Remarks on the abundance conjecture

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Abstract: We prove the abundance theorem for log canonical n-folds such that the boundary divisor is big assuming the abundance conjecture for log canonical $(n-1)$-folds. We also discuss the log minimal model program for log canonical 4-folds.

Key words: Abundance theorem; big boundary divisor; good minimal model; finite generation of adjoint ring.

1. Introduction. One of the most important open problems in the minimal model theory for higher-dimensional algebraic varieties is the abundance conjecture. The three-dimensional case of the conjecture was completely solved (cf. [18] for log canonical threefolds and [7] for semi log canonical threefolds). However it is still open in dimension ≥ 4. In this paper, we deal with the conjecture in relative setting.

Conjecture 1.1 (Relative abundance). *Let $\pi: X \to U$ be a projective morphism of varieties and (X, Δ) be a (semi) log canonical pair. If $K_X + \Delta$ is π-nef, then it is π-semi-ample.*

Hacon and Xu [14] proved that Conjecture 1.1 for log canonical pairs and Conjecture 1.1 for semi log canonical pairs are equivalent (see also [12]). In [1] Ambro studied the base point free theorem for quasi-log varieties, and as a consequence of the base point free theorem for log canonical pairs, which is a special case of his result, Conjecture 1.1 follows for any log canonical pair $(X, \Delta = \Delta' + A)$ where $\Delta' \geq 0$ and $A \geq 0$ is ample (see, for example, [9]). If (X, Δ) is Kawamata log terminal and Δ is big, then Conjecture 1.1 follows from the usual Kawamata–Shokurov base point free theorem in any dimension. This special case of Conjecture 1.1 plays a crucial role in [5]. In this way, it is natural to consider Conjecture 1.1 for log canonical pairs (X, Δ) under the assumption that Δ is big.

In this paper, we prove the following theorem.

Theorem 1.2 (Main theorem). *Assume Conjecture 1.1 for log canonical $(n-1)$-folds. Then Conjecture 1.1 holds for any projective morphism $\pi: X \to U$ and any log canonical n-fold (X, Δ) such that Δ is a π-big $\mathbf{R}$-divisor.*

We prove it by using the log minimal model program (log MMP, for short) with scaling. A key gradient is termination of the log minimal model program with scaling for Kawamata log terminal pairs such that the boundary divisor is big (cf. [5]). A similar technique in the proof of Theorem 1.2 was also used in [16]. For details, see Section 3.

By Theorem 1.2, we obtain the following results in the minimal model theory for 4-folds.

Theorem 1.3 (Relative abundance theorem). *Let $\pi: X \to U$ be a projective morphism from a normal variety to a variety, where the dimension of X is four. Let (X, Δ) be a log canonical pair such that Δ is a π-big $\mathbf{R}$-divisor. If $K_X + \Delta$ is π-nef, then it is π-semi-ample.*

Corollary 1.4 (Log minimal model program). *Let $\pi: X \to U$ be a projective morphism of normal quasi-projective varieties, where the dimension of X is four. Let (X, Δ) be a log canonical pair such that Δ is a π-big $\mathbf{R}$-divisor. Then any log MMP on $K_X + \Delta$ with scaling over U terminates with a good minimal model or a Mori fiber space of (X, Δ) over U. Moreover, if $K_X + \Delta$ is π-pseudo-effective, then any log MMP on $K_X + \Delta$ over U terminates.*

Corollary 1.5 (Finite generation of adjoint ring). *Let $\pi: X \to U$ be a projective morphism from a normal variety to a variety, where the dimension of X is four. Let $\Delta^{\bullet} = (\Delta_1, \cdots, \Delta_n)$ be an n-tuple of π-big $\mathbf{Q}$-divisors such that (X, Δ_i) is log canonical for any $1 \leq i \leq n$. Then the adjoint ring*

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$$\mathcal{R}(\pi,\Delta^{\bullet})=\bigoplus_{m_1,\cdots,m_n\in\mathbf{Z}_{\geq 0}}\pi_*\mathcal{O}_X\left(\llcorner\sum_{i=1}^{n}m_i(K_X+\Delta_i)\lrcorner\right)$$

is a finitely generated $\mathcal{O}_U$-algebra.

