Remarks on Symplectic Connections

Simone Gutt
Université Libre de Bruxelles, Campus Plaine CP 218,
Bvd du Triomphe, B-1050 Brussels, Belgium
Université de Metz, Département de mathématiques,
Ile du Saulcy, F-57045 Metz Cedex 01, France
sgutt@ulb.ac.be

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Abstract

This note contains a short survey on some recent work on symplectic connections: properties and models for symplectic connections whose curvature is determined by the Ricci tensor, and a procedure to build examples of Ricci-flat connections. For a more extensive survey, see [5]. This note also includes a moment map for the action of the group of symplectomorphisms on the space of symplectic connections, an algebraic construction of a large class of Ricci flat symmetric symplectic spaces, and an example of global reduction in a non symmetric case.

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Introduction

In view of Darboux's theorem, symplectic geometry is by essence global. Considering symplectic connections (which, on a given symplectic manifold (M,ω) form an infinite dimensional affine space) is nevertheless justified by their importance in deformation quantization and by deep links between conditions on a symplectic connection (curvature conditions for example) and the geometry of the manifold.

The paper is organized as follows. In section 1, I define a symplectic structure on the space of symplectic connections and a moment map for the action of the group of symplectomorphisms on this space. Section 2 is devoted to an algebraic study of the symplectic curvature tensor and to the definition of two types of symplectic connections: Ricci-type and Ricci-flat. In section 3, I recall how all local models of Ricci-type connections can be built by a Marsden-Weinstein reduction procedure of a constraint surface given by a quadratic equation in a standard symplectic vector space. This reduction procedure is global for a class of quadratic polynomials, giving rise to all Ricci-type symmetric symplectic spaces. I give an example of global reduction giving a non symmetric connection. An induction procedure (which is a sort of inverse of the reduction procedure above), described in section 4, gives examples of Ricci-flat connections. I also give a purely algebraic construction of Ricci flat symmetric symplectic spaces.

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1 The space of symplectic connections

A **symplectic connection** [15, 17] on a symplectic manifold (M,ω) is a torsion free linear connection ∇ on M for which the symplectic 2–form ω is parallel.

To see the existence of such a connection, take ∇^0 to be any torsion free linear connection (for instance, the Levi Civita connection associated to a metric g on M). Consider the tensor N on M defined by $\nabla^0_X\omega(Y,Z) =: \omega(N(X,Y),Z)$ where X,Y,Z are vector fields on M (i.e. $\in \chi(M)$). Since ω is closed, one has $\underset{XYZ}{\circlearrowleft\!\!\!\!\!\!\oplus}\ \omega(N(X,Y),Z) = 0$, where $\circlearrowleft\!\!\!\!\!\!\oplus$ denotes the sum over the cyclic permutations of the listed set of elements. Define

$$\nabla_X Y := \nabla^0_X Y + \frac{1}{3}N(X,Y) + \frac{1}{3}N(Y,X).$$

Then ∇ is a symplectic connection on (M,ω).

To see how (non)-unique is a symplectic connection, take ∇ symplectic; then any other linear connection reads $\nabla'_X Y := \nabla_X Y + A(X)Y$ where A is a 1–form with values in the endomorphisms of the tangent bundle. The connection ∇' is torsion free iff $A(X)Y = A(Y)X$ and is symplectic if furthermore $0 = \nabla'_X\omega(Y,Z) = -\omega(A(X)Y,Z) - \omega(Y,A(X)Z)$ hence iff

$$\underline{A} := \omega(A(X)Y,Z)$$

is totally symmetric.

This says that **the space** $\mathscr{E}(M,\omega)$ **of symplectic connections** on (M,ω) is an affine space modelled on the space of symmetric covariant 3–tensorfields on M; the choice of a particular symplectic connection $\tilde{\nabla}$, i.e. a base point in $\mathscr{E}(M,\omega)$, allows us the identification:

$$\mathscr{E}(M,\omega) = \tilde{\nabla} + \Gamma^\infty(S^3T^*M).$$

The space of symplectic connections has thus a structure of linear Fréchet space.

1.1 Moment map on the space of symplectic connections

There is a natural symplectic structure on the space $\mathscr{E}(M,\omega)$ of symplectic connections on M. The tangent space to $\mathscr{E}(M,\omega)$ at a "point" ∇ is identified with the space $\Gamma^\infty(S^3T^*M)$ of smooth symmetric covariant 3–tensorfields on M:

$$\underline{A}(X,Y,Z) = \frac{d}{dt}_{|_0} \omega(\nabla^t_X Y, Z).$$

If M is compact, we may define at each point ∇ of $\mathscr{E}(M,\omega)$ an alternate 2-form Ω_∇ on the tangent space $\Gamma^\infty(S^3T^*M)$ by

$$\Omega_\nabla(\underline{A},\underline{B}) := \int_M (\underline{A},\underline{B}) \frac{\omega^n}{n!} \tag{1}$$

where $(\cdot,\cdot)$ denotes the pairing of symmetric covariant 3–tensorfields induced by ω; thus, in a chart, $(\underline{A},\underline{B})(x) = (\omega_x^{-1})^{i_1 j_1}(\omega_x^{-1})^{i_2 j_2}(\omega_x^{-1})^{i_3 j_3}\underline{A}_{i_1 i_2 i_3}(x)\underline{B}_{j_1 j_2 j_3}(x)$.

If M is not compact, we can still give a meaning to the above expression. Let J be a smooth almost complex structure on M compatible with ω (i.e. $\omega(JX,JY) = \omega(X,Y)$ and $\omega(X,JX) > 0$ if $X \neq 0$); this always exists. Let g be the corresponding Riemannian structure (i.e. $g(X,Y) = \omega(X,JY)$). Then, if $\underline{A}^J$ is the 3-form:

$$\underline{A}^J(X,Y,Z) = \underline{A}(JX,JY,JZ), \tag{2}$$

the pairing is given by

$$(\underline{A},\underline{B}) = \underline{A}^J \cdot \underline{B} \tag{3}$$

where $\cdot$ indicates the scalar product of 3-covariant tensors induced by g. If $\underline{A}$ and $\underline{B}$ are smooth tensor fields which are L^2 (in the sense that $\|\underline{A}\|^2 = \int_M \underline{A} \cdot \underline{A} \frac{\omega^n}{n!} < \infty$), then formula (1) has a meaning using Cauchy-Schwarz and the fact that $\|\underline{A}^J\|^2 = \|\underline{A}\|^2$. Thus the expression makes sense provided one restricts to elements $\underline{A}$ in $T_\nabla\mathscr{E}(M,\omega)$ which decrease "sufficiently fast at ∞ on M".

In any case the 2–form Ω defines a **symplectic structure on the space** $\mathscr{E}(M,\omega)$ in the following sense: if $\Omega_\nabla(\underline{A},\underline{B}) = 0\ \forall \underline{B}$ then $\underline{A} = 0$; and Ω_∇ is a constant 2–form, hence closed.

The group $\mathscr{G}$ of symplectic diffeomorphisms of (M,ω) acts naturally on $\mathscr{E}(M,\omega)$:

$$(g\cdot\nabla)_X Y(x) := g_{*_{g^{-1}x}}\left(\nabla_{g_*^{-1}X} g_*^{-1} Y\right). \tag{4}$$

It clearly preserves the symplectic 2-form Ω.
We want to study when the action of $\mathscr{G}$ on $\mathscr{E}(M,\omega)$ posseses a moment map. Let us recall what a moment map is in the finite dimensional context. Let G be a finite dimensional Lie group acting by symplectomorphisms on a finite dimensional symplectic manifold (M',ω'). For any $X \in \mathfrak{g}$, we denote by $X^{*M'}$ the corresponding

fundamental vector field on M', i.e. $X_x^{*M'} = \frac{d}{dt} \exp -tX \cdot x_{|_0}$. The action has a moment map if there exists a map from M' in the dual of the Lie algebra of G,

$$J : M' \to \mathfrak{g}^*, \quad \text{such that} \quad < J(x), X >= \lambda_X(x) \quad \forall X \in \mathfrak{g} \tag{5}$$

where λ_X is a smooth function on M' so that

$$d\lambda_X = i(X^{*M})\omega' \qquad \text{and} \qquad \{\lambda_\mathrm{X}, \lambda_\mathrm{Y}\} = \lambda_{[\mathrm{X},\mathrm{Y}]} \tag{6}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket on $C^\infty(M')$ induced by ω', i.e.

$$\{f, g\} := \omega'(X_f, X_g)$$

where X_f is the Hamiltonian vector field corresponding to f, i.e. such that $i(X_f)\omega' = df$.

