

$$s_{(k)}(q^\rho) = \frac{q^{k(k-1)/4}}{[1][2]\cdots[k]},$$

recall the definition (34) of $[k]$. Thus, after some more algebra, the following expression of the coefficients a_k for $k > 0$ can be obtained:

$$a_k = \frac{q^{k(k-1)/4}}{[1]\cdots[k]} \prod_{1 \leq m < n} \prod_{i=1}^{k} (1 - Q_{m,n-1} q^{\sigma_n(k-i)})^{-\sigma_m \sigma_n} \times$$

$$\times \prod_{n < m \leq N} \prod_{i=1}^{k} (1 - Q_{n,m-1} q^{-\sigma_n(k-i)})^{-\sigma_n \sigma_m}. \tag{40}$$

One can rewrite this somewhat complicated expression of a_k's as

$$a_k = \frac{q^{k(k-1)/4}}{[1]\cdots[k]} \frac{C_n(1)C_n(q)\cdots C_n(q^{k-1})}{B_n(1)B_n(q)\cdots B_n(q^{k-1})}, \tag{41}$$

where $B_n(y)$ and $C_n(y)$ are Laurent polynomials of y:

$$B_n(y) = \prod_{1 \leq m < n,\, \sigma_m \sigma_n > 0} (1 - Q_{m,n-1} y^{\sigma_n}) \prod_{n < m \leq N,\, \sigma_m \sigma_n > 0} (1 - Q_{n,m-1} y^{-\sigma_n}),$$

$$C_n(y) = \prod_{1 \leq m < n,\, \sigma_m \sigma_n < 0} (1 - Q_{m,n-1} y^{\sigma_n}) \prod_{n < m \leq N,\, \sigma_m \sigma_n < 0} (1 - Q_{n,m-1} y^{-\sigma_n}).$$

Thus a_k's turn out to satisfy the recurrence relations

$$a_k = a_{k-1} \frac{q^{(k-1)/2} C_n(q^{k-1})}{[k] B_n(q^{k-1})}. \tag{42}$$

One can thereby derive the q-difference equation

$$[x\partial_x]\Phi_n(x) = x \frac{C_n(q^{x\partial_x})}{B_n(q^{x\partial_x})} q^{x\partial_x/2} \Phi_n(x). \tag{43}$$

Note that $q^{x\partial_x}$ and $[x\partial_x]$, where $\partial_x = \partial/\partial x$, act as q-shift and q-difference operators:

$$q^{x\partial_x} f(x) = f(qx), \quad [x\partial_x]f(x) = (q^{x\partial_x/2} - q^{-x\partial_x/2})f(x) = f(q^{1/2}x) - f(q^{-1/2}x).$$

(43) shows a precise form of the q-difference equations conjectured by Gukov and Sułkowski in a vague form.

In much the same way, the following q-difference equation for $\Psi_n(x)$ can be derived:

$$[-x\partial_x]\Psi_n(x) = x \frac{C_n(q^{-x\partial_x})}{B_n(q^{-x\partial_x})} q^{-x\partial_x/2} \Psi_n(x). \tag{44}$$

Note that this equation is formally related to (43) by the inversion $q \to q^{-1}$ of the parameter q.

Let us illustrate the q-difference equations in the case of the resolved conifold. $\Phi_1(x)$ and $\Phi_2(x)$ in this case are identical and become a power series of the form

$$\Phi(x) = 1 + \sum_{k=1}^{\infty} \frac{q^{k(k-1)/4}(1-Q)(1-Qq^{-1})\cdots(1-Qq^{1-k})}{[1][2]\cdots[k]} x^k. \tag{45}$$

This power series satisfies the q-difference equation

$$\Phi(q^{1/2}x) - \Phi(q^{-1/2}x) = x(1 - Qq^{-x\partial_x})\Phi(q^{1/2}x), \tag{46}$$

which can be rewritten as

$$\Phi(x) = \frac{1 - Qq^{-1/2}x}{1 - q^{-1/2}x}\Phi(q^{-1}x).$$

The last equation implies that that $\Phi(x)$ is an infinite product of the form

$$\Phi(x) = \prod_{n=1}^{\infty} \frac{1 - Qq^{-n+1/2}x}{1 - q^{-n+1/2}x}, \tag{47}$$

hence a quotient of two quantum dilogarithmic functions defined in (32) with q being replaced by q^{-1}.