We note that we need to construct log flips for log canonical pairs to run the log minimal model program. Fortunately, the existence of log flips for log canonical pairs is known for all dimensions (cf. [22] for threefolds, [8] for 4-folds and [4] and [13] for all higher dimensions). Therefore we can run the log minimal model program for log canonical pairs in all dimensions. By the above corollaries, we can establish almost completely the minimal model theory for any log canonical 4-fold (X,Δ) such that Δ is big.

The contents of this paper are as follows: In Section 2, we collect some notations and definitions for reader's convenience. In Section 3, we prove Theorem 1.2. In Section 4, we discuss the log minimal model program for log canonical 4-folds and prove Theorem 1.3, Corollary 1.4 and Corollary 1.5.

Throughout this paper, we work over the complex number field.

2. Notations and definitions. In this section, we collect some notations and definitions. We will freely use the standard notations in [5]. Here we write down some important notations and definitions for reader's convenience.

2.1 (Divisors)**.** Let X be a normal variety. $\mathrm{WDiv}_{\mathbf{R}}(X)$ is the $\mathbf{R}$-vector space with canonical basis given by the prime divisors of X. A variety X is called *$\mathbf{Q}$-factorial* if every Weil divisor is $\mathbf{Q}$-Cartier. Let $\pi: X\to U$ be a morphism from a normal variety to a variety and let $D=\sum a_iD_i$ be an $\mathbf{R}$-divisor on X. Then D is a *boundary $\mathbf{R}$-divisor* if $0\leq a_i\leq 1$ for any i. The *round down* of D, denoted by $\llcorner D\lrcorner$, is $\sum\llcorner a_i\lrcorner D_i$ where $\llcorner a_i\lrcorner$ is the largest integer which is not greater than a_i. D is *pseudo-effective over U* (or *π-pseudo-effective*) if D is π-numerically equivalent to the limit of effective $\mathbf{R}$-divisors modulo numerical equivalence over U. D is *nef over U* (or *π-nef*) if it is $\mathbf{R}$-Cartier and $(D\cdot C)\geq 0$ for every proper curve C on X contained in a fiber of π. D is *big over U* (or *π-big*) if there exists a π-ample divisor A and an effective divisor E such that $D\sim_{\mathbf{R},U}A+E$. D is *semi-ample over U* (or *π-semi-ample*) if D is an $\mathbf{R}_{\geq 0}$-linear combination of semi-ample Cartier divisors over U, or equivalently, there exists a morphism $f: X\to Y$ to a variety Y over U such that D is $\mathbf{R}$-linearly equivalent to the pullback of an ample $\mathbf{R}$-divisor over U.

2.2 (Singularities of pairs)**.** Let X be a normal variety and Δ be an effective $\mathbf{R}$-divisor such that $K_X+\Delta$ is $\mathbf{R}$-Cartier. Let $f: Y\to X$ be a birational morphism. Then f is called a *log resolution* of the pair (X,Δ) if f is projective, Y is smooth, the exceptional locus $\mathrm{Ex}(f)$ is pure codimension one and $\mathrm{Supp}\, f_*^{-1}\Delta\cup\mathrm{Ex}(f)$ is simple normal crossing. Suppose that f is a log resolution of the pair (X,Δ). Then we may write

$$K_Y=f^*(K_X+\Delta)+\sum b_iE_i$$

where E_i are prime divisors on Y. Then the *log discrepancy* $a(E_i,X,\Delta)$ of E_i with respect to (X,Δ) is $1+b_i$. The pair (X,Δ) is called *Kawamata log terminal* (*klt*, for short) if $a(E_i,X,\Delta)>0$ for any log resolution f of (X,Δ) and any E_i on Y. (X,Δ) is called *log canonical* (*lc*, for short) if $a(E_i,X,\Delta)\geq 0$ for any log resolution f of (X,Δ) and any E_i on Y. (X,Δ) is called *divisorially log terminal* (*dlt*, for short) if Δ is a boundary $\mathbf{R}$-divisor and there is a log resolution $f: Y\to X$ of (X,Δ) such that $a(E,X,\Delta)>0$ for any f-exceptional divisor E on Y.

