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Remarks on Singularities, Dimension and Energy Dissipation for Ideal Hydrodynamics and MHD

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Abstract: For weak solutions of the incompressible Euler equations, there is energy conservation if the velocity is in the Besov space B^3_s with s greater than $1/3$. B^p_s consists of functions that are $Lip(s)$ (i.e., Hölder continuous with exponent s) measured in the L^p norm. Here this result is applied to a velocity field that is $\mathrm{Lip}(\alpha_0)$ except on a set of co-dimension κ_1 on which it is $\mathrm{Lip}(\alpha_1)$, with uniformity that will be made precise below. We show that the Frisch-Parisi multifractal formalism is valid (at least in one direction) for such a function, and that there is energy conservation if $\min_\alpha(3\alpha + \kappa(\alpha)) > 1$. Analogous conservation results are derived for the equations of incompressible ideal MHD (i.e., zero viscosity and resistivity) for both energy and helicity . In addition, a necessary condition is derived for singularity development in ideal MHD generalizing the Beale-Kato-Majda condition for ideal hydrodynamics.

1. Introduction

In turbulent flow at high Reynolds number, the energy dissipation rate is observed to be approximately independent of the coefficient of viscosity. If the Euler equations for ideal hydrodynamics are to correctly describe the infinite Reynolds number limit for turbulent flow, which is a major open question of fluid mechanics, then energy dissipation and singularities must occur in their solutions.

The situation is similar for magneto-hydrodynamics (MHD) at high Reynolds and magnetic Reynolds number [2]. Although the available evidence is not as clear-cut,

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energy dissipation is apparently constant in the ideal limit. In contrast, according to the Taylor conjecture, magnetic helicity does not dissipate in the ideal limit. If the ideal MHD equations are to allow reasonable limits of incompressible MHD, these two observations must be reflected in properties of the solutions.

In 1949 Onsager [12] stated that energy is conserved for weak solutions $\boldsymbol{u} \in Lip(\alpha)$ with $\alpha > 1/3$. This result is contained in a famous paper that initiated the statistical theory of point vortices, and received little attention until the work of Eyink [7], which gave it a rigorous mathematical proof in a certain function class. The proof was considerably simplified and extended to the Besov function space $B_s^p (= B_s^{p,\infty})$ in subsequent work of Constantin, E and Titi [5].

In this note, we shall specialize the result of [5] to explicitly show the dependence on both the degree of singularity of the velocity and the dimension of the singular set. In particular, we consider a velocity which is Lip(α_0) everywhere (i.e. with co-dimension $\kappa_0 = 0$) except on a set of co-dimension κ_1 on which it is Lip(α_1).

Our main result for ideal hydrodynamics, which is stated formally in Corollary 3.1 below, is that there is energy conservation for weak solutions of the Euler equations if

$$\inf_{\alpha}(3\alpha + \kappa(\alpha)) > 1 \tag{1.1}$$

in which the inf is taken over $\alpha = \alpha_0, \alpha_1$. As shown below, this criterion is valid for negative, as well as positive, values of α.

In fact, we show that for this class of functions, the multifractal formalism of Frisch-Parisi [9] is valid (at least in one-direction), and that the functions are in the Besov space B_s^p for any $s > s_p = \inf_\alpha p\alpha + \kappa(\alpha)$ and for all $1 \le p < \infty$. So the energy conservation criterion (1.1), using $p = 3$, then follows from [5]. In fact, the criterion (1.1) is correct whenever the Frisch-Parisi formalism is valid.

The energy conservation criterion (1.1) is implicit in the work of Eyink [8] on multifractals and Besov spaces. Nevertheless, we believe that an explicit statement of this criterion and its validation for a particular class of velocities is noteworthy. In particular, it should be helpful in predicting the type of singularities for Euler flows, and in assessing their fluid dynamic significance if they do occur. Note, however, that there is no proof that the Euler velocity field will have the smoothness described above.

We then present two results on singularities and energy dissipation for ideal incompressible MHD. First, we derive criteria for energy conservation and helicity conservation for weak solutions of ideal MHD. Second, we show that if smooth initial data for the ideal MHD equations leads to a singularity at a finite time t_*, then

$$\int_0^{t_*} ||\boldsymbol{\omega}\,||_\infty + ||\boldsymbol{J}||_\infty dt = \infty \tag{1.2}$$

in which $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is the fluid vorticity, $\boldsymbol{J} = \nabla \times \boldsymbol{B}$ is the electrical current, and $||\ ||_\infty$ is the L^∞ norm in space. This result is analogous to the theorem of Beale-Kato-Majda [1] for singularity formation in ideal hydrodynamics.

2. Singularity and Dimension

Consider a function f, defined on a set $D \subset R^m$, and assume that f is smooth except on a manifold S_0 of co-dimension κ (an integer) on which it is $Lip(\alpha)$; e.g., $f(x) = dist(x, S_0)^\alpha$. Define sets $S(r)$ consisting of points in D within distance r of S_0. Then

$$|S(r)| \equiv vol(S(r)) \leq ar^{\kappa} \tag{2.1}$$

for some constant a, which will be adjusted for use in subsequent bounds. Next consider the difference of $f(x)$ and $f(x+y)$ for two points x and $x+y$ that are at least distance r from S_0, i.e. with $x, x+y \in D - S(r)$. Since the derivative of f blows up like $r^{-(1-\alpha)}$ then

$$|f(x) - f(x+y)| \leq ar^{-(1-\alpha)}|y|. \tag{2.2}$$

Alternatively, f is everywhere $Lip(\alpha)$ if $\alpha \geq 0$, while f is of size r^{α} if $\alpha < 0$; i.e.

$$|f(x) - f(x+y)| \leq \begin{cases} a|y|^{\alpha} & \text{if } \alpha \geq 0 \\ ar^{\alpha} & \text{if } \alpha < 0 \end{cases} \tag{2.3}$$

for $x, x+y \in D - S(r)$.