In our situation the Lie algebra of $\mathscr{G}$ consists of smooth symplectic vector fields on M; if X is such a vector field $\mathscr{L}_X\omega = 0 \Leftrightarrow di(X)\omega = 0$. The corresponding vector field $X^{*\mathscr{E}}$ on $\mathscr{E}(M,\omega)$ is such that

$$\underline{X}_\nabla^{*\mathscr{E}}(Y,Z,U)(x) = \omega_x((\mathscr{L}_X\nabla)_Y Z, U), \tag{7}$$

where $(\mathscr{L}_X\nabla)_Y Z = [X, \nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y[X,Z]$ and one checks easily that $\underline{X}_\nabla^{*\mathscr{E}}$ is indeed a completely symmetric covariant 3–tensor.

We look for a moment map, at least in a formal sense, for the symplectic action of $\mathscr{G}$ on $\mathscr{E}(M,\omega)$.

Assume first that M is simply connected and compact; if X is a symplectic vector field on M, then there exists a function f_X on M so that

$$i(X)\omega = df_X.$$

The function f_X is defined up to an additive constant and one can choose the constant so that

$$\int_M f_X \frac{\omega^n}{n!} = 0.$$

Thus there is a linear isomorphism between the space of symplectic vector fields and the space $\mathscr{C}^\infty_{(0)}(M)$ of smooth functions on M having 0 mean.
Now, if X, Y are symplectic vector fields:

$$i([X,Y])\omega = [i(X), \mathscr{L}_Y]\omega = -\mathscr{L}_Y i(X)\omega = -\mathscr{L}_Y df_X = -d\mathscr{L}_Y f_X = d\{f_X, f_Y\}$$

and clearly

$$\int_M \{f_X, f_Y\} \frac{\omega^n}{n!} = \int_M \mathscr{L}_X(f_Y \frac{\omega^n}{n!}) = 0$$

so the linear isomorphism is a Lie algebra isomorphism.

If M is simply connected and non compact, let us consider the group $\mathscr{G}_1$ of symplectic diffeomorphisms of (M,ω) which are the identity outside a compact set. The algebra of this group is the space of compactly supported symplectic vector fields on M. If X is such a field

$$i(X)\omega = d f_X$$

where the function f_X is defined up to an additive constant; there exists a compact set K such that $d f_X = 0$ on $M \setminus K$. Assume that M is "simple at ∞"; by this we mean that M satisfies the following topological condition: for any compact set K', there exists a compact set $\tilde{K}$ such that

$$(i)\ \tilde{K} \supset K' \qquad\qquad (ii)\ M \setminus \tilde{K} \text{ is connected.}$$

With this assumption, we can choose the constant in such a way that f_X is compactly supported.
Hence we have a linear isomorphism between the space of compactly supported symplectic vector fields on (M,ω) and the space $\mathscr{C}_0^\infty(M)$ of compactly supported smooth functions on M.

We consider both cases: M compact and simply connected or, resp. M non compact, simply connected, "simple at ∞". If X is a symplectic vector field on M (resp. a compactly supported symplectic vector field on M):

$$\left(i(X_\nabla^{*\mathscr{E}})\Omega_\nabla\right)(\underline{A}) = \int_M \left(\underline{\mathscr{L}_X \nabla}, \underline{A}\right) \frac{\omega^n}{n!} \tag{8}$$

and the integral makes sense.

A moment map should be a map $\mathscr{J}$ from the space $\mathscr{E}(M,\omega)$ of symplectic connections, with values in the dual of the algebra $\mathscr{C}_{(0)}^\infty(M)$ (resp. $\mathscr{C}_0^\infty(M)$). To avoid difficulties, we shall look for a moment map with values in a subspace of the dual. The space $\mathscr{C}^\infty(M)$ can be identified with a subspace of the dual of $\mathscr{C}_{(0)}^\infty(M)$ (resp. $\mathscr{C}_0^\infty(M)$). Thus we are looking for a map

$$\mathscr{J} : \mathscr{E}(M,\omega) \to \mathscr{C}^\infty(M) \tag{9}$$

so that, for any symplectic vector field X (resp. any compactly supported vector field X) on M, the smooth function $\mathscr{J}(\nabla)$ on M satisfies

$$< \mathscr{J}(\nabla), f_X > := \int_M \mathscr{J}(\nabla) f_X \frac{\omega^n}{n!} = \phi_X(\nabla) \tag{10}$$

where ϕ_X is a real function on $\mathscr{E}(M,\omega)$ such that

$$\frac{d}{dt}\phi_X(\nabla + tA)_{|_0} = \Omega_\nabla(\underline{X}^{*\mathscr{E}}, \underline{A}) = \int_M \left(\underline{\mathscr{L}_X \nabla}, \underline{A}\right) \frac{\omega^n}{n!}. \tag{11}$$

Observe that

$$\begin{aligned}(\mathscr{L}_X\nabla)_Y Z &= [X,\nabla_Y Z]-\nabla_{[X,Y]}Z-\nabla_Y[X,Z]\\ &= \nabla_X\nabla_Y Z-\nabla_{\nabla_Y Z}X-\nabla_{[X,Y]}Z-\nabla_Y\nabla_X Z+\nabla_Y\nabla_Z X\\ &= R^{\nabla}(X,Y)Z+(\nabla^2_{(Y,Z)}X),\end{aligned}$$

with R^{∇} the curvature tensor of ∇ (i.e. $R^{\nabla}(X,Y)Z=\left(\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}\right)Z$). We want to solve the relation

$$\begin{aligned}\frac{d}{dt}\int_M \mathscr{J}(\nabla+tA)\,f_X\frac{\omega^n}{n!}|_0 &= \int_M\left(\underline{\mathscr{L}_X\nabla},\underline{A}\right)\frac{\omega^n}{n!}=-\int_M(\omega^{-1})^{uv}((\mathscr{L}_X\nabla)_{\partial_q}\partial_u)^t A^q_{vt}\frac{\omega^n}{n!}\\ &= -\int_M(\omega^{-1})^{uv}(X^sR^{\nabla t}_{squ}+(\nabla^2_{qu}X)^t)A^q_{vt}\frac{\omega^n}{n!}\end{aligned}$$

with $\nabla_{\partial_j}f_X=\omega_{rj}X^r$ and with sumation over repeated indices.

Proposition 1.1 *(see also [13]) Let (M,ω) be a simply connected compact symplectic manifold (resp. a simply connected symplectic manifold "simple at ∞").*
Consider the space $\mathscr{E}(M,\omega)$ of smooth symplectic connections on (M,ω) (resp. choose a symplectic connection $\tilde{\nabla}$ on (M,ω) and look at the space of connections of the form $\tilde{\nabla}+A$ where A is smooth and in L^2). This space admits a natural symplectic structure.
Consider the group $\mathscr{G}$ of symplectic diffeomorphisms of (M,ω) (resp. the group of symplectic diffeomorphisms reducing to the identity outside a compact set).
Then the action of $\mathscr{G}$ on $\mathscr{E}(M,\omega)$ is symplectic and admits a moment map $\mathscr{J}$, with values in the space of smooth functions on M, given by formula

$$\mathscr{J}(\nabla)=-\frac{1}{2}r^{\nabla}_{pq}r^{\nabla pq}+\frac{1}{4}R^{\nabla}_{pqrs}R^{\nabla pqrs}-(\nabla^2_{pq}r^{\nabla})^{pq}. \tag{12}$$

