REMARKS ON PRIMITIVE IDEMPOTENTS IN COMPACT SEMIGROUPS WITH ZERO[1]

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We recall that a mob is a Hausdorff space together with a continuous associative multiplication. A nonempty subset A of a mob X is a submob if $AA \subset A$. This note consists of an amplification of results of Numakura dealing with primitive idempotents in a compact mob X with zero (see definitions below). We discuss the properties of certain "fundamental" sets determined by primitive idempotents, namely the sets XeX, Xe, eX, and eXe, where e is a primitive idempotent. These are, respectively, the smallest (two-sided, left, right, bi-) ideal containing e. Included in Theorem 1 is a characterization of a primitive idempotent in terms of its "fundamental" sets. There then follow some remarks on the structure of the smallest ideal containing the set of all primitive idempotents.

Finally, if e is a nonzero primitive idempotent of the compact connected mob X with zero, then the set of nilpotent elements of X is dense in each of the "fundamental" sets determined by e.

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We shall assume throughout most of this note that X is a compact mob with zero (0). For $a \in X$ we denote by $\Gamma(a)$ the closure of the set of positive powers of a, and by $K(a)$ the minimal (closed) ideal of $\Gamma(a)$. $K(a)$ is known to be a (topological) group and consists of the cluster points of the set of powers of a ([**3**; **5**]; these results depend only on the compactness of $\Gamma(a)$). Also $\Gamma(a)$ contains exactly one idempotent, e, and if $e=0$ then the powers of a converge to 0. An element a is termed nilpotent if its powers converge to 0, and we denote by N the set of all nilpotent elements of X. A subset A of X is termed *nil* if $A \subset N$. An idempotent e of X is *primitive* if $g=g^2 \in eXe$ implies $g=0$ or $g=e$. Recall that a subset A of X is a *bi-ideal* if (1) $AA \subset A$ and (2) $AXA \subset A$ [**2**; **3**].

LEMMA 1. *Let e be an idempotent of the compact mob X and denote by $\mathcal{M}(e)$ the collection of sets Xf_α, where f_α is an idempotent of XeX. Let $\mathcal{M}(e)$ be partially ordered by inclusion; then Xe is a maximal member of $\mathcal{M}(e)$.*

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PROOF. Suppose f is an idempotent in XeX and $Xe\subset Xf$. Then there are elements a, b of X so that $f=aeb$ and we may assume $ae=a$, $eb=b$. Now $Xe\subset Xf$ implies $e\in Xf$, $ef=e$; hence for each positive integer n, $a^neb^n=a^{n-1}eaebb^{n-1}=a^{n-1}efb^{n-1}=a^{n-1}eb^{n-1}=\cdots=aeb=f$. Hence there is an idempotent $g\in\Gamma(a)$ and an element $h\in\Gamma(b)$ so that $f=geh$ [**5**]. We note $ge=g$, hence $f=gf=(ge)f=g(ef)=ge=g$ and $f=g=ge=fe$. Hence $f\in Xe$ and $Xf\subset Xe$, completing the proof.

We remark that the arguments used in the proof of the lemma are due to Rees [**7**] and Numakura [**5**].

COROLLARY. *Let e be an idempotent of the compact mob X; then eXe is maximal among the sets fXf where $f^2=f\in XeX$.*

PROOF. Suppose $f\in XeX$ and $eXe\subset fXf$. Then $ef=e=fe$ and $e\in Xf$; hence $Xe\subset Xf$ so $Xe=Xf$ by the theorem. Hence $f=fe=e$ and $eXe=fXf$.

LEMMA 2. *Let M be a left (right, two-sided, bi-) ideal of a mob X and suppose $a\in M$ with $\Gamma(a)$ compact; then $\Gamma(a)\subset M$.*

PROOF. We give the proof for a bi-ideal M; the other proofs are similar. Since M is a mob, the set of powers of a is contained in M. Now $aK(a)a\subset MXM\subset M$ and $aK(a)a\subset K(a)$ since $K(a)$ is an ideal in $\Gamma(a)$. Hence $K(a)\cap M\neq\emptyset$, so that $K(a)\subset M$ since no group can properly contain a bi-ideal. It follows that $\Gamma(a)\subset M$.

