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Thierry GALLAY and Vladimír ŠVERÁK

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REMARKS ON THE CAUCHY PROBLEM FOR THE AXISYMMETRIC NAVIER-STOKES EQUATIONS

THIERRY GALLAY AND VLADIMÍR ŠVERÁK

Abstract. Motivated by applications to vortex rings, we study the Cauchy problem for the three-dimensional axisymmetric Navier-Stokes equations without swirl, using scale invariant function spaces. If the axisymmetric vorticity ω_θ is integrable with respect to the two-dimensional measure $\mathrm{d}r\,\mathrm{d}z$, where (r,θ,z) denote the cylindrical coordinates in $\mathbb{R}^3$, we show the existence of a unique global solution, which converges to zero in L^1 norm as $t \to \infty$. The proof of local well-posedness follows exactly the same lines as in the two-dimensional case, and our approach emphasizes the similarity between both situations. The solutions we construct have infinite energy in general, so that energy dissipation cannot be invoked to control the long-time behavior. We also treat the more general case where the initial vorticity is a finite measure whose atomic part is small enough compared to viscosity. Such data include point masses, which correspond to vortex filaments in the three-dimensional picture.

This article is dedicated to Denis Serre on the occasion of his 60th birthday.

1. Introduction

Among all three-dimensional incompressible flows, *axisymmetric flows without swirl* form a particular class that is relatively simple to study and yet contains interesting examples, such as circular vortex filaments or toroidal vortex rings. For the evolutions defined by both the Euler and the Navier-Stokes equations, global well-posedness in that class was established almost fifty years ago by Ladyzhenskaya [17] and Ukhovksii & Yudovich [23]. In the viscous case, the original approach of [17, 23] applies to velocity fields in the Sobolev space $H^2(\mathbb{R}^3)$, see [18], but it is possible to obtain the same conclusions under the weaker assumption that the initial velocity belongs to $H^{1/2}(\mathbb{R}^3)$ [1]. In all these works, global existence for arbitrary large data is shown by combining the standard energy estimate, which holds for general solutions of the Navier-Stokes equations, with a priori bounds on the vorticity that are specific to the axisymmetric case.

In this paper, we revisit the Cauchy problem for the axisymmetric Navier-Stokes equations (without swirl) for the following reasons. First, motivated by a future study of vortex filaments, we wish to formulate a global well-posedness result involving *scale invariant* function spaces only. As was already mentioned, all previous works deal with finite energy solutions, and energy is not a scale invariant quantity for three-dimensional viscous flows. In particular, the long-time behavior of axisymmetric solutions has not been studied in terms of scale invariant norms. Our second motivation is to emphasize the analogy between the axisymmetric case and the two-dimensional situation where the velocity field is planar and depends on two variables only. Indeed, we shall see that, if appropriate function spaces are used, local existence of solutions can be established in the axisymmetric case using literally the same proof as in the two-dimensional situation, which has been studied by many authors [3, 11, 14]. However, significant differences appear when one considers a priori estimates and long-time asymptotics.

To formulate our results, we introduce some notation. The time evolution of viscous incompressible flows is described by the Navier-Stokes equations

$$\partial_t u + (u\cdot\nabla)u \,=\, \Delta u - \nabla p\ , \qquad \operatorname{div} u \,=\, 0\ , \tag{1.1}$$

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where $u = u(x,t) \in \mathbb{R}^3$ denotes the velocity field and $p = p(x,t) \in \mathbb{R}$ the pressure field. For simplicity, we assume throughout this paper that the kinematic viscosity and the fluid density are both equal to 1. We restrict ourselves to axisymmetric solutions without swirl for which the velocity field has the following particular form :

$$u(x,t) \;=\; u_r(r,z,t)e_r + u_z(r,z,t)e_z\ . \tag{1.2}$$

Here (r,θ,z) are the usual cylindrical coordinates in $\mathbb{R}^3$, defined by setting $x = (r\cos\theta, r\sin\theta, z)$ for any $x \in \mathbb{R}^3$, and e_r, e_θ, e_z denote the unit vectors in the radial, toroidal, and vertical directions, respectively :

$$e_r \;=\; \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}\ , \qquad e_\theta \;=\; \begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}\ , \qquad e_z \;=\; \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\ . \tag{1.3}$$

We emphasize that the "swirl" $u \cdot e_\theta$ is assumed to vanish identically. This means that the velocity field (1.2) is not only invariant under rotations about the vertical axis, but also under reflections by any plane containing the vertical axis.