Definition 2.3 (Log minimal models)**.** Let $\pi: X\to U$ be a projective morphism from a normal variety to a variety and let (X,Δ) be a log canonical pair. Let $\pi': Y\to U$ be a projective morphism from a normal variety to U and $\phi: X\dashrightarrow Y$ be a birational map over U such that ϕ^{-1} does not contract any divisors. Set $\Delta_Y=\phi_*\Delta$. Then the pair (Y,Δ_Y) is a *log minimal model* of (X,Δ) over U if

(1) $K_Y+\Delta_Y$ is nef over U, and
(2) for any ϕ-exceptional prime divisor D on X, we have

$$a(D,X,\Delta)<a(D,Y,\Delta_Y).$$

A log minimal model (Y,Δ_Y) of (X,Δ) over U is called a *good minimal model* if $K_Y+\Delta_Y$ is semi-ample over U.

Finally, let us recall the definition of semi log canonical pairs.

Definition 2.4 (Semi log canonical pairs, cf. [11, Definition 4.11.3])**.** Let X be a reduced S_2 scheme. We assume that it is pure n-dimensional and normal crossing in codimension one. Let $X=\cup X_i$ be the irreducible decomposition and let $\nu: X'=\amalg X'_i\to X=\cup X_i$ be the normalization. Then the *conductor ideal* of X is defined by

$$\mathfrak{cond}_X = \mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_{X'}, \mathcal{O}_X) \subset \mathcal{O}_X$$

and the *conductor* $\mathcal{C}_X$ of X is the subscheme defined by $\mathfrak{cond}_X$. Since X is S_2 scheme and normal crossing in codimension one, $\mathcal{C}_X$ is a reduced closed subscheme of pure codimension one in X.

Let Δ be a boundary $\mathbf{R}$-divisor on X such that $K_X + \Delta$ is $\mathbf{R}$-Cartier and $\operatorname{Supp}\Delta$ does not contain any irreducible components of $\mathcal{C}_X$. An $\mathbf{R}$-divisor Θ on X' is defined by $K_{X'} + \Theta = \nu^*(K_X + \Delta)$ and we set $\Theta_i = \Theta|_{X'_i}$. Then (X, Δ) is called *semi log canonical* (*slc*, for short) if (X'_i, Θ_i) is lc for any i.

3. Proof of the main theorem. In this section, we prove Theorem 1.2. Before the proof, let us recall the useful theorem called dlt blow-up by Hacon.

Theorem 3.1 (cf. [10, Theorem 10.4], [20, Theorem 3.1])**.** *Let X be a normal quasi-projective variety of dimension n and let Δ be an $\mathbf{R}$-divisor such that (X, Δ) is log canonical. Then there exists a projective birational morphism $f : Y \to X$ from a normal quasi-projective variety Y such that*
(1) *Y is $\mathbf{Q}$-factorial, and*
(2) *if we set*

$$\Delta_Y = f_*^{-1}\Delta + \sum_{E:f\text{-exceptional}} E,$$

then (Y, Δ_Y) is dlt and $K_Y + \Delta_Y = f^*(K_X + \Delta)$.

Proof of Theorem 1.2. Without loss of generality, we can assume that U is affine. Let $f : (Y, \Delta_Y) \to (X, \Delta)$ be a dlt blow-up of (X, Δ). Then Δ_Y is big over U. Indeed, $\Delta_Y = f_*^{-1}\Delta + E$ where E is the sum of all f-exceptional prime divisors with coefficient one. By the definition of π-big divisors, there is a π-ample divisor A and an $\mathbf{R}$-divisor $G = \sum d_i(g_i)$ on X, where (g_i) are principal divisors, and an $\mathbf{R}$-Cartier divisor D on U such that Δ is the sum of $A + G + \pi^*D$ and an effective divisor. Set $A' = A + G + \pi^*D$. Then A' is π-ample and $\Delta - A'$ is effective, and therefore f^*A' is big over U and $f_*^{-1}\Delta - f_*^{-1}A'$ is effective. We can write $f^*A' = f_*^{-1}A' + E'$ for some f-exceptional divisor E'. Pick a positive integer m satisfying that $mE - E'$ is effective. Then

$$m\Delta_Y - f^*A = (mf_*^{-1}\Delta - f_*^{-1}A) + (mE - E')$$

is effective. Since f^*A' is big over U, Δ_Y is also big over U. Thus we can replace (X, Δ) with (Y, Δ_Y) and assume that (X, Δ) is a $\mathbf{Q}$-factorial dlt pair.