LEMMA 3. *Let X be a mob with zero and suppose $a\in X$ with $\Gamma(a)$ locally compact; then $a\notin N$ implies $\Gamma(a)\cap N=\emptyset$.*

PROOF. Suppose $x\in\Gamma(a)\cap N$; then $\{x^n\}$ converges to 0, so $0\in\Gamma(x)\subset\Gamma(a)$. Hence $\Gamma(a)$ has a minimal ideal and we have from [**3**] that $\Gamma(a)$ is compact. Since $\Gamma(a)\cap N\neq\emptyset$, it follows that $K(a)\cap N\neq\emptyset$, hence $K(a)=0$ and $a\in N$, a contradiction.

THEOREM 1. *Let e be a nonzero idempotent of the compact mob X with zero; then these are equivalent:*

(1) *e is primitive.*
(2) *$(eXe)\backslash N$ is a group.*
(3) *eXe is a minimal non-nil bi-ideal.*
(4) *Xe is a minimal non-nil left ideal.*
(5) *XeX is a minimal non-nil ideal.*
(6) *each idempotent of XeX is primitive.*

PROOF. (1) implies (2). We first show $(eXe)\backslash N$ is a mob. Since e is assumed primitive, $(eXe)\backslash N$ has a unit e and no other idempotents. Suppose $a, b\in(eXe)\backslash N$ and $ab\in N$; we claim $Xa=Xb=Xe$. According to [3] there is an idempotent $f \in \Gamma(a)$ such that $\bigcap_n Xa^n = Xf$. Now $a \notin N$ implies (Lemma 3) $\Gamma(a) \cap N = \varnothing$, hence $e \in \Gamma(a)$ and $e = f$. Therefore $\bigcap_n Xa^n = Xe$, so $Xa \supset Xe$. Since $a \in eXe \subset Xe$, $Xa \subset Xe$ and the claim is established for a. Similar arguments establish the claim for b. Now using the claim and the fact that e is a unit for a and b one may verify that $X(ab)^n = Xe$ for each positive integer n. Hence $Xe = \bigcap_n X(ab)^n = Xf$ for some idempotent f in $\Gamma(ab)$ (see [3]); but $ab \in N$ implies $f = 0$, $Xe = 0$, and $e = 0$, a contradiction. This shows $(eXe)\backslash N$ is a mob. For $y \in (eXe)\backslash N$ we conclude as above that $e \in \Gamma(y)$; since e is a unit for y it follows that $K(y) = \Gamma(y)$ is a group [3] contained in $(eXe)\backslash N$ by Lemmas 3 and 2. Hence y has an inverse in $(eXe)\backslash N$, completing the proof of (2).

(2) implies (3). Let M be a non-nil bi-ideal of X contained in eXe and choose $a \in M\backslash N$; then $aXa \subset M \subset eXe$. Let f be a nonzero idempotent in aXa; then since $(eXe)\backslash N$ is a group and $f \notin N$, $f = e \in M$. Hence $eXe \subset MXM \subset M$.

(3) implies (4). Let P be a non-nil left ideal of X contained in Xe and choose $a \in P\backslash N$. Then there is a nonzero idempotent $f \in \Gamma(a)$, and $Xf \subset P$. Hence $eXf \subset eP = ePe \subset eXe$. Now since $f \in Xe$, $f = fe$ and $(ef)(ef) = e(fe)f = eff = ef$ so that ef is idempotent. Note that $ef \notin N$, for otherwise $ef = 0$ and $f = (fe)f = f(ef) = 0$. Therefore eXf is a non-nil bi-ideal and hence coincides with eXe. Since $f \in P$, $eXe = eXf \subset P$ and we conclude $e \in P$, $Xe \subset P$.