This can be generalized to a function that is $Lip(\alpha_0)$ in $D - S_0$, , with $\alpha_0 > \alpha_1$, in which case the bounds can be combined as

$$|f(x) - f(x+y)| \leq \Delta(r, \alpha_0, \alpha_1) \tag{2.4}$$

if $x, x+y \in D - S(r)$, in which

$$\Delta(r, \alpha_0, \alpha_1) = \begin{cases} a|y|^{\alpha_0} r^{-(\alpha_0-\alpha_1)} & \text{if } |y| \leq r \\ a|y|^{\alpha_1} & \text{if } r < |y| \text{ and } \alpha_1 \geq 0 \\ ar^{\alpha_1} & \text{if } r < |y| \text{ and } \alpha_1 < 0 \end{cases} \tag{2.5}$$

Definition. *A function f satisfying the bounds (2.4) with $\alpha_0 > \alpha_1$ and $0 < \kappa$ will be said to be in class $Lip(\alpha_0, \alpha_1, 0, \kappa)$.*

Next we derive L^p estimates for any function in $Lip(\alpha_0, \alpha_1, 0, \kappa)$. These estimates show that such functions are in Besov space.

Lemma 2.1. *Let $f \in Lip(\alpha_0, \alpha_1, 0, \kappa_1)$, let $1 \leq p \leq \infty$, and denote $\kappa_0 = 0$. Define*

$$s_p = \min_{i=0,1}(\alpha_i + \kappa_i/p) \tag{2.6}$$

and assume that $s_p > 0$. Then for any $s_p > s > 0$ there is a constant b (depending on $s_p - s$) such that

$$\|f(\cdot + y) - f(\cdot)\|_{L^p} < b|y|^s. \tag{2.7}$$

Proof of Lemma 2.1. First assume that $\alpha_1 \geq 0$ and rewrite the defining inequality (2.4) in a smooth way as

$$|f(x+y) - f(x)| \leq \Delta(r) \equiv a(r + |y|)^{-\alpha_0+\alpha_1}|y|^{\alpha_0} \tag{2.8}$$

for $x, x+y \in D - S(r)$. Also denote

$$\begin{aligned} V(r) &= \text{vol}(S(r)) \leq a(r+|y|)^{\kappa_1} \\ \tilde{V}(r) &= \text{vol}(S(r) \cup (S(r) - y)) \leq 2V(r). \end{aligned} \tag{2.9}$$

Write the integral of the Hölder difference as a Stieljes integral over r, then integrate by parts to estimate (omitting constant factors)

$$\begin{aligned}\int_D |f(x+y)-f(x)|^p dx &\le \int_D \Delta(r)^p dx\\ &= \int_0^1 \Delta(r)^p d\tilde{V}(r)\\ &= -\int_0^1 \frac{\partial}{\partial r}(\Delta(r)^p)\tilde{V}(r)dr + \Delta(1)^p\tilde{V}(1)\\ &\le |y|^{\alpha_0 p}\left\{\int_0^1 (r+|y|)^{-1-p(\alpha_0-\alpha_1)+\kappa_1}dr + 1\right\}\\ &\le |y|^{sp}\begin{cases}\log|y| & \text{if } \alpha_1+\kappa_1/p=\alpha_0\\ 1 & \text{otherwise}\end{cases}\end{aligned} \tag{2.10}$$

in which $s = \min(\alpha_0, \alpha_1 + \kappa_1/p)$. This proves (2.7) for $\alpha_1 \ge 0$.

On the other hand, if $\alpha_1 < 0$ then

$$\Delta(r) = \min(r^{-(\alpha_0-\alpha_1)}|y|^{\alpha_0}, r^{\alpha_1}) \tag{2.11}$$

Then, repeating the first few steps of the previous estimation, the bound becomes

$$\begin{aligned}\int_D |f(x+y)-f(x)|^p dx &= -2\int_0^1 \frac{\partial}{\partial r}(\Delta(r)^p)V(r)dr + 2\Delta(1)^p V(1)\\ &\le \int_0^{|y|} r^{-1+p\alpha_1+\kappa_1}dr + |y|^{\alpha_0 p}\int_{|y|}^1 r^{-1-p(\alpha_0-\alpha_1)+\kappa_1}dr + a|y|^{\alpha_0 p}\\ &\le |y|^{sp}\begin{cases}\log|y| \text{ if } \alpha_1+\kappa_1/p=\alpha_0\\ 1 \text{ otherwise}\end{cases}\end{aligned} \tag{2.12}$$

in which $s = \min(\alpha_0, \alpha_1 + \kappa_1/p) > 0$. This proves (2.7) for $f \in Lip(\alpha_0, \alpha_1, 0, \kappa_1)$.

The Besov spaces are characterized by the L^p bounds proved in Lemma 2.1, which leads to the following result:

Corollary 2.1. *Assume that function $f \in Lip(\alpha_0, \alpha_1, 0, \kappa_1)$ and that $1 \le p < \infty$. Define*

$$s_p = \min(\alpha + \kappa(\alpha)/p). \tag{2.13}$$

If $s_p > 0$, then $f \in B_s^p$ for any $s_p \ge s > 0$.