Let V be the finite dimensional subspace in $\mathrm{WDiv}_{\mathbf{R}}(X)$ spanned by all components of Δ, and set

$$\mathcal{N} = \{B \in V \mid (X, B) \text{ is lc and } K_X + B \text{ is } \pi\text{-nef}\}.$$

Then $\mathcal{N}$ is a rational polytope in V (cf. [11, Theorem 4.7.2 (3)], [23, 6.2. First Main Theorem]). Therefore we can find finitely many π-big $\mathbf{Q}$-divisors $\Delta_1, \cdots, \Delta_l \in \mathcal{N}$ which are sufficiently close to Δ and positive real numbers $r_1, \cdots, r_l$ such that $\sum_{i=1}^l r_i = 1$ and $\sum_{i=1}^l r_i\Delta_i = \Delta$. Then (X, Δ_i) are dlt because $\operatorname{Supp}\Delta_i \subset \operatorname{Supp}\Delta$ and Δ_i are sufficiently close to Δ. Since we have $K_X + \Delta = \sum_{i=1}^l r_i(K_X + \Delta_i)$, it is sufficient to prove that $K_X + \Delta_i$ is π-semi-ample for any i. Thus we may assume that Δ is a $\mathbf{Q}$-divisor.

If $\llcorner\Delta\lrcorner = 0$, then (X, Δ) is klt and Theorem 1.2 follows from [5, Corollary 3.9.2]. Thus we may assume that $\llcorner\Delta\lrcorner \neq 0$. Let k be a positive integer such that $k(K_X + \Delta)$ is Cartier. Pick a sufficiently small positive rational number ϵ such that $\Delta - \epsilon\llcorner\Delta\lrcorner$ is big over U and $(2k\epsilon \cdot \dim X)/(1 - \epsilon) < 1$. Since (X, Δ) is dlt, $(X, \Delta - \epsilon\llcorner\Delta\lrcorner)$ is klt. By [5, Corollary 1.4.2], the log MMP on $K_X + \Delta - \epsilon\llcorner\Delta\lrcorner$ with scaling of a π-ample divisor

$$X = X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots$$

over U terminates. Let Δ_i be the birational transform of Δ on X_i. Then $(X_m, \Delta_m - \epsilon\llcorner\Delta_m\lrcorner)$ is a log minimal model or a Mori fiber space $h : X_m \to Z$ of $(X, \Delta - \epsilon\llcorner\Delta\lrcorner)$ over U for some $m \in \mathbf{Z}_{>0}$.

Let $f_i : X_i \to V_i$ be the contraction morphism of the i-th step of the log MMP over U, that is, $X_{i+1} = V_i$ or $X_{i+1} \to V_i$ is the flip of f_i over U. Then $K_{X_i} + \Delta_i$ is nef over U and f_i-trivial for any $i \geq 1$. Indeed, by the induction on i, it is sufficient to prove that $K_X + \Delta$ is f_1-trivial and $K_{X_2} + \Delta_2$ is nef over U. Recall that k is a positive integer such that $k(K_X + \Delta)$ is Cartier. We show that $K_X + \Delta$ is f_1-trivial and $k(K_{X_2} + \Delta_2)$ is a nef Cartier divisor over U. Since $K_X + \Delta$ is nef over U, for any $(K_X + \Delta - \epsilon\llcorner\Delta\lrcorner)$-negative extremal ray over U, it is also a $(K_X + \Delta - \llcorner\Delta\lrcorner)$-negative extremal ray over U. Then we can find a rational curve C on X contracted by f_1 such that $0 < -(K_X + \Delta - \llcorner\Delta\lrcorner) \cdot C \leq 2\dim X$ by [10, Theorem 18.2]. By the choice of ϵ, we have

$$\begin{aligned} 0 &\leq k(K_X + \Delta) \cdot C \\ &= \frac{k}{1 - \epsilon}(K_X + \Delta - \epsilon\llcorner\Delta\lrcorner - \epsilon(K_X + \Delta - \llcorner\Delta\lrcorner)) \cdot C \end{aligned}$$