A direct calculation shows that the vorticity $\omega = \operatorname{curl} u$ associated with the velocity field (1.2) is purely toroidal :

$$\omega(r,z,t) \;=\; \omega_\theta(r,z,t)e_\theta\ , \qquad \text{where} \quad \omega_\theta = \partial_z u_r - \partial_r u_z\ . \tag{1.4}$$

The flow is entirely determined by the single quantity ω_θ, because the velocity field u can be reconstructed by solving the linear elliptic system

$$\partial_r u_r + \frac{1}{r}u_r + \partial_z u_z \;=\; 0\ , \qquad \partial_z u_r - \partial_r u_z \;=\; \omega_\theta\ , \tag{1.5}$$

in the half space $\Omega = \{(r,z) \in \mathbb{R}^2 \,|\, r > 0\,,\ z \in \mathbb{R}\}$, with boundary conditions $u_r = \partial_r u_z = 0$ at $r = 0$. System (1.5) is the differential formulation of the axisymmetric Biot-Savart law, which will be studied in more detail in Section 2 below.

The evolution equation for ω_θ reads

$$\partial_t \omega_\theta + u \cdot \nabla \omega_\theta - \frac{u_r}{r}\omega_\theta \;=\; \Delta\omega_\theta - \frac{\omega_\theta}{r^2}\ , \tag{1.6}$$

where $u \cdot \nabla = u_r\partial_r + u_z\partial_z$ and $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$ denotes the Laplace operator in cylindrical coordinates. Equation (1.6) is considered in the half-plane Ω with homogeneous Dirichlet condition at the boundary $r = 0$. As was already observed in [17, 23], it is useful to consider also the related quantity

$$\eta(r,z,t) \;=\; \frac{\omega_\theta(r,z,t)}{r}\ , \tag{1.7}$$

which satisfies the advection-diffusion equation

$$\partial_t \eta + u \cdot \nabla \eta \;=\; \Delta\eta + \frac{2}{r}\partial_r \eta\ , \tag{1.8}$$

with homogeneous Neumann condition at the boundary $r = 0$. Systems (1.6) and (1.8) are, of course, perfectly equivalent. In what follows we find it more convenient to work with the axisymmetric vorticity equation (1.6), at least to prove local existence of solutions, but equation (1.8) will be useful to derive a priori estimates and to study the long-time behavior.

Throughout this paper, to emphasize the similarity with the two-dimensional case, we equip the half-plane $\Omega = \{(r,z) \in \mathbb{R}^2 \,|\, r > 0\,,\ z \in \mathbb{R}\}$ with the *two-dimensional* measure $\mathrm{d}r\,\mathrm{d}z$, as opposed to the 3D measure $r\,\mathrm{d}r\,\mathrm{d}z$ which could appear more natural for axisymmetric problems. Thus, given any $p \in [1,\infty)$, we denote by $L^p(\Omega)$ the space of measurable functions $\omega_\theta : \Omega \to \mathbb{R}$ for which the following norm is finite :

$$\|\omega_\theta\|_{L^p(\Omega)} \;=\; \Bigl(\int_\Omega |\omega_\theta(r,z)|^p \,\mathrm{d}r\,\mathrm{d}z\Bigr)^{1/p}\ , \qquad 1 \leqslant p < \infty\ .$$

The space $L^\infty(\Omega)$ is defined similarly. Sometimes, however, it is more convenient to use the 3D measure $r\,\mathrm{d}r\,\mathrm{d}z$, and the corresponding spaces are then denoted by $L^p(\mathbb{R}^3)$ to avoid confusion. For instance, we define

$$\|\eta\|_{L^p(\mathbb{R}^3)} \;=\; \left(\int_\Omega |\eta(r,z)|^p\, r\,\mathrm{d}r\,\mathrm{d}z\right)^{1/p}, \qquad 1 \leqslant p < \infty\ .$$

We are now in position to state our first main result.