This is exactly the formula for s_p in the Frisch-Parisi formalism, which shows one-sided validity of the Frisch-Parisi formalism for this function class.

3. Energy Conservation for Ideal Hydrodynamics

For simplicity assume that $D = [0,1]^3$ with periodic boundary conditions. A weak solution of the incompressible Euler equation is a function $\boldsymbol{u} = (u_1, u_2, u_3)$ satisfying

$$\begin{aligned}\int_0^T \int_D u_j\partial_t\psi_j + (\partial_i\psi_j)u_iu_j + (\partial_j\psi_j)p d\boldsymbol{x}\, dt &= \int_D u_j\psi_j(t=0)d\boldsymbol{x}\\ \int_D u_j(\partial_j\varphi)d\boldsymbol{x} &= 0\end{aligned} \tag{3.1}$$

for all test functions $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3) \in C^\infty(D \times \boldsymbol{R}^+)$ and $\varphi \in C^\infty(D)$ with compact support. Energy is conserved for an Euler solution if

$$\int_D |\boldsymbol{u}(x,t)|^2 d\boldsymbol{x} = \int_D |\boldsymbol{u}(x,0)|^2 d\boldsymbol{x} \tag{3.2}$$

for $t \in [0, T]$.

The following energy conservation theorem for ideal hydrodynamics is a consequence of Corollary 2.1 and the theorem of [5].

Corollary 3.1. *(Energy Conservation for Euler). Let $\boldsymbol{u}$ be a weak solution of the Euler equations on $D = [0,1]^3$. Suppose that $\boldsymbol{u} \in C([0,T], B(D))$ in which $B(D) = Lip(\alpha_0, \alpha_1, 0, \kappa_1))$. Then energy is conserved if*

$$\min_i (3\alpha_i + \kappa_i) > 1. \tag{3.3}$$

Note that here and in the next section, the function space $C([0,T], B(D))$ could be replaced by $L^3([0,T], B(D)) \cap C([0,T], L^2(D))$ or something similar, as in [5].

4. Energy Conservation for Ideal MHD

The energy conservation results of [5] can be extended to ideal MHD in a straightforward manner. The equations for ideal MHD are

$$\begin{aligned}
(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -\nabla p - \frac{1}{2}\nabla \boldsymbol{b}^2 + \boldsymbol{b} \cdot \nabla \boldsymbol{b} \\
(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{b} &= \boldsymbol{b} \cdot \nabla \boldsymbol{u} \\
\nabla \cdot \boldsymbol{u} &= \nabla \cdot \boldsymbol{b} = 0.
\end{aligned} \tag{4.1}$$

Actually, incompressibility of $\boldsymbol{b}$ ($\nabla \cdot \boldsymbol{b} = 0$) need only be required at $t = 0$, and it then holds for all t. Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ be functions satisfying the weak form of the ideal MHD equations, namely,

$$\int_0^T \int_D \left[u_j \partial_t \psi_j^{(1)} - (b_i b_j - u_i u_j)\partial_i \psi_j^{(1)} + (p + b^2/2)\partial_i \psi_i^{(1)} \right] d\boldsymbol{x}\, dt = \int_{D,t=0} u_j \psi_j^{(1)} d\boldsymbol{x}$$

$$\int_0^T \int_D \left[b_j \partial_t \psi_j^{(2)} + (\epsilon_{jkl} u_k b_l)(\epsilon_{jmn} \partial_m \psi_n^{(2)}) \right] d\boldsymbol{x}\, dt = \int_{D,t=0} b_j \psi_j^{(2)} d\boldsymbol{x}$$

$$\int_D u_j \partial_j \xi^{(1)} d\boldsymbol{x} = 0 \qquad \int_D b_j \partial_j \xi^{(2)} d\boldsymbol{x} = 0$$

for all test functions $\boldsymbol{\psi}^{(\beta)} = (\psi_1^{(\beta)}, \psi_2^{(\beta)}, \psi_3^{(\beta)}) \in C^\infty(D \times R^+)$ and $\xi^{(\beta)} \in C^\infty(D)$, with $\beta = 1, 2$. Again, the incompressibility condition on $\boldsymbol{b}$ need only be imposed at $t = 0$ and it then follows for all t. In analogy to the conservation of energy for the Euler equations, energy conservation for ideal MHD holds if

$$\int_D \left(|\boldsymbol{u}(\boldsymbol{x}, t)|^2 + |\boldsymbol{b}(\boldsymbol{x}, t)|^2 \right) d\boldsymbol{x} = \int_D \left(|\boldsymbol{u}(\boldsymbol{x}, 0)|^2 + |\boldsymbol{b}(\boldsymbol{x}, 0)|^2 \right) d\boldsymbol{x} \tag{4.2}$$

for $t \in [0, T]$. For simplicity we assume that $D = [0,1]^n$. Whereas singularity formation and energy dissipation is only possible for three-dimensional hydrodynamics, for MHD it is a possibility for dimension $n = 2$ or $n = 3$.

Theorem 4.1. *(Energy Conservation for Ideal MHD). Let $\boldsymbol{u}$ and $\boldsymbol{b}$ be a weak solution of the ideal MHD equations in $D = [0,1]^n$. Suppose that $\boldsymbol{u} \in C([0,T], B_3^{\alpha_1})$ and $\boldsymbol{b} \in C([0,T], B_3^{\alpha_2})$. If*

$$\begin{aligned} \alpha_1 &> 1/3, \\ \alpha_1 + 2\alpha_2 &> 1, \end{aligned} \tag{4.3}$$

then (4.2) holds.