$$< \frac{k\epsilon}{1-\epsilon} \cdot 2 \dim X < 1.$$

Since $k(K_X + \Delta)$ is Cartier, $k(K_X + \Delta) \cdot C$ is an integer. Then we have $k(K_X + \Delta) \cdot C = 0$ and thus $K_X + \Delta$ is f_1-trivial. By the cone theorem (cf. [11, Theorem 4.5.2]), there is a Cartier divisor D on V_1 such that $k(K_X + \Delta) \sim f_1^* D$. Since $k(K_X + \Delta)$ is nef over U, D is also nef over U. Let $g : W \to X$ and $g' : W \to X_2$ be a common resolution of $X \dashrightarrow X_2$. Then $g^*(K_X + \Delta) = g'^*(K_{X_2} + \Delta_2)$ by the negativity lemma because $K_X + \Delta$ is f_1-trivial. Then $k(K_{X_2} + \Delta_2)$ is linearly equivalent to the pullback of D and therefore it is a nef Cartier divisor over U. Thus, $K_{X_i} + \Delta_i$ is nef over U and f_i-trivial for any $i \geq 1$.

Like above, by taking a common resolution of $X_i \dashrightarrow X_{i+1}$ and the negativity lemma, we see that $K_{X_i} + \Delta_i$ is semi-ample over U if and only if $K_{X_{i+1}} + \Delta_{i+1}$ is semi-ample over U for any $1 \leq i \leq m-1$. Moreover X_m is $\mathbf{Q}$-factorial and Δ_m is big over U by the construction. Therefore we can replace (X, Δ) with (X_m, Δ_m) and assume that X is a log minimal model or a Mori fiber space $h : X \to Z$ of $(X, \Delta - \epsilon \llcorner \Delta \lrcorner)$ over U. We note that after replacing (X, Δ) with (X_m, Δ_m), (X, Δ) is lc but not necessarily dlt.

Case 1. X is a Mori fiber space $h : X \to Z$ of $(X, \Delta - \epsilon \llcorner \Delta \lrcorner)$ over U.

Proof of Case 1. First, note that in this case $K_X + \Delta$ is h-trivial by the above discussion. Moreover $\llcorner \Delta \lrcorner$ is ample over Z. By the cone theorem (cf. [11, Theorem 4.5.2]), there exists a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor Ξ on Z such that $K_X + \Delta \sim_{\mathbf{Q},U} h^* \Xi$. Since $\llcorner \Delta \lrcorner$ is ample over Z, $\mathrm{Supp} \llcorner \Delta \lrcorner$ dominates Z. In particular, there exists a component of $\llcorner \Delta \lrcorner$, which we denote T, such that T dominates Z. Let $f : (Y, \Delta_Y) \to (X, \Delta)$ be a dlt blow-up (see Theorem 3.1) and $\widetilde{T}$ be the strict transform of T on Y. Then $K_X + \Delta$ is semi-ample over U if and only if $K_Y + \Delta_Y$ is semi-ample over U. Furthermore, we have $K_{\widetilde{T}} + \mathrm{Diff}(\Delta_Y - \widetilde{T}) \sim_{\mathbf{Q},U} ((h \circ f)|_{\widetilde{T}})^* \Xi$ since $\widetilde{T}$ dominates Z. Thus it is sufficient to prove that $K_{\widetilde{T}} + \mathrm{Diff}(\Delta_Y - \widetilde{T})$ is semi-ample over U. Since (Y, Δ_Y) is dlt, $\widetilde{T}$ is normal by [21, Corollary 5.52]. By [19, 17.2. Theorem], we see that the pair $(\widetilde{T}, \mathrm{Diff}(\Delta_Y - \widetilde{T}))$ is lc. Then $K_{\widetilde{T}} + \mathrm{Diff}(\Delta_Y - \widetilde{T})$ is semi-ample over U by the relative abundance theorem for log canonical $(n-1)$-folds. So we are done. □

Case 2. X is a log minimal model of $(X, \Delta - \epsilon \llcorner \Delta \lrcorner)$ over U.