(4) implies (5). Let M be a non-nil ideal of X contained in XeX, and let f be a nonzero idempotent in M. Then there are elements a, b, of X so that $f = aeb$. Let $g = bae$; then $g^2 = baebae = bfae$ and $g^3 = g^2$. Note that $bf \neq 0$, since otherwise $f = aeb = aebf = 0$. Also $g^2bf = bfaebf = bf$; hence $g^2 \neq 0$, otherwise $bf = 0$. Now $g^2 \in XfX$ and $g^2 \in Xe$, so by (4), $Xe = Xg^2 \subset XfX$ and we conclude $e \in XfX$, $XeX \subset M$.

(5) implies (6). Let f be a nonzero idempotent of XeX and suppose g is a nonzero idempotent with $g \in fXf$ (hence $gXg \subset fXf$). Since $f, g \in XeX$ we have $XgX = XfX = XeX$ and $f \in XgX$. It follows from the corollary to Lemma 1, then, that $gXg = fXf$, hence $g = f$ and f is primitive.

(6) clearly implies (1), completing the proof of the theorem.

Several of the above implications have been demonstrated by Numakura [6].

COROLLARY 1. *Let e be a primitive idempotent of the compact mob X with zero. Then $(Xe)\backslash N$ and $(Xe) \cap N$ are submobs and $(Xe)\backslash N$ is the disjoint union of the maximal (closed) groups $(e_\alpha X e_\alpha)\backslash N$ where e_α runs over the nonzero idempotents of Xe.*

PROOF. Suppose a, $b\in(Xe)\backslash N$ and $ab\in N$. Since Xe is a minimal non-nil left ideal, we know that $Xa=Xe=Xb$. Then as in the proof of (1) implies (2) we conclude $Xe=0$, a contradiction.

Suppose a, $b\in(Xe)\cap N$ and $ab\notin N$. Then $(ab)^2\in Xab$ and $(ab)^2\notin N$, otherwise $ab\in N$ [5, Lemma 3]. Hence $Xab=Xe$ by the theorem; since $a\in Xe$, $Xa\subset Xe$. We have a right translate of Xa filling all of Xe, so according to [3, Corollary 2.2.1] there is an idempotent f in $\Gamma(b)$ which is a right unit for Xe. However $b\in N$ implies $f=0$ so that $Xe=0$, a contradiction.

Finally, pick $a\in(Xe)\backslash N$; by the theorem we have $Xa=Xe$ and by Lemma 3 we have $\Gamma(a)\cap N=\emptyset$. Choose an idempotent f in $\Gamma(a)$; then $Xe=Xf$ so that f is a right unit for Xe. Hence $\Gamma(a)$ is a group, showing that $Xe\backslash N$ is the union of groups. For any nonzero idempotent $e_\alpha\in Xe$, $Xe_\alpha=Xe$ so that e_α is primitive and $(e_\alpha Xe_\alpha)\backslash N$ is a group. Now the maximal group [9] containing e_α is contained in $e_\alpha Xe_\alpha$; moreover, since any group which meets N must be zero, we conclude that $(e_\alpha Xe_\alpha)\backslash N$ is a maximal group. This is closed by the compactness of X, completing the proof.

In [6] Numakura shows that if M is a minimal non-nil ideal, and if J is the largest ideal of X contained in N, then $M-(J\cap M)$, the difference semigroup in the sense of Rees [7], is completely simple (i.e. simple with each idempotent primitive). It follows that $M\backslash N$ is the disjoint union of isomorphic groups, and $M\backslash J=\bigcup[(Xe_\alpha)\backslash J]$ where e_α runs over the nonzero idempotents in M. It would be of interest to know more of the multiplication in $M\backslash J$. Corollary 1 aims in this direction. If $\tilde{E}$ represents the set of primitive idempotents of the compact mob X with zero, then $(X\tilde{E}X)\backslash J=(X\tilde{E})\backslash J$ and $(X\tilde{E}X)\backslash N$ is the disjoint union of groups. (In this connection, see also [1].) At this writing it is not known whether or not $\tilde{E}$ must be a closed set.