where one sums over repeated indices, where indices are lifted via the components of the inverse matrix of the one given by the components of ω, where r^{∇} is the Ricci tensor of ∇ (i.e. $r^{\nabla}(X,Y)=\mathrm{Tr}\left(Z\mapsto R^{\nabla}(X,Z)Y\right)$) and, where, for any tensor A, the second covariant derivative is defined by $\nabla^2_{pq}A=\nabla^2_{(\partial_p,\partial_q)}A$ and

$$\nabla^2_{(X,Y)}A=\nabla_X(\nabla_YA)-\nabla_{\nabla_XY}A.$$

Indeed one checks that

$$\begin{aligned}\frac{d}{dt}R^{\nabla+tA}(X,Y)Z|_0 &= \nabla_X(A(Y)Z)+A(X)\nabla_YZ\\ &\qquad -\nabla_Y(A(X)Z)-A(Y)\nabla_XZ-A([X,Y])Z\\ &= (\nabla_XA)(Y)Z-(\nabla_YA)(X)Z;\\ \frac{d}{dt}r^{\nabla+tA}(X,Y)|_0 &= \mathrm{Tr}\,(Z\mapsto(\nabla_XA)(Z)Y-(\nabla_ZA)(X)Y)\\ &= -\mathrm{Tr}\,(Z\mapsto(\nabla_ZA)(X)Y)\,.\end{aligned}$$

We also have:

$$\frac{d}{dt}(\nabla^2_{pq} r^{\nabla+tA})^{pq}|_0 = (\nabla_q A)^q_{tp} r^{\nabla tp} + A^q_{tp}(\nabla_q r^{\nabla})^{tp} - (\nabla^3_{pqr} A)^{pqr}$$

so that

$$\begin{aligned}\frac{d}{dt}\mathscr{J}(\nabla+tA)|_0 &= r^{\nabla pq}(\nabla_s A)^s_{pq} + \frac{1}{2} R^{\nabla pqrs}((\nabla_p A)_{qrs} - (\nabla_q A)_{prs}) \\ &\quad -((\nabla_q A)^q_{tp} r^{\nabla tp} + A^q_{tp}(\nabla_q r^{\nabla})^{tp} - (\nabla^3_{pqr} A)^{pqr}).\end{aligned}$$

Remark 1.2 This moment map is formal and has up to now no geometrical interpretation.

1.2 Symplectic connections and Deformation Quantization

Symplectic connections are closely related to natural formal deformation quantizations at order 2. Flato, Lichnerowicz and Sternheimer introduced deformation quantization in [11] (see also [2]); quantization of a classical system is a way to pass from classical to quantum results and they "suggest that quantization be understood as a deformation of the structure of the algebra of classical observables rather than a radical change in the nature of the observables." In that respect, they introduce a star product which is a formal deformation of the algebraic structure of the space of smooth functions on a symplectic (or more generally a Poisson) manifold; the associative structure given by the usual product of functions and the Lie structure given by the Poisson bracket are simultaneously deformed.

Definition 1.3 A **star product** on a symplectic manifold (M,ω) is a bilinear map

$$C^\infty(M)\times C^\infty(M)\to C^\infty(M)[[\nu]] \quad (u,v)\mapsto u *_\nu v := \sum_{r\geq 0} \nu^r C_r(u,v) \tag{13}$$

such that
$(u*v)*w = u*(v*w)$ (when extended $\mathbb{R}[[\nu]]$ linearly);
$C_0(u,v) = uv \qquad C_1(u,v) - C_1(v,u) = \{u,v\}$;
$1*u = u*1 = u$.
If all the C_r's are bidifferential operators, one speaks of a **differential star product**; if, furthermore, each C_r is of order $\leq r$ in each argument, one speaks of a **natural star product**.

The link between symplectic connections and star products already appears in the seminal paper [2] where the authors observe that if there is a flat symplectic connection ∇ on (M,ω), one can generalise the classical formula for Moyal star product

$*_M$ defined on $\mathbb{R}^{2n}$ with a constant symplectic 2-form. A construction of a star product on a symplectic manifold associated to any symplectic connection was given by Fedosov:

Theorem 1.4 *[10] Given a symplectic connection ∇ and a sequence $\Omega = \sum_{k=1}^{\infty} \nu^k \omega_k$ of closed 2-forms on a symplectic manifold (M, ω), one can build a star product $*_{\nabla,\Omega}$ on it.*

This is obtained by identifying the space $C^\infty(M)[[\nu]]$ of formal series in a parameter ν of smoonth functions on the manifold, with a subalgebra of the algebra of sections of a bundle in associative algebra called the Weyl bundle on M. The subalgebra is that of flat sections of the Weyl bundle, when this is endowed with a flat connection whose construction is related to the choices made of the connection on M and of the sequence of closed 2-forms on M.

Reciprocally a natural star product determines a symplectic connection. This was first observed by Lichnerowicz [16] for a restricted class of star products.

Theorem 1.5 *[12] A natural star product at order 2 determines a unique symplectic connection.*

2 Choice of particular symplectic connections

There are examples of symplectic manifolds where there is a natural choice of a unique symplectic connection, preserving some extra data on the manifold.

Pseudo Kähler manifolds

Given a symplectic manifold (M, ω), one can choose an almost complex compatible structure J i.e. as before $J : TM \to TM$ is a bundle endomorphism so that $J^2 = -\mathrm{Id} \quad \omega(JX, JY) = \omega(X,Y)$ and $\omega(X, JX) > 0$ if $X \neq 0$.

Lemma 2.1 *A symplectic connection ∇ preserves J (in the sense that $\nabla J = 0$) iff it is the Levi Civita connection associated to the pseudo Riemannian metric g defined by $g(X,Y) = \omega(X, JY)$; so it is unique and it only exits in a (pseudo)Kähler situation.*

Symmetric symplectic space

A symmetric symplectic space is a symplectic manifold with symmetries attached to each of its points. Precisely:

Definition 2.2 A **symmetric symplectic space** is a triple (M,ω,S) where (M,ω) is a symplectic manifold and where S is a smooth map $S : M\times M \to M$ such that, defining for any point $x\in M$ the map (called the symmetry at x):

$$s_x := S(x,\cdot) : M \to M, \tag{14}$$

each s_x squares to the identity [$s_x^2 = \mathrm{Id}$] and is a symplectomorphism of (M,ω) [$s_x^*\omega = \omega$], x is an isolated fixed point of s_x, and $s_x s_y s_x = s_{s_x y}$ for any $x,y\in M$.

Lemma 2.3 *On a symmetric symplectic space, there exists a unique symplectic connection for which each symmetry s_x is an affinity. It is given by*

$$\omega_x(\nabla_X Y, Z) = \frac{1}{2} X_x \omega(Y + s_{x\star}Y, Z). \tag{15}$$

On a symmetric symplectic space, the deformation quantisation constructed via Fedosov using this unique connection has the symmetries acting as automorphisms.

A natural way to select a subclass of symplectic connections on any symplectic manifold is to impose further conditions on its curvature.

2.1 Curvature tensor for a symplectic connection

The **curvature tensor** R^∇ of a symplectic connection ∇ can be viewed as a 2-form on M with values in the endomorphisms of the tangent bundle and $R_x^\nabla(X,Y)$ has values in the symplectic Lie algebra $sp(T_xM,\omega_x) = \{A\in \mathrm{End}(T_xM) \mid \omega_x(Au,v)+\omega_x(u,Av) = 0\ \forall u,v\in T_xM\}$.

The **Ricci tensor** r^∇ is the 2-tensor $r^\nabla(X,Y) = \mathrm{Tr}\big(Z\mapsto R^\nabla(X,Z)Y\big)$. The first Bianchi identity ($\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\bigoplus} R^\nabla(X,Y)Z = 0$) implies that the Ricci tensor r^∇ is symmetric.

The second possible trace of the curvature tensor $r'_x(X,Y) := \sum_i \omega(R_x^\nabla(e_i,e^i)X,Y)$, where the e_i constitute a basis of T_xM and the e^i constitute the dual basis of T_xM defined by $\omega(e_i,e^j) = \delta_i^j$, is proportional to the Ricci tensor. Indeed, the first Bianchi identity implies that $r' = -2r^\nabla$. Since the Ricci tensor is symmetric and one only has a skewsymmetric contravariant 2-tensor on M (the Poisson tensor related to the symplectic form) there is **no "scalar curvature"**.