Proof of Case 2. In this case, both $K_X + \Delta$ and $K_X + \Delta - \epsilon \llcorner \Delta \lrcorner$ are nef over U. Set $M = K_X + \Delta$ and $M' = K_X + \Delta - \epsilon \llcorner \Delta \lrcorner$. By [5, Corollary 3.9.2], M' is semi-ample over U. Therefore we may assume that $\llcorner \Delta \lrcorner \neq 0$, and there exists a sufficiently large and divisible positive integer l such that both lM and $l\epsilon \llcorner \Delta \lrcorner$ are Cartier and $\pi^* \pi_* \mathcal{O}_X(lM') \to \mathcal{O}_X(lM')$ is surjective. Then, in the following diagram,

$$\begin{tikzcd}
\pi^*\pi_*\mathcal{O}_X(lM')|_{X \setminus \llcorner \Delta \lrcorner} \arrow[r] \arrow[d] & \pi^*\pi_*\mathcal{O}_X(lM)|_{X \setminus \llcorner \Delta \lrcorner} \arrow[d] \\
\mathcal{O}_X(lM')|_{X \setminus \llcorner \Delta \lrcorner} \arrow[r, "\cong"] & \mathcal{O}_X(lM)|_{X \setminus \llcorner \Delta \lrcorner}
\end{tikzcd}$$

the left vertical morphism is surjective. Moreover, the lower horizontal morphism is an isomorphism. Therefore the right vertical morphism is surjective. Thus $\pi^* \pi_* \mathcal{O}_X(l(K_X + \Delta)) \to \mathcal{O}_X(l(K_X + \Delta))$ is surjective outside of $\llcorner \Delta \lrcorner$.

Next, set $D = \mathrm{Diff}(\Delta - \llcorner \Delta \lrcorner)$ and consider the following exact sequence

$$0 \to \mathcal{O}_X(l'M - \llcorner \Delta \lrcorner) \to \mathcal{O}_X(l'M) \to \mathcal{O}_{\llcorner \Delta \lrcorner}(l'(K_{\llcorner \Delta \lrcorner} + D)) \to 0$$

where l' is a sufficiently large and divisible positive integer such that $1/l' \leq \epsilon$. Then we have

$$l'M - \llcorner \Delta \lrcorner = l' \left(M - \frac{1}{l'} \llcorner \Delta \lrcorner \right).$$

Moreover, $\Delta - (1/l') \llcorner \Delta \lrcorner$ is big over U and $M - (1/l') \llcorner \Delta \lrcorner$ is nef over U. Since $(X, \Delta - (1/l') \llcorner \Delta \lrcorner)$ is klt, by [5, Lemma 3.7.5], we may find a π-big $\mathbf{Q}$-divisor $A + B$, where $A \geq 0$ is a general ample $\mathbf{Q}$-divisor over U and $B \geq 0$, such that $(X, A + B)$ is klt and $\Delta - (1/l') \llcorner \Delta \lrcorner \sim_{\mathbf{Q},U} A + B$. In particular, (X, B) is klt. Furthermore, $A + (l' - 1)(M - (1/l') \llcorner \Delta \lrcorner)$ is ample over U. Thus we have

$$l'M - \llcorner \Delta \lrcorner \sim_{\mathbf{Q},U} K_X + A + (l' - 1)\left(M - \frac{1}{l'} \llcorner \Delta \lrcorner \right) + B$$

and $R^1 \pi_* \mathcal{O}_X(l'M - \llcorner \Delta \lrcorner) = 0$ (cf. [17, Theorem 1-2-5]). Then $\pi_* \mathcal{O}_X(l'M) \to \pi_* \mathcal{O}_{\llcorner \Delta \lrcorner}(l'(K_{\llcorner \Delta \lrcorner} + D))$ is surjective and thus $\pi^* \pi_* \mathcal{O}_X(l'M) \otimes \mathcal{O}_{\llcorner \Delta \lrcorner} \to \pi^* \pi_* \mathcal{O}_{\llcorner \Delta \lrcorner}(l'(K_{\llcorner \Delta \lrcorner} + D))$ is surjective.