Proof. The proof follows that of [5] but will be briefly repeated here. Define $\varphi_\epsilon(\boldsymbol{x}) = (1/\epsilon^n)\varphi(\boldsymbol{x}/\epsilon)$ to be a positive, smooth mollifier with support in $B(0,1)$ and total mass 1. We make use of the definitions

$$r_\epsilon(f,g)(\boldsymbol{x}) = \int \varphi^\epsilon(y)(\delta_y f(\boldsymbol{x}) \otimes \delta_y g(\boldsymbol{x}))d\boldsymbol{y},$$

$$q_\epsilon(f,g)(\boldsymbol{x}) = \int \varphi^\epsilon(y)(\delta_y f(\boldsymbol{x}) \times \delta_y g(\boldsymbol{x}))d\boldsymbol{y},$$

where $\delta_y h(\boldsymbol{x}) = h(\boldsymbol{x} - \boldsymbol{y}) - h(\boldsymbol{x})$. The proof relies critically on the following identities (first observed in [5]):

$$(f \otimes g)^\epsilon = f^\epsilon \otimes g^\epsilon + r_\epsilon(f,g) - (f - f^\epsilon) \otimes (g - g^\epsilon) \tag{4.4}$$

$$(f \times g)^\epsilon = f^\epsilon \times g^\epsilon + q_\epsilon(f,g) - (f - f^\epsilon) \times (g - g^\epsilon). \tag{4.5}$$

In addition the following estimates hold for functions in B_3^α:

$$||f(\cdot + y) - f(\cdot)||_{L^3} \le c|y|^\alpha, \tag{4.6}$$

$$||\nabla f_\epsilon||_{L^3} \le C\epsilon^{\alpha-1}||f||_{L^3}, \tag{4.7}$$

$$||f - f_\epsilon||_{L^3} \le C\epsilon^\alpha ||f||_{L^3}. \tag{4.8}$$

Using $\psi^{(1)\epsilon}(\boldsymbol{x}) = \int \varphi^\epsilon(\boldsymbol{y} - \boldsymbol{x})\boldsymbol{u}^\epsilon(\boldsymbol{y}, t)d\boldsymbol{y}$ and $\psi^{(2)\epsilon}(\boldsymbol{x}) = \int \varphi^\epsilon(\boldsymbol{y} - \boldsymbol{x})\boldsymbol{b}^\epsilon(\boldsymbol{y}, t)d\boldsymbol{y}$ as test functions results in the equations

$$\begin{aligned} &\int_D |\boldsymbol{u}^\epsilon(\boldsymbol{x}, t)|^2 d\boldsymbol{x} - \int_D |\boldsymbol{u}^\epsilon(\boldsymbol{x}, 0)|^2 d\boldsymbol{x} \\ &\quad = \int_0^t \int_D Tr\left[(\boldsymbol{u} \otimes \boldsymbol{u})^\epsilon \nabla \boldsymbol{u}^\epsilon - (\boldsymbol{b} \otimes \boldsymbol{b})^\epsilon \nabla \boldsymbol{u}^\epsilon\right](\boldsymbol{x}, t)d\boldsymbol{x}\, dt \\ &\int_D |\boldsymbol{b}^\epsilon(\boldsymbol{x}, t)|^2 d\boldsymbol{x} - \int_D |\boldsymbol{b}^\epsilon(\boldsymbol{x}, 0)|^2 d\boldsymbol{x} \\ &\quad = \int_0^t \int_D \left[(\boldsymbol{u} \times \boldsymbol{b})^\epsilon \cdot \nabla \times \boldsymbol{b}^\epsilon\right](\boldsymbol{x}, t)d\boldsymbol{x}\, dt. \end{aligned}$$

The identities (4.6), (4.7) and (4.8) then yield the estimates

$$\begin{aligned} &\left| \int_D |\boldsymbol{u}^\epsilon(\boldsymbol{x}, t)|^2 + |\boldsymbol{b}^\epsilon(\boldsymbol{x}, t)|^2 d\boldsymbol{x} - \int_D |\boldsymbol{u}^\epsilon(\boldsymbol{x}, 0)|^2 + |\boldsymbol{b}^\epsilon(\boldsymbol{x}, 0)|^2 d\boldsymbol{x} \right| \\ &\quad \le \int_0^t \int_D \Big| Tr\Big[\big(r_\epsilon(\boldsymbol{u}, \boldsymbol{u}) - r_\epsilon(\boldsymbol{b}, \boldsymbol{b}) - (\boldsymbol{u} - \boldsymbol{u}^\epsilon) \otimes (\boldsymbol{u} - \boldsymbol{u}^\epsilon) \\ &\qquad + (\boldsymbol{b} - \boldsymbol{b}^\epsilon) \otimes (\boldsymbol{b} - \boldsymbol{b}^\epsilon)\big) \nabla \boldsymbol{u}^\epsilon\Big] \Big| d\boldsymbol{x}\, d\tau \end{aligned}$$