The **symplectic curvature tensor** is defined as

$$\underline{R}^\nabla(X,Y,Z,T) := \omega(R^\nabla(X,Y)Z,T); \tag{16}$$

it satisfies $\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\bigoplus} \underline{R}_x^\nabla(X,Y,Z,T) = 0$, it is skewsymmetric in its first two arguments and symmetric in its last two. We denote by $\underline{\mathscr{R}}_x$ the space of 4-tensors on T_xM satisfying those algebraic identities. The group $Sp(T_xM,\omega_x) = \{\ A\in \mathrm{End}(V) \mid \omega_x(Au,Av) = \omega_x(u,v)\ \forall u,v\in V\ \}$ acts on T_xM, and thus on $\underline{\mathscr{R}}_x$. Under this action the space $\underline{\mathscr{R}}_x$ in dimension $2n \geq 4$, decomposes into two irreducible subspaces [18] $\underline{\mathscr{R}}_x = \underline{\mathscr{E}}_x \oplus \underline{\mathscr{W}}_x$. The coresponding decomposition of the curvature tensor reads

$$R^{\nabla}_x = E^{\nabla}_x + W^{\nabla}_x, \tag{17}$$

where

$$\begin{aligned} E^{\nabla}(X,Y)Z \quad = \quad & \tfrac{1}{2n+2}\Big(2\omega(X,Y)\rho^{\nabla}Z + \omega(X,Z)\rho^{\nabla}Y - \omega(Y,Z)\rho^{\nabla}X \\ & +\omega(X,\rho^{\nabla}Z)Y - \omega(Y,\rho^{\nabla}Z)X\Big) \end{aligned} \tag{18}$$

with r^{∇} converted into an endomorphism $\omega(X,\rho^{\nabla}Y) = r^{\nabla}(X,Y)$.

Definition 2.4 A symplectic connection ∇ on (M,ω) will be said to be **of Ricci-type** if $W^{\nabla} = 0$; it will be said to be **Ricci-flat** if $E^{\nabla} = 0$ (hence iff $r^{\nabla} = 0$).

A twistorial interpretation of the Ricci-type condition:

Let $J(M) \to M$ be the bundle of compatible positive almost complex structures on the tangent bundle; a symplectic connection on M defines an almost complex structure on $J(M)$; it is integrable iff the connection is of Ricci-type. [19]

2.2 Variational principle to select some symplectic connections

If one tries to select symplectic connections through a variational principle [6], one way is to build a Lagrangian $L(R^{\nabla})$, which is a polynomial in the curvature of the connection, invariant under the action of the symplectic group, and consider the functional

$$\int_M L(R^{\nabla})\omega^n.$$

There is no invariant polynomial of degree 1 in the curvature, so the easiest choice is a polynomial of degree 2 in R^{∇}. The space of degree 2 polynomials in the curvature which are invariant under the action of the symplectic group is 2-dimensional and spanned by (E^{∇},E^{∇}) and (W^{∇},W^{∇}) (or, equivalently by (R^{∇},R^{∇}) and (r^{∇},r^{∇})) where $(\cdot,\cdot)$ denotes, as before, the (symmetric) function-valued product of (even) tensors induced by ω.
Precisely, if S and T are even tensors-fields on M of the same type, (S,T) is given in local coordinates as

$$(S,T) = (\omega^{-1})^{i_1 i'_1} \cdots (\omega^{-1})^{i_p i'_p} \omega_{j_1 j'_1} \cdots \omega_{j_q j'_q} S^{j_1 \dots j_q}_{i_1 \dots i_p} T^{j'_1 \dots j'_q}_{i'_1 \dots i'_p}.$$

One oberves that a combinaison of the corresponding Lagrangian multiplied by the volume form (ω^n) gives

$$P_1(\nabla) \wedge \omega^{n-2} = \frac{1}{16\pi^2}[(r^\nabla, r^\nabla) - \tfrac{1}{2}(R^\nabla, R^\nabla)]\omega^n$$

where P_1 is the first Pontryagin class of the manifold M. Hence, all non trivial Euler equations related to a variational principle built from a second order invariant polynomial in the curvature are the same:

$$\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\bigoplus} (\nabla_X r^\nabla)(Y,Z) = 0. \tag{19}$$

Definition 2.5 A symplectic connection ∇ is said to be **preferred** if it is a solution of the equation 19.

The space of preferred connections on a 2-dimensional symplectic manifold has been studied in [6]. In higher dimension, only partial results are known.

3 Reduction and Ricci-type connections

3.1 A construction by reduction

Let $(M = \mathbb{R}^{2n+2}, \Omega')$ be the standard symplectic vector space. Let A be a nonzero element in the symplectic Lie algebra $sp(\mathbb{R}^{2n+2}, \Omega')$. Let Σ_A be the closed hypersurface $\Sigma_A \subset \mathbb{R}^{2n+2}$ defined by

$$\Sigma_A = \{x \in \mathbb{R}^{2n+2} \,|\, \Omega'(x, Ax) = 1\}. \tag{20}$$

(In order for Σ_A to be non empty we replace, if necessary, A, by $-A$.)

The 1-parameter subgroup $\exp tA$ of the symplectic group acts on $\mathbb{R}^{2n+2}$, preserving Ω' and Σ_A; the corresponding fundamental vector field A^* on $\mathbb{R}^{2n+2}$ (defined by $A^*_x := \frac{d}{dt}\exp -tAx_{|_0} = -Ax$) is Hamiltonian; indeed $i(A^*)\Omega' = dH_A$, with $H_A(x) = \frac{1}{2}\Omega'(x, Ax)$. The hypersurface Σ_A is a level set of this Hamiltonian.
We shall consider the **reduced space** $M^{red} := \Sigma_A/_{\exp tA}$ with the canonical projection $\pi : \Sigma_A \to M^{red}$.
Since the vector field Ax is nowhere 0 on Σ_A, this can always be locally defined. Indeed, for any $x_0 \in \Sigma_A$, one can find local coordinates $\{y^1, \ldots, y^{2n+2}\}$ in which the vector field $Ax = \frac{\partial}{\partial y^1}$ so there exists a neighborhood $U_{x_0} (\subset \Sigma_A)$, a ball $D^{red} \subset \mathbb{R}^{2n}$ of radius r_0, centered at the origin, a real interval $I = (-\varepsilon, \varepsilon)$ and a diffeomorphism

$$\chi : D^{red} \times I \to U_{x_0} \tag{21}$$

such that $\chi(0,0) = x_0$ and $\chi(y,t) = \exp -tA(\chi(y,0))$.

Then $U_{x_0}/_{\{\exp tA \quad t\in I\}} \sim D^{red}$, and one defines $\pi : U_{x_0} \to D^{red}$ by $\pi = p_1 \otimes \chi^{-1}$. The space D^{red} is a local version of the Marsden-Weinstein reduction of Σ_A around the point x_0.

If $x \in \Sigma_A$, the tangent space is given by $T_x\Sigma_A = \{ X \in \mathbb{R}^{2n+2} \,|\, \Omega'(X,Ax) = 0 \}$; one defines $\mathscr{H}_x (\subset T_x\Sigma_A) = \{ X \in \mathbb{R}^{2n+2} \,|\, \Omega'(X,Ax) = 0\,,\, \Omega'(X,x) = 0 \}$; then π_{*x} defines an isomorphism between $\mathscr{H}_x$ and the tangent space T_yD^{red} for $y = \pi(x)$.

A **reduced symplectic form** on D^{red}, ω^{red}, is defined by

$$\omega^{red}_{y=\pi(x)}(X,Y) := \Omega'_x(\overline{X}_x, \overline{Y}_x) \tag{22}$$

where $\overline{Z}$ denotes the horizontal lift of $Z \in T_yD^{red}$; i.e. $\overline{Z} \in \mathscr{H}_x$ and $\pi_{*x}(\overline{X}) = X$.