THEOREM 1.1. — *For any initial data $\omega_0 \in L^1(\Omega)$, the axisymmetric vorticity equation* (1.6) *has a unique global mild solution*

$$\omega_\theta \in C^0([0,\infty), L^1(\Omega)) \cap C^0((0,\infty), L^\infty(\Omega))\ . \tag{1.9}$$

The solution satisfies $\|\omega_\theta(t)\|_{L^1(\Omega)} \leqslant \|\omega_0\|_{L^1(\Omega)}$ for all $t > 0$, and

$$\lim_{t\to 0} t^{1-\frac{1}{p}}\|\omega_\theta(t)\|_{L^p(\Omega)} \;=\; 0\ , \qquad \textit{for} \quad 1 < p \leqslant \infty\ , \tag{1.10}$$

$$\lim_{t\to \infty} t^{1-\frac{1}{p}}\|\omega_\theta(t)\|_{L^p(\Omega)} \;=\; 0\ , \qquad \textit{for} \quad 1 \leqslant p \leqslant \infty\ . \tag{1.11}$$

If, in addition, the axisymmetric vorticity is non-negative and has finite impulse:

$$\mathcal{I} \;=\; \int_\Omega r^2\omega_0(r,z)\,\mathrm{d}r\,\mathrm{d}z \;<\; \infty\ , \tag{1.12}$$

then

$$\lim_{t\to\infty} t^2\omega_\theta(r\sqrt{t}, z\sqrt{t}, t) \;=\; \frac{\mathcal{I}}{16\sqrt{\pi}}\, r\, e^{-\frac{r^2+z^2}{4}}\ , \qquad (r,z)\in\Omega\ , \tag{1.13}$$

where convergence holds in $L^p(\Omega)$ for $1 \leqslant p \leqslant \infty$. In particular $\|\omega_\theta(t)\|_{L^p(\Omega)} = \mathcal{O}(t^{-2+\frac{1}{p}})$ as $t \to \infty$ in that case.

Remark 1.2. — A mild solution of (1.6) on $\mathbb{R}_+ = [0,\infty)$ is a solution of the associated integral equation, namely Eq. (4.2) below. As will be verified in Section 4, both sides of (4.2) are well-defined if ω_θ satisfies (1.9) and (1.10). In the uniqueness claim, we only assume that ω_θ satisfies (1.9) and is a mild solution of (1.6) for strictly positive times. The proof then shows that (1.10) automatically holds.

The statement of Theorem 1.1 has several aspects, and it is worth discussing them separately. The *local well-posedness* claim is certainly not surprising, because the class of initial data we consider is covered by at least two existence results in the literature. Indeed, if $\omega_\theta \in L^1(\Omega)$, it is easy to verify that the vorticity $\omega = \omega_\theta e_\theta$ belongs to the Morrey space $M^{3/2}(\mathbb{R}^3)$ defined by the norm

$$\|\omega\|_{M^{3/2}} \;=\; \sup_{x\in\mathbb{R}^3}\,\sup_{R>0}\,\frac{1}{R}\int_{B(x,R)} |\omega(x)|\,\mathrm{d}x\ ,$$

where $B(x,R) \subset \mathbb{R}^3$ denotes the ball of radius $R > 0$ centered at $x \in \mathbb{R}^3$. In addition ω can be approximated in $M^{3/2}(\mathbb{R}^3)$ by smooth and compactly supported functions. As was proved by Giga & Miyakawa [12], the Navier-Stokes equations in $\mathbb{R}^3$ thus have a unique local solution with initial vorticity ω, which is even global in time if the norm $\|\omega\|_{M^{3/2}}$ is sufficiently small. On the other hand, under the same assumption on the vorticity, one can show that the velocity field u given by the Biot-Savart law in $\mathbb{R}^3$ belongs to the space $\mathrm{BMO}^{-1}(\mathbb{R}^3)$ defined by the norm

$$\|u\|_{\mathrm{BMO}^{-1}} \;=\; \sup_{x\in\mathbb{R}^3}\,\sup_{R>0}\left(\frac{1}{R^3}\int_{B(x,R)}\int_0^{R^2} |e^{t\Delta}u|^2\,\mathrm{d}t\,\mathrm{d}x\right)^{1/2},$$