$$+\int_0^t \int_D | \left(q_\epsilon(\boldsymbol{u}, \boldsymbol{b}) - (\boldsymbol{u} - \boldsymbol{u}^\epsilon) \times (\boldsymbol{b} - \boldsymbol{b}^\epsilon)\right) \cdot \nabla \times \boldsymbol{b}^\epsilon | d\boldsymbol{x}\, d\tau$$
$$\le \int_0^t \Big[\Big(||r_\epsilon(\boldsymbol{u}, \boldsymbol{u})||_{3/2}^{2/3} + ||r_\epsilon(\boldsymbol{b}, \boldsymbol{b})||_{3/2}^{2/3}$$
$$+ ||\boldsymbol{u} - \boldsymbol{u}^\epsilon||_{3/2}^{2/3} + ||\boldsymbol{b} - \boldsymbol{b}^\epsilon||_{3/2}^{2/3} \Big) ||\nabla \boldsymbol{u}^\epsilon||_3^{1/3}$$
$$+ \Big(||q_\epsilon(\boldsymbol{u}, \boldsymbol{b})||_{3/2}^{2/3} + ||\boldsymbol{u} - \boldsymbol{u}^\epsilon||_{3/2}^{1/3} ||\boldsymbol{b} - \boldsymbol{b}^\epsilon||_{3/2}^{1/3} \Big) ||\nabla \boldsymbol{u}^\epsilon||_3^{1/3} \Big] d\tau$$
$$\le C_1 \epsilon^{3\alpha_1 - 1} + C_2 \epsilon^{\alpha_1 + 2\alpha_2 - 1}.$$

The result (4.2) follows in the limit $\epsilon \to 0$, which finishes the proof of Theorem 4.1.

A similar theorem for magnetic helicity can be proven. The time evolution of the magnetic helicity for smooth ideal MHD is given by

$$\int_D [\boldsymbol{a}_t \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{b}_t]\, d\boldsymbol{x}$$
$$= \int_D [\boldsymbol{b} \cdot (\boldsymbol{u} \times \boldsymbol{b}) + \boldsymbol{b} \cdot \nabla \alpha + \boldsymbol{a} \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{b})]\, d\boldsymbol{x}$$
$$= \int_D [\boldsymbol{b} \cdot \nabla \alpha + \boldsymbol{a} \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{b})]\, d\boldsymbol{x},$$
$$= 0$$

where α is some smooth function and $\boldsymbol{b} = \nabla \times \boldsymbol{a}$. Then for $\boldsymbol{\psi} \in C^\infty(D \times R^+)$,

$$\int_0^T \int_D (\nabla \times \boldsymbol{\psi}(\boldsymbol{x}, t)) \cdot (\boldsymbol{u}(\boldsymbol{x}, t) \times \boldsymbol{b}(\boldsymbol{x}, t)) d\boldsymbol{x}\, dt = 0 \tag{4.9}$$

implies weak conservation of helicity. Using arguments identical to those of the previous proof we obtain

Theorem 4.2. *(Magnetic Helicity Conservation for Ideal MHD). Let $\boldsymbol{u}$ and $\boldsymbol{b}$ be a weak solution of the ideal MHD equations in $D = [0,1]^n$. Suppose that $\boldsymbol{u} \in C([0,T], B_3^{\alpha_1})$ and $\boldsymbol{b} \in C([0,T], B_3^{\alpha_2})$. If $\alpha_1 + 2\alpha_2 > 0$, then (4.9) holds.*

In 2 dimensions the magnetic helicity vanishes identically. In its place the quantity $\int_D \boldsymbol{a}^2 d\boldsymbol{x}$ serves as a non-trivial invariant. In 2 dimensions, $\boldsymbol{a}$ satisfies (up to a gradient)

$$\boldsymbol{a}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{a} = 0$$

and we have

Theorem 4.3. *Let $\boldsymbol{u}$ and $\boldsymbol{b}$ be a weak solution of the ideal MHD equations in $D = [0,1]^2$. Suppose that $\boldsymbol{u} \in C([0,T], B_3^{\alpha_1})$and $\boldsymbol{a} \in C([0,T], B_3^{\alpha_2+1})$. If $\alpha_1 + 2\alpha_2 > -1$, then $\int_D \boldsymbol{a}^2 d\boldsymbol{x}$ is conserved.*

We remark that Theorems 4.1, 4.2, and 4.3 specialize easily to functions $\boldsymbol{u}$ and $\boldsymbol{b}$ in Lip$(\alpha_0, \alpha_1, 0, \kappa_1)$, as in Corollary 3.1 . In these cases the bounds of Theorem 4.1 become $s_1 > 1/3$, $s_1 + 2s_2 > 1$, the bound for Theorem 4.2 becomes $s_1 + 2s_2 > 0$, and the bound for Theorem 4.3 becomes $s_1 + 2s_2 > -1$. Here

$$s_1 = \min_{\alpha_1}(\alpha_1 + \kappa_1(\alpha_1)/3),$$
$$s_2 = \min_{\alpha_2}(\alpha_2 + \kappa_2(\alpha_2)/3),$$

where κ_1, κ_2 are defined as in the introduction. For the commonly observed phenomenon of codimension 1 current sheets, $\kappa_2 = 1$ so that $s_1 + 2\alpha_2 > 1/3$ implies energy conservation and $s_1 + 2\alpha_2 > -2/3$ implies helicity conservation ($-5/3$ in 2D). We also remark that while the fluid result (1.1) picks out the Kolmogorov exponent $1/3$ naturally, the classical MHD exponent (namely $1/4$ [10, 11]), while consistent with the bounds of Theorems 4.1 and 4.2, does not drop out as naturally. This should not be a surprise since important non-local MHD effects are not included in the argument. Additionally, Theorems 4.1 and 4.2 are consistent with recent intermittency models (see, e.g., [3]).