Let ∇ be the standard flat symplectic affine connection on $\mathbb{R}^{2n+2}$. The **reduced symplectic connection** ∇^{red} on D^{red} is defined by

$$(\nabla^{red}_X Y)_y := \pi_{*_x}(\nabla_{\overline{X}}\overline{Y} - \Omega'(A\overline{X}, \overline{Y})x + \Omega'(\overline{X}, \overline{Y})Ax). \tag{23}$$

Proposition 3.1 *[1] The manifold (D^{red}, ω^{red}) is a symplectic manifold and ∇^{red} is a symplectic connection of Ricci-type on this manifold.*

3.2 Local models for Ricci type connections

Let (M, ω) be a smooth symplectic manifold of dim $2n$ ($n \geq 2$) endowed with a smooth Ricci-type symplectic connection ∇. Then the curvature endomorphism reads

$$R^\nabla(X,Y) = -\frac{1}{2(n+1)}[-2\omega(X,Y)\rho^\nabla - \rho^\nabla Y \otimes \underline{X} + \rho^\nabla X \otimes \underline{Y} - X \otimes \underline{\rho^\nabla Y} + Y \otimes \underline{\rho^\nabla X}] \tag{24}$$

where $\underline{X}$ denotes the 1-form $i(X)\omega$ $(forX$ a vector field on $M)$ and where, as before, ρ^∇ is the endomorphism associated to the Ricci tensor [$r^\nabla(U,V) = \omega(U, \rho^\nabla V)$].
Bianchi's second identity ($\underset{X,Y,Z}{\oplus} (\nabla_X R^\nabla)(Y,Z) = 0$) shows that there exists a vector field U^∇ such that

$$\nabla_X \rho^\nabla = -\frac{1}{2n+1}[X \otimes \underline{U}^\nabla + U^\nabla \otimes \underline{X}]; \tag{25}$$

thus any Ricci-type connection is preferred in the sense of equation (19); further derivation proves the existence of a function f^∇ such that

$$\nabla_X U^\nabla = -\frac{2n+1}{2(n+1)}(\rho^\nabla)^2 X + f^\nabla X; \tag{26}$$

and there exists a real number K^∇ such that

$$tr(\rho^\nabla)^2 + \frac{4(n+1)}{2n+1} f^\nabla = K^\nabla. \tag{27}$$

The second covariant derivative $\nabla^2 R^\nabla$ is determined by ∇U^∇ hence by ρ^∇ and f^∇ (26). Since f^∇ satisfies equation 27, all succesive covariant derivative of the curvature tensor are determined by ρ^∇, U^∇ and K^∇.

Hence, given a point p_0 in a smooth symplectic manifold (M,ω) of dimension $2n$ $(n \geq 2)$ endowed with a smooth Ricci-type connection ∇, the curvature $R^\nabla_{p_0}$ and its covariant derivatives $(\nabla^k R^\nabla)_{p_0}$ (for all k) are determined by $(\rho^\nabla_{x_0}, U^\nabla_{x_0}, K^\nabla)$. This implies:

Corollary 3.2 *Let (M,ω,∇) (resp. (M',ω',∇')) be two real analytic symplectic manifolds of the same dimension $2n$ $(n \geq 2)$ each of them endowed with a symplectic connection of Ricci-type.*

Assume that there exists a linear map $b : T_{x_0}M \to T_{x'_0}M'$ such that (i) $b^*\omega'_{x'_0} = \omega_{x_0}$ (ii) $bu^\nabla_{x_0} = u^{\nabla'}_{x'_0}$ (iii) $b \circ \rho^\nabla_{x_0} \circ b^{-1} = \rho^{\nabla'}_{x'_0}$. Assume further that $K^\nabla = K^{\nabla'}$.

Then the manifolds are locally affinely symplectically isomorphic, i. e. there exists a normal neighborhood of x_0 (resp. x'_0) U_{x_0} (resp. $U'_{x'_0}$) and a symplectic affine diffeomorphism $\varphi : (U_{x_0},\omega,\nabla) \to (U'_{x'_0},\omega',\nabla')$ such that $\varphi(x_0) = x'_0$ and $\varphi_{*x_0} = b$.

A direct computation shows that in the reduction procedure described above, the Ricci type symplectic connection ∇^{red} on (D^{red},ω^{red}) has corresponding $\rho^{\nabla^{red}}, U^{\nabla^{red}}$ and $f^{\nabla^{red}}$ given by:

$$\overline{\rho^{\nabla^{red}} X}(x) = -2(n+1)\overline{A_x}\overline{X} \tag{28}$$

$$\overline{U^{\nabla^{red}}}(x) = -2(n+1)(2n+1)\overline{A_x^2}x \tag{29}$$

$$(\pi^* f^{\nabla^{red}})(x) = 2(n+1)(2n+1)\Omega'(A^2x, Ax) \tag{30}$$

where $\overline{A_x^k}$ is the map induced by A^k with values in $\mathscr{H}_x$:

$$\overline{A_x^k}(X) = A^kX + \Omega'(A^kX,x)Ax - \Omega'(A^kX,Ax)x.$$

Combining this with corollary 3.2 we get:

Theorem 3.3 *[8] Any real analytic symplectic manifold with a Ricci-type connection is locally symplectically affinely isomorphic to the symplectic manifold with a Ricci-type connection obtained by a local reduction procedure around $e_0 = (1,0,\dots,0)$ from a constraint surface Σ_A defined by a second order polynomial H_A for $A \in sp(\mathbb{R}^{2n+2},\Omega')$ in the standard symplectic manifold $(\mathbb{R}^{2n+2},\Omega')$ endowed with the standard flat connection.*

Indeed if $p \in M$ and if ξ is a symplectic frame at p (i.e. $\xi : (\mathbb{R}^{2n},\Omega^{(2n)}) \to (T_p,\omega_p)$ is a symplectic isomorphism of vector spaces), one defines

$$\tilde{u}(\xi) = (\xi)^{-1}\, U^\nabla(p), \quad \tilde{\rho}(\xi) = (\xi)^{-1}\, \rho^\nabla(p)\, \xi \tag{31}$$

and

$$\tilde{A}(\xi) = \begin{pmatrix} 0 & \frac{f(p)}{2(n+1)(2n+1)} & \frac{-\underline{\tilde{u}(\xi)}}{2(n+1)(2n+1)} \\ 1 & 0 & 0 \\ 0 & \frac{-\tilde{u}(\xi)}{2(n+1)(2n+1)} & \frac{-\tilde{\rho}(\xi)}{2(n+1)} \end{pmatrix} \tag{32}$$

with $\underline{u} := \Omega'(u, \cdot)$ and one looks at the reduction for this $A = \tilde{A}(\xi)$.

3.3 Global models for Ricci type connections

Theorem 3.4 *[8] If (M, ω, ∇) is of Ricci type with M simply connected there exists (P, ω^P) symplectic of dimension 2 higher with a flat connection ∇^P so that (M, ω, ∇) is obtained from (P, ω^P, ∇^P) by reduction.*

The manifold P is obtained as the product $P = N \times \mathbb{R}$ of a contact manifold N and the real line R. The manifold N is the holonomy bundle over M corresponding to a connection defined on the $Sp(\mathbb{R}^{2n+2}, \Omega')$- principal bundle

$$B'(M) = B(M) \times_{Sp(\mathbb{R}^{2n},\Omega)} Sp(\mathbb{R}^{2n+2}, \Omega')$$

with projection $\pi' : B(M)' \to M$, where $B(M) \xrightarrow{\pi} M$ is the $Sp(\mathbb{R}^{2n}, \Omega)$ principal bundle of symplectic frames over M and where we inject the symplectic group $Sp(\mathbb{R}^{2n}, \Omega)$ into $Sp(\mathbb{R}^{2n+2}, \Omega')$ as the set of matrices

$$\tilde{j}(A) = \begin{pmatrix} I_2 & 0 \\ 0 & A \end{pmatrix} \qquad A \in Sp(\mathbb{R}^{2n}, \Omega).$$

The connection $1-$form α' on $B'(M)$ is characterised by the fact that

$$\alpha'_{[\xi,1]}([\overline{X}^{hor}, 0]) = \alpha_\xi(\overline{X}^{hor}).$$

where

$$\alpha_\xi(\overline{X}^{hor}) = \begin{pmatrix} 0 & \frac{-\omega_x(u,X)}{2(n+1)(2n+1)} & \frac{-\underline{\widetilde{\rho(X)}(\xi)}}{2(n+1)} \\ 0 & 0 & -\underline{\tilde{X}(\xi)} \\ \tilde{X}(\xi) & \frac{-\widetilde{\rho(X)}(\xi)}{2(n+1)} & 0 \end{pmatrix} \tag{33}$$

where $X \in T_xM$ with $x = \pi(\xi)$ and $\overline{X}^{hor}$ is the horizontal lift of X in $T_\xi B(M)$.