where $e^{t\Delta}$ denotes the heat semigroup in $\mathbb{R}^3$. In fact u can be approximated by smooth compactly supported functions in $\mathrm{BMO}^{-1}(\mathbb{R}^3)$, so that $u \in \mathrm{VMO}^{-1}(\mathbb{R}^3)$. Thus we can also invoke the celebrated result by Koch & Tataru [16] to obtain the existence of a unique local solution to the Navier-Stokes equation in $\mathbb{R}^3$, which is again global in time if the norm $\|u\|_{\mathrm{BMO}^{-1}}$ is sufficiently small. In contrast to the general results in [12, 16], the approach we follow to solve the Cauchy problem for Eq. (1.6) uses specific features of the axisymmetric case. As we shall see in Section 4, it is elementary and completely parallel to the two-dimensional situation which was studied e.g. in [3, 7].

The assertion of *global well-posedness* in Theorem 1.1 is also quite natural in view of the historical results by Ladyzhenskaya [17] and Ukhovkii & Yudovich [23]. As in [17, 23] we use the structure of equation (1.8) to derive a priori estimates on the quantity η in $L^p(\mathbb{R}^3)$, for $1 \leqslant p \leqslant \infty$. However, since the solutions we consider do not have finite energy in general, we cannot apply the classical energy estimate to obtain a uniform bound on the velocity field in $L^2(\mathbb{R}^3)$. Instead we prove that any solution of (1.6) with initial data $\omega_0 \in L^1(\Omega)$ satisfies $\|\omega_\theta(t)\|_{L^1(\Omega)} \leqslant \|\omega_0\|_{L^1(\Omega)}$ for all $t > 0$ and

$$\sup_{t>0} t\|\omega_\theta(t)\|_{L^\infty(\Omega)} \leqslant C\big(\|\omega_0\|_{L^1(\Omega)}\big)~, \tag{1.14}$$

where $C(s) = \mathcal{O}(s)$ as $s \to 0$. This new a priori estimate is scale invariant, and implies that all solutions of (1.6) with initial data in $L^1(\Omega)$ are global. Using the axisymmetric Biot-Savart law, we also deduce the following optimal bound on the velocity field :

$$\sup_{t>0} t^{1/2}\|u(t)\|_{L^\infty(\Omega)} \leqslant C\big(\|\omega_0\|_{L^1(\Omega)}\big)~. \tag{1.15}$$

Our last comment on Theorem 1.1 concerns the *long-time behavior*, which differs significantly from what happens in the two-dimensional case. In the latter situation, the L^1 norm of the vorticity is non-increasing in time, but does not converge to zero in general (in particular, it is constant for solutions with a definite sign). The long-time behavior is described by self-similar solutions, called Oseen vortices, which have a nonzero total circulation [8, 9]. In contrast, the axisymmetric vorticity ω_θ vanishes on the boundary of the half-plane Ω, so that the L^1 norm is strictly decreasing for all non-trivial solutions. As we shall see in Section 6, this implies that $\|\omega_\theta(t)\|_{L^1(\Omega)} \to 0$ when $t \to \infty$, as asserted in (1.11). In other words, the long-time behavior of the axisymmetric vorticity is trivial when measured in scale invariant function spaces. More can be said when the initial data have a definite sign and a finite impulse, given by (1.12). In that case, the axisymmetric vorticity ω_θ inherits the same properties for all positive times and converges as $t \to \infty$ to a self-similar solution of the linearized equation (3.1), whose profile is explicitly determined in (1.13).

As in the two-dimensional case, it is possible to extend the local existence claim in Theorem 1.1 to a larger class of initial data, so as to include finite measures as initial vorticities. Let $\mathcal{M}(\Omega)$ denote the set of all real-valued finite measures on the half-plane Ω, equipped with the total variation norm

$$\|\mu\|_{\rm tv} = \sup\left\{\int_\Omega \phi\,{\rm d}\mu \,\Big|\, \phi \in C_0(\Omega)~,~ \|\phi\|_{L^\infty(\Omega)} \leqslant 1\right\}~, \qquad \text{for } \mu \in \mathcal{M}(\Omega)~,$$