Analogous results can be obtained in terms of the Elsasser (characteristic) variables $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$ for the MHD equations. The system (4.1) can be rewritten as

$$\begin{aligned} (\partial_t + \boldsymbol{z}^{+} \cdot \nabla)\boldsymbol{z}^{-} &= -\nabla \Pi, \\ (\partial_t + \boldsymbol{z}^{-} \cdot \nabla)\boldsymbol{z}^{+} &= -\nabla \Pi, \\ \nabla \cdot \boldsymbol{z}^{\pm} &= 0 \end{aligned} \tag{4.10}$$

in which $\Pi = p + \frac{1}{2}\boldsymbol{b}^{2}$.

The following theorem gives two variants of the previous energy conservation result for MHD.

Theorem 4.4. *(Energy Conservation for Ideal MHD in Characteristic Variables). For a weak solution of the MHD equations in* $[0,1]^n$*, there is energy conservation if either of the following conditions are satisfied:*

(i) For some p, q *with values in* $(1, \infty)$ *and with* $1/p + 2/q = 1$

$$\begin{aligned} \boldsymbol{u} &\in C([0,T], B_3^{\alpha_0} \cap B_p^{\alpha_1}), \\ \boldsymbol{b} &\in C([0,T], B_q^{\alpha_2}), \end{aligned} \tag{4.11}$$

in which

$$\begin{aligned} 3\alpha_0 &> 1, \\ \alpha_1 + 2\alpha_2 &> 1. \end{aligned} \tag{4.12}$$

(ii) For some p_i, q_i *(*$i = 1, 2$*) with values in* $(1, \infty)$ *and with* $1/p_1 + 2/q_1 = 2/p_2 + 1/q_2 = 1$*,*

$$\begin{aligned} \boldsymbol{z}^{+} &\in C([0,T], B_{p_1}^{\alpha_1} \cap B_{p_2}^{\alpha_2}), \\ \boldsymbol{z}^{-} &\in C([0,T], B_{q_1}^{\beta_1} \cap B_{q_2}^{\beta_2}) \end{aligned} \tag{4.13}$$

in which

$$\begin{aligned} \alpha_1 + 2\beta_1 &> 1, \\ 2\alpha_2 + \beta_2 &> 1. \end{aligned} \tag{4.14}$$

Similar statements can be made with regards to magnetic helicity.

5. Singularity Formation for Ideal MHD

We will show the analogue of the Beale-Kato-Majda theorem for ideal MHD.

Theorem 5.1. *For the system (4.1) with initial data* $\boldsymbol{u}_0, \boldsymbol{b}_0 \in H^s$*, with* $s \geq 3$*, the solution* $\boldsymbol{u}(t), \boldsymbol{b}(t)$ *is in the class*

$$C([0,T], H^s) \cap C^1([0,T], H^{s-1})$$

as long as

$$\int_0^T |\boldsymbol{\omega}(t)|_\infty + |\boldsymbol{j}(t)|_\infty dt < \infty,$$

and

$$\int_0^T |\nabla \times z^+|_\infty + |\nabla \times z^-|_\infty dt < \infty.$$

(The 2 inequalities are in fact equivalent.)

Here $\boldsymbol{j} = \nabla \times \boldsymbol{b}$ and H^s is the L_2 Sobolev space. The approach closely follows that of [1]. Assume that

$$\int_0^T |\nabla \times \boldsymbol{z}^+|_\infty + |\nabla \times \boldsymbol{z}^-|_\infty dt = M < \infty. \tag{5.1}$$

The proof consists of three parts: First, we derive energy estimates on $|\boldsymbol{z}^\pm|_s$ in terms of $|\nabla \boldsymbol{z}^\pm|_\infty$. Second, we estimate $|\nabla \times \boldsymbol{z}^\pm|_{L_2}$. Finally, we utilize an inequality derived in [1] and Gronwall's lemma to bound $|\boldsymbol{z}^\pm|_s$.

5.1. Energy estimates. We begin by deriving energy estimates for the system (4.10) with $t \in [0,T]$. Let α be a multi-index with $|\alpha| \leq s$. Let $\boldsymbol{\eta} = D_x^\alpha \boldsymbol{z}^+$. Apply D_x^α to the second equation in (4.10) to obtain

$$(\partial_t + \boldsymbol{z}^- \cdot \nabla)\boldsymbol{\eta} = -\nabla \Pi' - \boldsymbol{F}$$

in which $\Pi' = D_x^\alpha \Pi$ and

$$\boldsymbol{F} = D^\alpha[(\boldsymbol{z}^- \cdot \nabla \boldsymbol{z}^+)] - \boldsymbol{z}^- \cdot D^\alpha \nabla \boldsymbol{z}^+.$$

A bound on $\boldsymbol{F}$ in the L_2 norm can be based on the general inequality

$$|D^\alpha(fg) - fD^\alpha g|_{L^2} \leq c(|f|_s |g|_\infty + |\nabla f|_\infty |g|_{s-1}),$$

which was derived in [1] based on the Gagliardo-Nirenberg inequalities. Application of this to $\boldsymbol{F}$ yields

$$|\boldsymbol{F}|_{L^2} \leq c(|\boldsymbol{z}^-|_s |\nabla \boldsymbol{z}^+|_\infty + |\nabla \boldsymbol{z}^-|_\infty |\nabla \boldsymbol{z}^+|_{s-1}). \tag{5.2}$$