One can consider the case of $\mathbf{C}^3$ as the "decoupling" limit letting $Q \to 0$. The foregoing results (45), (46) and (47) thereby reduce to the wave function

$$\Phi(x) = 1 + \sum_{k=1}^{\infty} \frac{q^{k(k-1)/4}}{[1][2]\cdots[k]} x^k, \tag{48}$$

the q-difference equation

$$\Phi(q^{1/2}x) - \Phi(q^{-1/2}x) = x\Phi(q^{1/2}x), \tag{49}$$

and the infinite product formula

$$\Phi(x) = \prod_{n=1}^{\infty} (1 - q^{-n+1/2}x)^{-1} \tag{50}$$

for $\mathbf{C}^3$. Thus the wave function in this case is the quantum dilogarithmic function itself.

Let us stress that the infinite product formulae (47) and (50) of the wave functions, which are well known to experts, are a special feature of the resolved conifold and $\mathbf{C}^3$. This feature stems from the simple structure of the q-difference equations (46) and (49). The q-difference equation (44) for other generalized conifolds are more complicated, and presumably do not imply an infinite product formula of solutions.

In the classical ($q \to 1$) limit, the q-difference equation (43) for $\Phi_n(x)$ turns into the equation

$$y^{1/2} - y^{-1/2} = x\frac{C_n(y)}{B_n(y)}y^{1/2} \tag{51}$$

of the "mirror curve". The q-difference equation (44) for $\Psi_n(x)$ yields the same equation with y replaced by y^{-1}. One can rewrite this equation in the form

$$x = \frac{(1 - y^{-1})B_n(y)}{C_n(y)}, \tag{52}$$

which almost agrees with the one conjectured by Gukov and Sułkowski. For example, this equation for the resolved conifold reads

$$x = \frac{1 - y^{-1}}{1 - Qy^{-1}}. \tag{53}$$

Although this equations is slightly different from the usual mirror curve, the discrepancy can be resolved by "framing transformations" [7, 8].

5 Conclusion

The partition functions $Z^{\alpha_0 \alpha_N}_{\beta_1 \cdots \beta_N}$ of topological string theory on a generalized conifold have rich mathematical contents. In this paper, we have mostly considered the case where the partitions α_0, α_N on the leftmost and rightmost external legs of the web diagram are specialized to $\alpha_0 = \alpha_N = \emptyset$.

Armed with the explicit formula (23) of these partition functions and the Cauchy identities, we have shown the following facts:

- The generating functions (or closed string partition function) $Z_n(x)$ defined in (25) are tau functions of the KP hierarchy. The x variables are linked with the KP time variables $t = (t_1, t_2, \ldots)$ as shown in (26). This is a piece of evidence indicating that the generating function $Z^{\emptyset\emptyset}(x^{(1)}, \ldots, x^{(N)})$ of $Z^{\emptyset\emptyset}_{\beta_1 \cdots \beta_N}$ will be a tau function of the N-component KP hierarchy.

- The wave functions $\Phi_n(x)$ and $\Psi_n(x)$ defined in (38) and (39) are the dual pair of Baker-Akhiezer functions of the KP hierarchy specialized to $t = 0$. These wave functions satisfy the q-difference equations (43) and (44). In the classical limit, these q-difference equations turn into the equation (51) or, equivalently, (52), of the mirror curve of the generalized conifold.

- The case of the resolved conifold is very special. Because of the special form of the q-difference equation (46), the wave function has an infinite product form (47). Moreover, the generating function $Z^{\emptyset\emptyset}(x^{(1)}, x^{(2)})$, too, can be factorized as shown in (35). From this factorized form, $Z^{\emptyset\emptyset}(x^{(1)}, x^{(2)})$ turns out to be a very special tau function of the 2-component KP hierarchy. This reasoning cannot be applied to generalized conifolds.

When α_0 and α_N are turned on, we can use the fermionic representation of $Z^{\alpha_0 \alpha_N}_{\beta_1 \cdots \beta_N}$ [14, 15] to show that the generating function $Z_{\beta_1 \cdots \beta_N}(y, z)$ defined by (27) is a tau function of the Toda hierarchy. This link with the Toda hierarchy will lead to a new perspective of quantum mirror curves. This issue will be reported elsewhere.

Acknowledgements

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