where $C_0(\Omega)$ is the set of all real-valued continuous functions on Ω that vanish at infinity and on the boundary $\partial\Omega$. If $\mu \in \mathcal{M}(\Omega)$ is absolutely continuous with respect to Lebesgue's measure, then $\mu = \omega_\theta\,{\rm d}r\,{\rm d}z$ for some $\omega_\theta \in L^1(\Omega)$, and $\|\mu\|_{\rm tv} = \|\omega_\theta\|_{L^1(\Omega)}$. More generally, one can decompose any $\mu \in \mathcal{M}(\Omega)$ as $\mu = \mu_{ac}+\mu_{sc}+\mu_{pp}$, where μ_{ac} is absolutely continuous with respect to Lebesgue's measure, μ_{pp} is a countable collection of Dirac masses, and μ_{sc} has no atoms but is supported on a set of zero Lebesgue measure. In the original three-dimensional picture, each Dirac mass in the atomic part μ_{pp} corresponds to a circular vortex filament, whereas vortex sheets are included in the singularly continuous part μ_{sc}.

The proof of Theorem 1.1 can be adapted to initial vorticities in $\mathcal{M}(\Omega)$, and gives the following statement, which is our second main result.

Theorem 1.3. — *There exist positive constants ϵ and C such that, for any initial data $\omega_0 \in \mathcal{M}(\Omega)$ with $\|(\omega_0)_{\rm pp}\|_{\rm tv} \leqslant \epsilon$, the axisymmetric vorticity equation*

(1.6) *has a unique global mild solution* $\omega_\theta \in C^0((0,\infty), L^1(\Omega) \cap L^\infty(\Omega))$ *satisfying*

$$\limsup_{t\to 0} \|\omega_\theta(t)\|_{L^1(\Omega)} < \infty \ , \qquad \limsup_{t\to 0} t^{1/4}\|\omega_\theta(t)\|_{L^{4/3}(\Omega)} \leqslant C\epsilon \ , \tag{1.16}$$

and such that $\omega_\theta(t) \rightharpoonup \omega_0$ *as* $t \to 0$. *Moreover, the asymptotic estimates for* $t \to \infty$ *given in Theorem 1.1 hold without change.*

Observe that we now have a limitation on the size of the data, which however only affects the atomic part of the initial vorticity. This technical restriction inevitably occurs if local existence is established using a fixed point argument in scale invariant spaces, as we do in Section 4. In the two-dimensional case, early results by Giga, Miyakawa, & Osada [11] and by Kato [14] had a similar limitation, which was then relaxed in [9, 6] using completely different techniques. In the axisymmetric situation, existence of a global solution to (1.6) with a large Dirac mass as initial vorticity has recently been established by Feng and Šverák [5], using an approximation argument, but uniqueness is still under investigation. For a general initial vorticity $\omega_0 \in \mathcal{M}(\Omega)$, both existence and uniqueness are open.

Even if we restrict ourselves to initial vorticities with a small atomic part, the uniqueness claim in Theorem 1.3 is probably not optimal. Indeed, although the solutions we construct satisfy both estimates in (1.16), we believe that uniqueness should hold (as in the two-dimensional case) under the sole assumptions that $\omega_\theta(t)$ is uniformly bounded in $L^1(\Omega)$ for $t > 0$ and converges weakly to the initial vorticity ω_0 as $t \to 0$. However, technical difficulties arise when adapting the two-dimensional proof to the axisymmetric case, and for the moment we need an additional assumption, such as the second estimate in (1.16), to obtain uniqueness. We hope to clarify that question in a future work.

Remark 1.4. — It is interesting to ask whether our results can be extended to the case where a non-zero axisymmetric forcing is included in the Navier-Stokes equations. For example, one can add a forcing term $f = f(r,z,t)$ to the right-hand side of equation (1.6). If f is "sufficiently regular" and decays "sufficiently fast" as $r + |z| + t \to \infty$, our results remain true, with quite straightforward modifications of the proofs. However, if we wish to obtain results under optimal, scale invariant assumptions such as

$$\int_0^\infty \int_\Omega |f(r,z,t)| \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \ < \ \infty \ ,$$

non-trivial modifications seem to be needed. We thank the anonymous referee for raising this interesting point, which we plan to address in a future work.