This leads to the following bound on $\boldsymbol{\eta}$

$$\frac{d}{dt}|\boldsymbol{\eta}|_{L^2}^2 \leq c(|\boldsymbol{z}^-|_s |\nabla \boldsymbol{z}^+|_\infty + |\nabla \boldsymbol{z}^-|_\infty |\nabla \boldsymbol{z}^+|_{s-1})|\boldsymbol{\eta}|_{L^2}.$$

Summing over α leads to

$$\frac{d}{dt}|\boldsymbol{z}^{\,+}|_s^2 \le c(|z^-|_s|\nabla \boldsymbol{z}^{\,+}|_\infty + |\nabla \boldsymbol{z}^{\,-}|_\infty |\boldsymbol{z}^{\,+}|_s)|\boldsymbol{z}^{\,+}|_s. \tag{5.3}$$

There is a similar result for $\boldsymbol{z}^{\,-}$; i.e.,

$$\frac{d}{dt}|\boldsymbol{z}^{\,-}|_s^2 \le c(|\boldsymbol{z}^{\,+}|_s|\nabla \boldsymbol{z}^{\,-}|_\infty + |\nabla \boldsymbol{z}^{\,+}|_\infty |\boldsymbol{z}^{\,-}|_s)|\boldsymbol{z}^{\,-}|_s. \tag{5.4}$$

Add these two inequalities to obtain

$$\frac{d}{dt}(|\boldsymbol{z}^{\,-}|_s^2 + |\boldsymbol{z}^{\,+}|_s^2) \le c(|\nabla \boldsymbol{z}^{\,+}|_\infty + |\nabla \boldsymbol{z}^{\,-}|_\infty)(|\boldsymbol{z}^{\,+}|_s^2 + |\boldsymbol{z}^{\,-}|_s^2), \tag{5.5}$$

and thus

$$|\boldsymbol{z}^{\,+}|_s^2 + |\boldsymbol{z}^{\,-}|_s^2 \le (|\boldsymbol{z}_0^{\,+}|_s^2 + |\boldsymbol{z}_0^{\,-}|_s^2)\exp\left(C\int_0^t (|\nabla \boldsymbol{z}^{\,+}|_\infty + |\nabla \boldsymbol{z}^{\,-}|_\infty) d\tau\right). \tag{5.6}$$

5.2. L^2 bounds on $\nabla \times z_\pm$. Take the curl of (4.10) to obtain

$$\begin{aligned}(\partial_t + \boldsymbol{z}^{\,+}\cdot\nabla)\boldsymbol{\zeta}^{\,-} &= \nabla \boldsymbol{z}^{\,+} A \nabla \boldsymbol{z}^{\,-},\\ (\partial_t + \boldsymbol{z}^{\,-}\cdot\nabla)\boldsymbol{\zeta}^{\,+} &= \nabla \boldsymbol{z}^{\,-} A \nabla \boldsymbol{z}^{\,+}.\end{aligned} \tag{5.7}$$

where $\boldsymbol{\zeta}^{\,\pm} = \nabla \times \boldsymbol{z}^{\,\pm}$ and A is a constant matrix. Multiplying the first equation in (5.7) by $\boldsymbol{\zeta}^{\,-}$ and integrating gives

$$\begin{aligned}\frac{d}{dt}|\boldsymbol{\zeta}^{\,-}|_{L^2}^2 &\le C\int |\nabla \boldsymbol{z}^{\,+}|\,|\nabla \boldsymbol{z}^{\,-}|\,|\boldsymbol{\zeta}^{\,-}|\,dx\\ &\le C|\boldsymbol{\zeta}^{\,-}|_\infty(|\nabla \boldsymbol{z}^{\,+}|_{L^2}|\nabla \boldsymbol{z}^{\,-}|_{L^2})\\ &\le C|\boldsymbol{\zeta}^{\,-}|_\infty(|\nabla \boldsymbol{z}^{\,+}|_{L^2}^2 + |\nabla \boldsymbol{z}^{\,-}|_{L^2}^2).\end{aligned} \tag{5.8}$$

Since $\nabla\cdot\boldsymbol{z}^{\,\pm} = 0,\ \boldsymbol{z}^{\,\pm}$ and $\boldsymbol{\zeta}^{\,\pm}$ are related by

$$\boldsymbol{z}^{\,\pm} = -\nabla \times (\Delta^{-1}\boldsymbol{\zeta}^{\,\pm})$$

and their Fourier transforms are related by $(\nabla \boldsymbol{z}^{\,\pm})(k) = S(k)\boldsymbol{\zeta}^{\,\pm}(k)$ where $S(k)$ is bounded independent of k. Thus $|\nabla \boldsymbol{z}^{\,\pm}|_{L^2} \le C|\boldsymbol{\zeta}^{\,\pm}|_{L^2}$, so that (5.8) leads to

$$\frac{d}{dt}|\boldsymbol{\zeta}^{\,-}|_{L^2}^2 \le C|\boldsymbol{\zeta}^{\,-}|_\infty(|\boldsymbol{\zeta}^{\,+}|_{L^2}^2 + |\boldsymbol{\zeta}^{\,-}|_{L^2}^2).$$

We obtain a similar result for $\boldsymbol{\zeta}^{\,+}$; that is

$$\frac{d}{dt}|\boldsymbol{\zeta}^{\,+}|_{L^2}^2 \le C|\boldsymbol{\zeta}^{\,+}|_\infty(|\boldsymbol{\zeta}^{\,+}|_{L^2}^2 + |\boldsymbol{\zeta}^{\,-}|_{L^2}^2).$$