The equations on a Ricci-type connection imply that the curvature $2-$form of the connection $1-$form α' is equal to $-2\tilde{A}'\pi'^*\omega$ with $\tilde{A}'$ the $Sp(\mathbb{R}^{2n+2}, \Omega')-$equivariant extension of $\tilde{A}$ to $B'(M)$; and this curvature $2-$form is invariant by parallel transport

$(d^{\alpha'} curv(\alpha') = 0)$.
Thus the holonomy algebra of α' is of dimension 1. Assume M is simply connected. The holonomy bundle of α' is a circle or a line bundle over M, $N \xrightarrow{\pi'} M$. This bundle has a natural contact structure ν given by the restriction to $N \subset B(M)'$ of the $1-$form $-\frac{1}{2}\alpha'$ (viewed as real valued since it is valued in a $1-$dimensional algebra). One has $d\nu = \pi'^{*}\omega$.

The symplectic manifold with connection (P, ω^P, ∇^P) is then obtained by an induction procedure that appears in a more general setting and will be defined in section 4.1.

3.4 Global reduction and symmetric spaces

The reduction construction described in section 3.1 yields a locally symmetric symplectic space (i.e. such that the curvature tensor is parallel) if and only if the element $0 \neq A \in sp(\mathbb{R}^{2n+2}, \Omega')$ satisfies $A^2 = \lambda I$ for a constant $\lambda \in \mathbb{R}$. The study of symmetric symplectic spaces was initiated in [4] and [3]. A case by case analysis shows that:

Proposition 3.5 *[8] If $0 \neq A \in sp(\mathbb{R}^{2n+2}, \Omega')$ satisfies $A^2 = \lambda I$ for a constant $\lambda \in \mathbb{R}$, the quotient of Σ_A by the action of $\exp tA$ is a manifold M^{red} and the natural projection map $\Sigma_A \to M^{red}$ is a submersion. The manifold M^{red} is a symmetric symplectic space, and the connection obtained by reduction (which is of Ricci type) is the canonical symmetric connection.*

One can show that any simply connected symmetric symplectic space (M, ω, S) whose canonical symmetric connection is of Ricci type [7] can be obtained by such a reduction procedure.

Proposition 3.6 *There are examples which are not symmetric (i.e. A^2 is not a multiple of the identity) and where the quotient $M^{red} := \Sigma_A/_{\exp tA}$ is globally defined .*

For example consider on $\mathbb{R}^6$ the symplectic 2-form Ω' and the element A in the symplectic Lie algebra defined in terms of blocs of 2 by 2 matrices as:

$$\Omega' = \begin{pmatrix} 0 & 0 & I \\ 0 & D & 0 \\ -I & 0 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & D & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \tag{34}$$

where I is the 2 by 2 identity matrix and $D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We denote by (x,y,z) an element in $\mathbb{R}^6$ with x,y and z in $\mathbb{R}^2$. The hypersurface Σ_A is given by

$$\Sigma_A = \{\, (x,y,z) \in \mathbb{R}^6 \,|\, y^1 z^2 - y^2 z^1 = \frac{1}{2} \,\} = \{\, (x,y,z) \in \mathbb{R}^6 \,|\, z \neq 0, y = \frac{Dz}{\|z\|^2} + \lambda z \,\text{with}\, \lambda \in \mathbb{R} \,\}. \tag{35}$$

It is globally diffeomorphic to $\mathbb{R}^3 \times (\mathbb{R}^2 \setminus \{0\})$ We choose $(x, \lambda = \frac{y \cdot z}{\|z\|^2}, z)$ for coordinates on Σ_A. An element of the group $g = \{\exp tA\}$ is given by

$$\exp tA = \begin{pmatrix} I & tD & \frac{t^2}{2}D \\ 0 & I & tI \\ 0 & 0 & I \end{pmatrix}.$$

Its action on $\Sigma_A \subset \mathbb{R}^6$ is given by $(x', y', z') := \exp tA \cdot (x, y, z) = (x + tDy + \frac{t^2}{2}Dz, y + tz, z)$. Remark that the scalar product of x and z is modified by $x' \cdot z' = x \cdot z - \frac{1}{2}t$.One can choose on any orbit of the group the unique point $(\tilde{x}, \tilde{y}, \tilde{z})$ where $\tilde{x} \cdot \tilde{z} = 0$. This shows that the reduced space

$$M^{red} := \Sigma_A /_{\{\exp tA \mid t \in \mathbb{R}\}} = \{ (\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^6 \,|\, \tilde{y}^1\tilde{z}^2 - \tilde{y}^2\tilde{z}^1 = \frac{1}{2} \text{ and } \tilde{x} \cdot \tilde{z} = 0 \} \tag{36}$$

is globally diffeomorphic to $\mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\})$. We choose $(\tilde{\mu} = \frac{\tilde{x}^1\tilde{z}^2 - \tilde{x}^2\tilde{z}^1}{\|\tilde{z}\|^2}, \tilde{\lambda} = \frac{\tilde{y} \cdot \tilde{z}}{\|\tilde{z}\|^2}, \tilde{z})$ as coordinates on M^{red}.

The projection $\pi : \Sigma_A \longrightarrow M^{red}$ maps the point of coordinates (x, λ, z) to the point of coordinates $(\tilde{\mu} = \frac{x^1z^2 - x^2z^1}{\|z\|^2} + 2(x \cdot z)\lambda + 2(x \cdot z)^2, \tilde{\lambda} = \lambda + 2(x \cdot z), \tilde{z} = z)$. It is a submersion.

3.5 Special connections

The striking rigidity results on Ricci-type connections turn out to be a special case of a much more general phenomenon. As we saw, a connection of Ricci-type can be obtained by a symplectic reduction of a symplectic vector space with a flat symplectic connection. This implies, for example, that the local moduli space of such connections is finite dimensional.

Another point of view of the above reduction is to consider in the symplectic Lie algebre $\mathfrak{g} = sp(\mathbb{R}^{2n+2}, \Omega)$ the minimal nilpotent orbit in the adjoint representation on $\mathfrak{g} \cong S^2(\mathbb{R}^{2n+2})$:

$$\hat{\mathscr{C}} := \{x^2 \mid x \neq 0, x \in \mathbb{R}^{2n+2}\} = (\mathbb{R}^{2n+2} \setminus \{0\})/\mathbb{Z}_2.$$

It is an orbit, hence a symplectic manifold, and it is a cone. Projectivizing by positive real numbers, one obtains the manifold $\mathscr{C} = \mathbb{RP}^{2n+1}$ which has an invariant contact structure. Given $A \neq 0 \in sp(\mathbb{R}^{2n+2}, \Omega)$ one looks at the points in this projectivized manifold for which the fundamental vector field associated to the action of A is positively transversal to the contact distrbution

$$\mathscr{C}_A = \{[x] \in \mathscr{C} \mid \Omega(Ax, x) > 0\}$$

This is clearly in bijection with $\hat{\mathscr{C}}_A = \{x \in \hat{\mathscr{C}} \mid \Omega(Ax, x) = 1\}$.