We can check that the pair $(\llcorner \Delta \lrcorner, D)$ is semi log canonical. Indeed, since $(X, \Delta - \epsilon \llcorner \Delta \lrcorner)$ is klt and since X is $\mathbf{Q}$-factorial, by [21, Corollary 5.25], $\llcorner \Delta \lrcorner$ is Cohen–Macaulay. In particular, $\llcorner \Delta \lrcorner$ satisfies the S_2 condition. Moreover, since (X, Δ) is lc, $\llcorner \Delta \lrcorner$

is normal crossing in codimension one. We also see that D does not contain any irreducible components of $\mathcal{C}_{\llcorner\Delta\lrcorner}$ by [6, 16.6 Proposition]. Therefore $(\llcorner\Delta\lrcorner, D)$ is semi log canonical by [19, 17.2 Theorem]. Since $K_{\llcorner\Delta\lrcorner}+D=M|_{\llcorner\Delta\lrcorner}$ is nef over U, $K_{\llcorner\Delta\lrcorner}+D$ is semi-ample over U by [14, Theorem 2] and the relative abundance theorem for log canonical $(n-1)$-folds.

By these facts, in the following diagram,

$$\begin{tikzcd} \pi^*\pi_*\mathcal{O}_X(l'M)\otimes\mathcal{O}_{\llcorner\Delta\lrcorner} \arrow[r] \arrow[d] & \pi^*\pi_*\mathcal{O}_{\llcorner\Delta\lrcorner}(l'(K_{\llcorner\Delta\lrcorner}+D)) \arrow[d] \\ \mathcal{O}_X(l'M)\otimes\mathcal{O}_{\llcorner\Delta\lrcorner} \arrow[r, "\cong"] & \mathcal{O}_{\llcorner\Delta\lrcorner}(l'(K_{\llcorner\Delta\lrcorner}+D)) \end{tikzcd}$$

the right vertical morphism and the upper horizontal morphism are both surjective. Furthermore, the lower horizontal morphism is an isomorphism. Therefore the left vertical morphism is surjective. Then $\pi^*\pi_*\mathcal{O}_X(l'(K_X+\Delta))\to\mathcal{O}_X(l'(K_X+\Delta))$ is surjective in a neighborhood of $\llcorner\Delta\lrcorner$.

Therefore, $\pi^*\pi_*\mathcal{O}_X(l(K_X+\Delta))\to\mathcal{O}_X(l(K_X+\Delta))$ is surjective for some sufficiently large and divisible positive integer l. So we are done. $\square$

Thus, in both cases, $K_X+\Delta$ is semi-ample over U. Therefore we complete the proof. $\square$

4. Minimal model program in dimension four. In this section, we discuss the log minimal model program for log canonical 4-folds and prove Theorem 1.3, Corollary 1.4 and Corollary 1.5.

Proof of Theorem 1.3. It immediately follows from Theorem 1.2 since the relative abundance theorem for log canonical threefolds holds (cf. [18, 1.1 Theorem]). $\square$

Proposition 4.1. *Let $\pi: X\to U$ be a projective morphism of normal quasi-projective varieties, where the dimension of X is four. Let (X,Δ) be a log canonical pair and let A be an effective $\mathbf{R}$-divisor such that $(X,\Delta+A)$ is log canonical and $K_X+\Delta+A$ is π-nef. Then we can run the log MMP on $K_X+\Delta$ with scaling of A over U and this log MMP with scaling terminates.*

Proof. We can run the log MMP on $K_X+\Delta$ with scaling of A over U by [11, Remark 4.9.2]. Therefore we only have to prove the termination of the log MMP with scaling.

Suppose by contradiction that we get an infinite sequence of birational maps by running the log MMP with scaling of A

$$(X=X_1,\Delta=\Delta_1)\dashrightarrow\cdots\dashrightarrow(X_i,\Delta_i)\dashrightarrow\cdots$$

over U. Let A_i be the birational transform of A on X_i and set

$$\lambda_i=\inf\{\mu\in\mathbf{R}_{\geq 0}\mid K_{X_i}+\Delta_i+\mu A_i \text{ is nef over } U\}$$

for every $i\geq 1$. Let $X_i\to V_i$ be the contraction morphism of the i-th step of the log MMP on $K_X+\Delta$ with scaling of A over U. Note that by [3, Lemma 3.8], the log MMP with scaling terminates for all $\mathbf{Q}$-factorial dlt 4-folds. By the same argument as in the proof of [11, Lemma 4.9.3], we obtain the following diagram