Add these two equations to obtain

$$\frac{d}{dt}(|\boldsymbol{\zeta}^{\,+}|_{L^2}^2 + |\boldsymbol{\zeta}^{\,-}|_{L^2}^2) \le c(|\boldsymbol{\zeta}^{\,+}|_\infty + |\boldsymbol{\zeta}^{\,-}|_\infty)(|\boldsymbol{\zeta}^{\,+}|_{L^2}^2 + |\boldsymbol{\zeta}^{\,-}|_{L^2}^2)$$

so that

$$|\zeta^+|_{L^2}^2 + |\zeta^-|_{L^2}^2 \leq (|\zeta_0^+|_{L^2}^2 + |\zeta_0^-|_{L^2}^2 \exp\left(C\int_0^t (|\zeta^+(\tau)|_\infty + |\zeta^-(\tau)|_\infty) d\tau \right).$$

By Assumption (5.1) we have

$$|\zeta^+|_{L^2}^2 + |\zeta^-|_{L^2}^2 \leq \overline{M}(|\zeta_0^+|_{L^2}^2 + |\zeta_0^-|_{L^2}^2), \tag{5.9}$$

where $\overline{M} = \exp(CM)$.

5.3. Final estimates. In [1] it was proved, via the Biot-Savart law, that

$$|\nabla f|_\infty \leq C\{1 + (1 + \log^+ |f|_3)|\nabla \times f|_\infty + |\nabla \times f|_{L^2}\} \tag{5.10}$$

where

$$\log^+ a = \begin{cases} \log a & \text{if } a \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.11}$$

Thus

$$\begin{aligned} |\nabla z^+|_\infty + |\nabla z^-|_\infty \leq C\{1 &+ (1 + \log^+ |z^+|_3)|\zeta^+|_\infty + |\zeta^+|_{L^2} \\ &+ (1 + \log^+ |z^-|_3)|\zeta^-|_\infty + |\zeta^-|_{L^2}\}. \end{aligned}$$

Using the result from Section 5.2, we have

$$\begin{aligned} &|\nabla z^+|_\infty + |\nabla z^-|_\infty \\ &\leq C\{1 + (|\zeta^+|_\infty + |\zeta^-|_\infty)(\log^+ |z^+|_3 + \log^+ |z^-|_3 + 2). \end{aligned}$$

Combining this with the result from Sect. 5.2 gives

$$\begin{aligned} |z^+|_s + |z^-|_s \leq c(|z_0^+|_s + |z_0^-|_s) \exp\Big\{C\int_0^t &\big[(1 + (|\zeta^+|_\infty + |\zeta^-|_\infty)) \\ &(\log(|z^+|_3 + e) + \log(|z^-|_3 + e))\big]\, d\tau\Big\}. \end{aligned}$$

Let $y^\pm(t) = \log(|z^\pm|_s + e)$ then

$$\begin{aligned} y^+(t) + y^-(t) \leq \log c(|z_0^+|_s + |z_0^-|_s) \\ + C\int_0^t (1 + (|\zeta^+|_\infty + |\zeta^-|_\infty)(y^+(\tau) + y^-(\tau))d\tau. \end{aligned}$$

Application of Gronwall's lemma then shows that $y^+(t)+y^-(t)$ is bounded by a constant which depends only on M, T and $|z^\pm(0,\cdot)|_s$. This concludes the proof of Theorem 5.1.

6. Conclusions

At present, there are only a few analytical results on singularities in ideal hydrodynamics: The Beale-Kato-Majda theorem is a necessary condition for the formation of singularities from smooth initial data. Constantin [4] and Constantin & Fefferman [6] have obtained additional necessary conditions in terms of the geometry of the vorticity field. Finally, Onsager's energy conservation criterion provides a necessary condition for energy dissipation due to singularities in an ideal fluid.

The first part of this paper has refined Onsager's criterion by explicitly showing the effect of singularity type and dimension on the necessary condition for energy dissipation. The result is an example of the Frisch-Parisi multi-fractal formalism, which has been proved to be valid for functions in the class Lip$(\alpha_0, \alpha_1, 0, \kappa_1)$.

In the remaining parts of the paper two analytical results–the Beale-Kato-Majda theorem and Onsager's energy conservation theorem–have been extended to ideal MHD. Since energy dissipation but helicity conservation are expected, this suggests a limited range of values for the uniform singularity spectrum in MHD. The appearance of the Elsasser variables $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ in the extension of the Beale-Kato-Majda inequality should also be noted.

We expect these results to be useful in two ways: First, as a sufficient condition for regularity of ideal hydrodynamic and MHD solutions. They should also serve as a guide in investigation of possible singularities and their physical significance. For example in 3D hydrodynamics with singularities of type α on a smooth set S, nonzero energy dissipation requires $\alpha \leq 0$ for a 2D singularity surface (κ = 1), $\alpha \leq -1/3$ for a curve of singularities (κ = 2), and $\alpha \leq -2/3$ for a point singularity (κ = 3). In particular, in the point and curve cases, infinite velocities are required.

These results also help to indicate the relation between the smoothness of $\boldsymbol{b}$ and that of $\boldsymbol{u}$. Theorem 5.1 suggests that $\boldsymbol{b}$ and $\boldsymbol{u}$ should have the same degree of smoothness, while Theorems 4.1, 4.2, and 4.3 suggest a tradeoff between smoothness of $\boldsymbol{u}$ and that of $\boldsymbol{b}$.

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