This was generalised by M. Cahen and L. Schwachhöfer in [9]; consider $\mathfrak{g}$ a complex simple Lie algebra (or a real simple Lie algebra which contains a long root space), define $\hat{\mathscr{C}} \subset \mathfrak{g} \cong \mathfrak{g}^*$ to be the orbit of a long root vector and its projectivisation $\mathscr{C} = \mathbb{P}_+(\hat{\mathscr{C}}) \subset S^d \subset \mathfrak{g}$ which carries a contact distribution. Given $A \neq 0 \in \mathfrak{g}$ one looks at $\mathscr{C}_A \subset \mathscr{C}$ consisting of the points where the action field A^* on $\mathscr{C}$ is positively transversal to the contact distribution and one proceeds with standard reduction. The reduced connection is obtained from a connection defined in terms of the Maurer Cartan form on $\mathfrak{g}$.

Theorem 3.7 *[9] In this way one obtains local models for all special connections of the following type: Bochner-Kähler connexions, Bochner Bi-Lagrangian connexions, Ricci-type connexions and connexions with special holonomies.*

4 Ricci-flat connections

4.1 A construction by induction

Definition 4.1 A **contact quadruple** (M, N, α, π) is a $2n$ dimensional smooth manifold M, a $2n+1$ dimensional smooth manifold N, a co-oriented contact structure α on N (i.e. α is a 1-form on N such that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing), and a smooth submersion $\pi : N \to M$ with $d\alpha = \pi^*\omega$ where ω is a symplectic 2-form on M.

Definition 4.2 Given a contact quadruple (M, N, α, π) the **induced symplectic manifold** is the $2n+2$ dimensional manifold

$$P := N \times \mathbb{R} \tag{37}$$

endowed with the (exact) symplectic structure

$$\mu := 2e^{2s}\, ds \wedge p_1^*\alpha + e^{2s}\, dp_1^*\alpha = d(e^{2s}\, p_1^*\alpha) \tag{38}$$

where s denotes the variable along $\mathbb{R}$ and $p_1 : P \to N$ the projection on the first factor.

Induction in the sense of building a $(2n+2)$-dimensional symplectic manifold from a symplectic manifold of dimension $2n$ is also considered by Kostant in [14].

Notations: Let p denote the projection $p = \pi \circ p_1 : P \to M$. Let E be the vector field on P such that $ds(E) = 0$ and $p_{1*}E = Z$ where Z is the Reeb vector field on N (i.e. the vector field Z on N such that $i(Z)d\alpha = 0$ and $i(Z)\alpha = 1$). If X is a vector field on M, denote by $\overline{\overline{X}}$ the vector field on P such that

$$(i)\;\; p_*\overline{\overline{X}} = X \qquad (ii)\;\; (p_1^*\alpha)(\overline{\overline{X}}) = 0 \qquad (iii)\;\; ds(\overline{\overline{X}}) = 0. \tag{39}$$

Proposition 4.3 *Let (M,ω) be the first term of a contact quadruple (M,N,α,π) and let (P,μ) be the associated induced symplectic manifold. Then*

(i) If G is a connected Lie group acting in a strongly Hamiltonian way on (M,ω), this action lifts to a strongly Hamiltonian action of $\tilde{G}$ (= universal cover of G) on (P,μ); the lift $\tilde{X}$ of a Hamiltonian vector field X on (M,ω) with $i(X)\omega = d f_X$ is the vector field on P defined by $\tilde{X} = \overline{\overline{X}} - (p_1^*\pi^* f_X)\cdot E$.

(ii) If C is a conformal vector field on (M,ω) ($L_C\omega = \omega$), it admits a conformal -and a symplectic- lift to (P,μ) if the closed 1-form $\pi^*(i(X)\omega) - \alpha$ is exact ($\alpha - \pi^* i(C)\omega = db$). The conformal lift is $\tilde{C}_1 = \overline{\overline{C}} + p_1^* bE$. The symplectic lift is $\tilde{C}_2 = \overline{\overline{C}} + p_1^* bE - 1/2\partial_s$; it is Hamiltonian and $f_{\tilde{C}_2} = -p_1^* b e^{2s}$.

(iii) The vector field E on P is Hamiltonian and the vector field ∂_s is conformal. If (M,ω) admits a transitive Hamiltonian action (P,μ) admits a transitive conformal action. (The Lie group G is said to **act conformally** *if $\forall g \in G$, $g^*\omega = c(g)\omega$ and there exists an element $g \in G$ such that $c(g) \neq 1$). If (M,ω) admits a transitive conformal-Hamiltonian action then so does (P,μ). (A conformal-Hamiltonian action is a conformal action so that all fundamental vector fields associated to elements in the Lie algebra $\mathfrak{g}_1$ of the subgroup $G_1 := \mathrm{Ker}\, c$ are Hamiltonian.)*

Any symplectic connection ∇ on (M,ω) can be lifted to a symplectic connection ∇^P on (P,ω^P) in the following way: the values at any point of P of the vector fields $\overline{\overline{X}}, E, S = \partial_s$ span the tangent space to P at that point and we have $[E,\partial_s] = 0 \quad [E,\overline{\overline{X}}] = 0 \quad [\partial_s,\overline{\overline{X}}] = 0 \quad [\overline{\overline{X}},\overline{\overline{Y}}] = \overline{\overline{[X,Y]}} - p^*\omega(X,Y)E$. We define ∇^P by the formulas:

$$\begin{aligned}
&\nabla^P_{\overline{\overline{X}}}\overline{\overline{Y}} = \overline{\overline{\nabla_X Y}} - \tfrac{1}{2}p^*(\omega(X,Y))E - p^*(\hat{s}(X,Y))\partial_s \\
&\nabla^P_E\overline{\overline{X}} = \nabla^P_{\overline{\overline{X}}}E = 2\overline{\overline{\sigma X}} + p^*(\omega(X,u))\partial_s \qquad \nabla^P_E E = p^* f\,\partial_s - 2\overline{\overline{U}} \\
&\nabla^P_{\overline{\overline{X}}}\partial_s = \nabla^P_{\partial_s}\overline{\overline{X}} == \overline{\overline{X}} \qquad \nabla^P_E\partial_s = \nabla^P_{\partial_s}E = E \qquad \nabla^P_{\partial_s}\partial_s = \partial_s
\end{aligned} \tag{40}$$

where f is a function on M, U is a vector field on M, $\hat{s}$ is a symmetric 2-tensor on M, and σ is the endomorphism of TM associated to s, hence $\hat{s}(X,Y) = \omega(X,\sigma Y)$.

Theorem 4.4 *[5] With the formulas above, ∇^P is a symplectic connection on (P,μ) for any choice of $\hat{s}$,U and f. The vector field E on P is affine ($L_{\tilde{E}}\nabla^P = 0$) and symplectic ($L_{\tilde{E}}\mu = 0$); the vector field ∂_s on P is affine and conformal ($L_{\partial_s}\mu = 2\mu$).*

Furthermore, choosing

$$\begin{aligned}
\hat{s} &= \frac{-1}{2(n+1)} r^\nabla \\
\underline{U}: &= \omega(U,\cdot) = \frac{2}{2n+1}\mathrm{Tr}[Y \to \nabla_Y \sigma]
\end{aligned}$$

$$f = \frac{1}{2n(n+1)^2}\operatorname{Tr}(\rho^{\nabla})^2 + \frac{1}{n}\operatorname{Tr}[X \to \nabla_X U]. \quad (41)$$

we have:

- *the connection ∇^P on (P,μ) is Ricci-flat;*
- *if the symplectic connection ∇ on (M,ω) is of Ricci-type, then the connection ∇^P on (P,μ) is flat.*
- *if the connection ∇^P is locally symmetric, the connection ∇ is of Ricci-type, hence ∇^P is flat.*

4.2 A construction of Ricci flat symmetric symplectic spaces

Algebraic desciption of simply-connected symmetric symplectic space (see [3, 4])

Definition 4.5 A **symmetric symplectic triple** is a triple $(\mathfrak{g},\sigma,\Omega)$ where $\mathfrak{g}$ is a finite dimensional real Lie algebra, σ is an involutive automorphism of $\mathfrak{g}$ such that if we write $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ with $\sigma = \mathrm{Id}_{\mathfrak{k}} \oplus -\mathrm{Id}_{\mathfrak{p}}$, then $[\mathfrak{p},\mathfrak{p}] = \mathfrak{k}$ and the action of $\mathfrak{k}$ on $\mathfrak{p}$ is faithful, and where Ω is a non degenerate skewsymmetric 2-form on $\mathfrak{p}$, invariant by $\mathfrak{k}$ under the adjoint action.