As shown in [6], if N is open then there exists a nonzero primitive idempotent. According to Corollary 1, the condition that N be open may be weakened as follows:

COROLLARY 2. *Let X be a compact mob with zero; then X contains a nonzero primitive idempotent if and only if there is a nonzero idempotent e with $(eXe)\backslash N$ closed.*

PROOF. If f is a nonzero primitive idempotent of X, then $(fXf)\backslash N$ is a maximal group and hence is closed. On the other hand, if $(eXe)\backslash N$ is closed and $e\neq 0$, then since the set of nilpotent elements of eXe is $(eXe)\cap N$, we conclude from [6] that eXe contains a nonzero primitive idempotent. Hence so does X, completing the proof.

A five element example due to R. P. Rich [8] serves to illustrate these results; J. G. Wendel has given the following matrix representation of Rich's example:

$$0=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad a=\begin{pmatrix}0&0\\-1&0\end{pmatrix},\quad l=\begin{pmatrix}1&0\\0&0\end{pmatrix},$$

$$r=\begin{pmatrix}0&0\\-1&1\end{pmatrix},\quad s=\begin{pmatrix}1&-1\\0&0\end{pmatrix}.$$

This can be modified to furnish a compact connected example, as follows. Let

$$X=\left\{\begin{pmatrix}0&0\\\delta&\omega\end{pmatrix}\right\}\cup\left\{\begin{pmatrix}t&v\\0&0\end{pmatrix}\right\},$$

where the entries are real numbers between -1 and 1 inclusive. Here N is the totality of those matrices with main diagonal entries in the open interval $(-1, 1)$; J, the largest ideal of X contained in N, is the totality of those matrices with every entry lying in the open interval $(-1, 1)$. It can be shown that $X-J$ is completely simple. If e is one of the four idempotents

$$\begin{pmatrix}1&1\\0&0\end{pmatrix},\quad\begin{pmatrix}1&-1\\0&0\end{pmatrix},\quad\begin{pmatrix}0&0\\1&1\end{pmatrix},\quad\begin{pmatrix}0&0\\-1&1\end{pmatrix},$$

then $(Xe)\backslash J$ consists of two disjoint two-element groups. If e is any other nonzero idempotent, then $(Xe)\backslash J$ consists of one two-element group and one two-element subset whose square lies in J.

THEOREM 2. *Let e be a primitive idempotent of the compact connected mob X with zero; then $(eXe)\cap N$ is dense in eXe (hence $(Xe)\cap N$ is dense in Xe and $(X\tilde{E})\cap N$ is dense in $X\tilde{E}$).*

PROOF. We denote the compact mob eXe by Y, let $N_1=Y\cap N$, and note that N_1 is open in Y in view of Corollary 1 of Theorem 1. Denote by L the largest left ideal of Y contained in N_1; since N_1 is open in Y, so is L [4]. Since L^* (stars denote closure) is a left ideal of Y it follows from the connectedness of Y that $L^*\cap(Y\backslash N)\neq\emptyset$. Hence there is a nonzero idempotent in $L^*\cap(Y\backslash N)$, and this must be e. Therefore $(eXe)e=eXe\subset L^*\subset N_1^*$ so that $(eXe)\cap N$ is dense in eXe. The remainder of the theorem follows from Corollary 1 and the remarks which follow it.

In conclusion we remark that the results of this note can be extended as follows. Let M be an ideal of the mob X. We define an idempotent e to be *M-primitive* if the only idempotents in eXe either coincide with e or else belong to M. Then by replacing N by $N_M \equiv \{a: \Gamma(a) \cap M \neq \emptyset\}$, the results obtained here for primitive idempotents hold for M-primitive idempotents with obvious modifications in statements and proofs; here we need not assume the existence of a zero.

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