$$\begin{tikzcd} (Y_1^1,\Psi_1^1) \arrow[r, dashed] \arrow[d, "\alpha_1"'] & \cdots \arrow[r, dashed] & (Y_i^1,\Psi_i^1) \arrow[r, dashed] \arrow[d, "\alpha_i"'] & \cdots \\ (X_1,\Delta_1) \arrow[r, dashed] & \cdots \arrow[r, dashed] & (X_i,\Delta_i) \arrow[r, dashed] & \cdots \end{tikzcd}$$

such that

(1) (Y_i^1,Ψ_i^1) is $\mathbf{Q}$-factorial dlt and $K_{Y_i^1}+\Psi_i^1=\alpha_i^*(K_{X_i}+\Delta_i)$,

(2) the sequence of birational maps
$$(Y_i^1,\Psi_i^1)\dashrightarrow\cdots\dashrightarrow(Y_i^{k_i},\Psi_i^{k_i})=(Y_{i+1}^1,\Psi_{i+1}^1)$$
is a finitely many steps of the log MMP on $K_{Y_i^1}+\Psi_i^1$ over V_i

for any $i\geq 1$, and

(3) the sequence of the upper horizontal birational maps is an infinite sequence of divisorial contractions and log flips of the log MMP on $K_{Y_1^1}+\Psi_1^1$ over U.

For every $i\geq 1$ and $1\leq j<k_i$, let A_i^j be the birational transform of α_1^*A on Y_i^j and set

$$\lambda_i^j=\inf\{\mu\in\mathbf{R}_{\geq 0}\mid K_{Y_i^j}+\Psi_i^j+\mu A_i^j \text{ is nef over } U\}.$$

Then we have $\lambda_i^j=\lambda_i$ for any $i\geq 1$ and $1\leq j<k_i$. Indeed, since $K_{X_i}+\Delta_i+\lambda_iA_i$ is nef over U and it is also trivial over V_i, there is an $\mathbf{R}$-Cartier divisor D, which is nef over U, on V_i such that $K_{X_i}+\Delta_i+\lambda_iA_i$ is $\mathbf{R}$-linearly equivalent to the pullback of D. Since $A_i^1=\alpha_i^*A_i$, by the condition (1), $K_{Y_i^1}+\Psi_i^1+\lambda_iA_i^1$ is also $\mathbf{R}$-linearly equivalent to the pullback of D. Thus $K_{Y_i^1}+\Psi_i^1+\lambda_iA_i^1$ is nef over U. Moreover, by the condition (2), $K_{Y_i^j}+\Psi_i^j+\lambda_iA_i^j$ is also $\mathbf{R}$-linearly equivalent to the pullback of D. Therefore $K_{Y_i^j}+\Psi_i^j+\lambda_iA_i^j$ is nef over U and trivial over V_i for any $0\leq j<k_i$. We also see that $K_{Y_i^j}+\Psi_i^j+\mu A_i^j$ is not nef over V_i for any $\mu\in[0,\lambda_i)$ by the condition (2). In particular it is not nef over U. Therefore we have $\lambda_i^j=\lambda_i$ for any $i\geq 1$ and $1\leq j<k_i$.

By these facts, we can identify the sequence of birational maps

$$(Y_1^1, \Psi_1^1) \dashrightarrow \cdots \dashrightarrow (Y_i^j, \Psi_i^j) \dashrightarrow \cdots$$

with an infinite sequence of birational maps of the log MMP on $K_{Y_1^1} + \Psi_1^1$ with scaling of $A_1^1 = \alpha_1^* A$ over U. But then it must terminate by [3, Lemma 3.8]. It contradicts our assumption. So we are done. □

Proof of Corollary 1.4. The first half of the assertions immediately follows from Proposition 4.1 and Theorem 1.3. For the latter half, if $K_X + \Delta$ is π-pseudo-effective then it is π-effective by the first half of this corollary. By [2, Theorem 1.3], termination of any log MMP follows. So we are done. □

Proof of Corollary 1.5. Without loss of generality, we can assume that U is affine. Then the assertion follows from Proposition 4.1 and Theorem 1.3 with the same argument as in the proof of [15, Lemma 3.2] and the discussion of [15, Section 4]. □

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