To any connected symmetric symplectic manifold (M,ω,S) one associates a symmetric symplectic triple $(\mathfrak{g},\sigma,\Omega)$ in the following way:

- $\mathfrak{g}$ is the Lie algebra of its transvection group (i.e. the group generated by the composition of an even number of symmetries);
- σ is the differential at the identity of the conjugation by the symmetry s_{p_0} where p_0 is a point chosen on the manifold;
- $\Omega = \omega_{p_0}$ with the identification between $T_{p_0}M$ and $\mathfrak{p}$ induced by the projection $\pi\colon G \to M : g \mapsto \pi(g) = g\cdot p_0$.

Denoting by X^* the vector field on M which is the image under π_* of the right invariant vector field on G, the canonical symmetric connection has the form

$$(\nabla_{X^*}Y^*)_{g\cdot p_0} = ([Y,\mathrm{Ad}g(\mathrm{Ad}g^{-1}X)_{\mathfrak{p}}])^*_{g\cdot p_0} \quad (42)$$

where $Z_{\mathfrak{p}}$ denotes the component in $\mathfrak{p}$ of $Z\in\mathfrak{g}$ relatively to the decomposition $\mathfrak{g} = \mathfrak{p}\oplus\mathfrak{k}$ and where $[\ ,\ ]$ is the bracket in $\mathfrak{g}$.
The curvature tensor is given by

$$R_{p_0}(X^*_{p_0},Y^*_{p_0})Z^*_{p_0} = -([[X,Y],Z])^*_{p_0}. \quad (43)$$

The value at p_0 of the symplectic curvature of a symmetric connection is thus given by the symplectic curvature $\underline{S}$ of the corresponding symmetric symplectic triple, where $\underline{S}$ is defined as the map

$$\underline{S} : \mathfrak{p} \times \mathfrak{p} \times \mathfrak{p} \times \mathfrak{p} \to \mathbb{R} \quad X,Y,Z,T \mapsto \underline{S}(X,Y,Z,T) := -\Omega([[X,Y],Z],T).$$

Similarly, the Ricci tensor $s : \mathfrak{p} \times \mathfrak{p} \to \mathbb{R}$ of a symmetric symplectic triple is defined by $s(A,B) := \mathrm{Tr}[C \to S(A,C)B]$ with $\underline{S}(A,B,C,D) =: \omega(S(A,B)C,D)$.

Reciprocally, given a symmetric symplectic triple $(\mathfrak{g},\sigma,\Omega)$, one builds a simply-connected symmetric symplectic space (M,ω,s) with $M = G/K$ where G is the simply-connected Lie group with Lie algebra $\mathfrak{g}$ and K is its connected subgroup with Lie algebra $\mathfrak{k}$, with ω the G-invariant 2-form on M whose value at eK is given by Ω (identifying $T_{eK}M$ and $\mathfrak{p}$ via the differential of the canonical projection $\pi\colon G \to G/K$) and with symmetries defined by $s_{\pi(g)}\pi(g') = \pi(g\widetilde{\sigma}(g^{-1}g'))$ where $\widetilde{\sigma}$ is the automorphism of G whose differential at e is σ.

Construction of Ricci-flat symmetric symplectic triples

Let (V,Ω') be a symplectic vector space and let $\underline{R}$ be an "algebraic curvature tensor" on this space, i.e. $\underline{R} \in \Lambda^2 V^* \otimes S^2 V^*$ and $\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\!\bigoplus} \underline{R}(X,Y,Z,T) = 0$.

Define on the vector space

$$\mathfrak{g}_1 = V \oplus \Lambda^2 V \oplus V^* \tag{44}$$

the structure of Lie algebra defined, for all $X,Y,Z \in V, \alpha,\beta \in V^*$ by:

$$\begin{gathered}[X,Y] = X \wedge Y, \quad [X,\alpha] = 0, \quad [X \wedge Y,Z] = \underline{R}(X,Y,Z,\cdot) \in V^*,\\ [X \wedge Y, Z \wedge T] = 0, \quad [X \wedge Y,\alpha] = 0, \quad [\alpha,\beta] = 0.\end{gathered} \tag{45}$$

Let σ be the involutive automorphism of the Lie algebra $\mathfrak{g}_1$ defined by:

$$\sigma_{|_V} = -\mathrm{Id}_{|_V} \qquad \sigma_{|_{\Lambda^2 V}} = \mathrm{Id}_{|_{\Lambda^2 V}} \qquad \sigma_{|_{V^*}} = -\mathrm{Id}_{|_{V^*}}. \tag{46}$$

Let ω be the non degenerate skewsymmetric 2-form on $p := V \oplus V^*$ defined by:

$$\omega(X,Y) = 0, \qquad \omega(\alpha,\beta) = 0, \qquad \omega(\alpha,X) = \alpha(X). \tag{47}$$

Since $\omega([X \wedge Y,Z],T) = \underline{R}(X,Y,Z,T) = \underline{R}(X,Y,T,Z) = -\omega(Z,[X \wedge Y,T])$ the 2-form ω is invariant under $k := \Lambda^2 V$.

Thus $(\mathfrak{g}_1,\sigma,\omega)$ is a symmetric symplectic triple if and only if the map $\phi : \Lambda^2 V \to \mathscr{L}(V,V^*)$ $X \wedge Y \to [Z \mapsto \underline{R}(X,Y,Z,\cdot)]$ is injective. If the map ϕ is not injective, its kernel is an ideal of $\mathfrak{g}_1$. Consider the Lie algebra $\mathfrak{g} := \mathfrak{g}_1/_{\mathrm{Ker}\,\phi}$. Let $\tilde{\sigma}$ be the involutive automorphism of $\mathfrak{g}$ induced by σ ($\tilde{\sigma}(A + \mathrm{Ker}\,\phi) := \sigma(A) + \mathrm{Ker}\,\phi$). The decomposition of $\mathfrak{g}$ into eigenspaces for $\tilde{\sigma}$ reads $\mathfrak{g} = \mathfrak{k}' + \mathfrak{p}'$ where $\mathfrak{k}' = \mathfrak{k}/_{\mathrm{Ker}\,\phi} = \Lambda^2 V/_{\mathrm{Ker}\,\phi}$ and $\mathfrak{p}' \simeq \mathfrak{p} = V \oplus V^*$. Let $\tilde{\omega}$ be the non degenerate skewsymmetric 2-form on $\mathfrak{p}'$, induced by ω on $\mathfrak{p}$. Then $(\mathfrak{g} := \mathfrak{g}_1/_{\mathrm{Ker}\,\phi}, \tilde{\sigma}, \tilde{\omega})$ is a symmetric symplectic triple.

The symplectic curvature of this triple is $\underline{S}(A,B,C,D) = -\omega([[A,B],C],D)$ for elements $A,B,C,D \in \mathfrak{p} = V \oplus V^*$; it vanishes as soon as one of the arguments is in V^* and

$$\underline{S}(X,Y,Z,T) = -\underline{R}(X,Y,Z,T) \tag{48}$$

for $X,Y,Z,T \in V$. Thus $S(A,B)C$ vanishes as soon as one of the arguments is in V^* and $S(X,Y)Z \in V^*$ for $X,Y,Z \in V$. The Ricci tensor $s(A,B) := \mathrm{Tr}[C \to S(A,C)B]$ of the symmetric triple is thus identically zero. Hence:

Proposition 4.6 *Let (V,Ω) be a symplectic vector space of dimension $2n$. Given any element $\underline{R} \in \Lambda^2 V^* \otimes S^2 V^*$ so that $\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\oplus}\ \underline{R}(X,Y,Z,T) = 0$, one can construct a $4n$-dimensional symmetric symplectic space whose canonical symmetric connection is Ricci-flat.*

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