The rest of this paper is organized as follows. In Section 2, which is devoted to the axisymmetric Biot-Savart law, we estimate various norms of the velocity field u in terms of the axisymmetric vorticity ω_θ. In Section 3, we show that the semigroup generated by the linearization of (1.6) about the origin satisfies the same $L^p - L^q$ estimates as the two-dimensional heat kernel. After these preliminaries, we prove the local existence claims in Theorems 1.1 and 1.3 in Sections 4.1 and 4.2, respectively. Global existence follows from a priori estimates which are established in Section 5. Finally, the long-time behavior is investigated in Section 6. We prove that all solutions of (1.6) converge to zero in $L^1(\Omega)$, and we also compute the leading term in the long-time asymptotics for vorticities with a definite sign and a finite impulse.

2. The axisymmetric Biot-Savart law

In this section, we assume that the axisymmetric vorticity $\omega_\theta : \Omega \to \mathbb{R}$ is given, and we study the properties of the velocity field $u = (u_r, u_z)$ satisfying the linear elliptic system (1.5). The divergence-free condition $\partial_r(ru_r) + \partial_z(ru_z) = 0$ implies that there exists a function $\psi : \Omega \to \mathbb{R}$ such that

$$u_r \;=\; -\frac{1}{r}\frac{\partial\psi}{\partial z}\;, \qquad u_z \;=\; \frac{1}{r}\frac{\partial\psi}{\partial r}\;. \tag{2.1}$$

As $\partial_z u_r - \partial_r u_z = \omega_\theta$, the *axisymmetric stream function* ψ satisfies the following linear elliptic equation in the half-space Ω :

$$-\,\partial_r^2\psi + \frac{1}{r}\partial_r\psi - \partial_z^2\psi \;=\; r\omega_\theta\;. \tag{2.2}$$

Boundary conditions for (2.2) are determined by observing that, for a smooth axisymmetric vector field $u : \mathbb{R}^3 \to \mathbb{R}^3$ with $\operatorname{div} u = 0$, the stream function defined by (2.1) satisfies the asymptotic expansion

$$\psi(r,z) \;=\; \psi_0 + r^2\psi_2(z) + \mathcal{O}(r^4)\;, \qquad \text{as } r \to 0\;, \tag{2.3}$$

see [19] for an extensive discussion of these regularity issues. Without loss of generality, we can assume that the constant ψ_0 in (2.3) is equal to zero, in which case we conclude that $\psi(0,z) = \partial_r\psi(0,z) = 0$.

The solution of (2.2) with these boundary conditions is well known, see e.g. [5]. If we assume that the vorticity ω_θ decays sufficiently fast at infinity, we have the explicit representation

$$\psi(r,z) \;=\; \frac{1}{2\pi}\int_\Omega \sqrt{r\bar r}\,F\left(\frac{(r-\bar r)^2 + (z-\bar z)^2}{r\bar r}\right)\omega_\theta(\bar r,\bar z)\,\mathrm{d}\bar r\,\mathrm{d}\bar z\;, \tag{2.4}$$

where the function $F : (0,\infty) \to \mathbb{R}$ is defined by

$$F(s) \;=\; \int_0^\pi \frac{\cos\phi\;\mathrm{d}\phi}{\big(2(1-\cos\phi)+s\big)^{1/2}} \;=\; \int_0^{\pi/2}\frac{\cos(2\phi)\;\mathrm{d}\phi}{\big(\sin^2\phi + s/4\big)^{1/2}}\;, \quad s > 0\;. \tag{2.5}$$

Useful properties of F are collected in the following lemma, whose proof can be found in [22, Section 19].

Lemma 2.1. — *The function $F : (0,\infty) \to \mathbb{R}$ defined by (2.5) is decreasing and satisfies the asymptotic expansions:*

i) $F(s) = \log\Big(\dfrac{8}{\sqrt{s}}\Big) - 2 + \mathcal{O}\Big(s\log\dfrac{1}{s}\Big)$ *and* $F'(s) = -\dfrac{1}{2s} + \mathcal{O}\Big(\log\dfrac{1}{s}\Big)$ *as* $s \to 0$*;*

ii) $F(s) = \dfrac{\pi}{2s^{3/2}} + \mathcal{O}\Big(\dfrac{1}{s^{5/2}}\Big)$ *and* $F'(s) = -\dfrac{3\pi}{4s^{5/2}} + \mathcal{O}\Big(\dfrac{1}{s^{7/2}}\Big)$ *as* $s \to \infty$*.*

Remark 2.2. — It follows in particular from Lemma 2.1 that the maps $s \mapsto s^\alpha F(s)$ and $s \mapsto s^\beta F'(s)$ are bounded if $0 < \alpha \leqslant 3/2$ and $1 \leqslant \beta \leqslant 5/2$. These observations will be constantly used in the subsequent proofs.

Combining (2.1) and (2.4), we obtain explicit formulas for the axisymmetric Biot-Savart law:

$$\begin{aligned} u_r(r,z) \;&=\; \int_\Omega G_r(r,z,\bar r,\bar z)\omega_\theta(\bar r,\bar z)\,\mathrm{d}\bar r\,\mathrm{d}\bar z\;,\\ u_z(r,z) \;&=\; \int_\Omega G_z(r,z,\bar r,\bar z)\omega_\theta(\bar r,\bar z)\,\mathrm{d}\bar r\,\mathrm{d}\bar z\;, \end{aligned} \tag{2.6}$$

where

$$G_r(r,z,\bar r,\bar z) \;=\; -\frac{1}{\pi}\frac{z-\bar z}{r^{3/2}\,\bar r^{1/2}}\,F'(\xi^2)\;, \qquad \xi^2 \;=\; \frac{(r-\bar r)^2 + (z-\bar z)^2}{r\bar r}\;, \tag{2.7}$$

$$G_z(r,z,\bar r,\bar z) \;=\; \frac{1}{\pi}\frac{r-\bar r}{r^{3/2}\,\bar r^{1/2}}\,F'(\xi^2) + \frac{1}{4\pi}\frac{\bar r^{1/2}}{r^{3/2}}\Big(F(\xi^2) - 2\xi^2F'(\xi^2)\Big)\;. \tag{2.8}$$

Our first result gives elementary estimates for the axisymmetric Biot-Savart law in usual Lebesgue spaces. We emphasize the striking similarity with the corresponding bounds for the two-dimensional Biot-Savart law in the plane $\mathbb{R}^2$, see e.g. [8, Lemma 2.1].

Proposition 2.3. — *The following properties hold for the velocity field u defined from the vorticity ω_θ via the axisymmetric Biot-Savart law* (2.6).
i) Assume that $1 < p < 2 < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. If $\omega_\theta \in L^p(\Omega)$, then $u \in L^q(\Omega)^2$ and

$$\|u\|_{L^q(\Omega)} \leqslant C\|\omega_\theta\|_{L^p(\Omega)} \, . \tag{2.9}$$

ii) If $1 \leqslant p < 2 < q \leqslant \infty$ and $\omega_\theta \in L^p(\Omega) \cap L^q(\Omega)$, then $u \in L^\infty(\Omega)^2$ and

$$\|u\|_{L^\infty(\Omega)} \leqslant C\|\omega_\theta\|_{L^p(\Omega)}^{\sigma} \|\omega_\theta\|_{L^q(\Omega)}^{1-\sigma} \, , \quad \textit{where} \quad \sigma = \frac{p}{2}\frac{q-2}{q-p} \in (0,1) \, . \tag{2.10}$$

Proof. — Both assertions follow from the basic estimate

$$|G_r(r,z,\bar r,\bar z)| + |G_z(r,z,\bar r,\bar z)| \leqslant \frac{C}{\big((r-\bar r)^2 + (z-\bar z)^2\big)^{1/2}} \, , \tag{2.11}$$

which holds for all $(r,z) \in \Omega$ and all $(\bar r,\bar z) \in \Omega$. At a heuristic level, estimate (2.11) follows quite naturally from the scaling properties of $G_r(r,z,\bar r,\bar z)$ and $G_z(r,z,\bar r,\bar z)$ if we observe that these functions behave for (r,z) close to $(\bar r,\bar z)$ like the two-dimensional velocity field generated by a Dirac mass located at $(\bar r,\bar z)$ in $\mathbb{R}^2$. To prove (2.11) rigorously, we first bound the radial component G_r. We distinguish two cases:
a) If $\bar r \leqslant 2r$, we use the fact that $\xi^2 F'(\xi^2)$ is bounded, and obtain the estimate

$$\begin{aligned} |G_r(r,z,\bar r,\bar z)| &\leqslant C\,\frac{|z-\bar z|}{r^{3/2}\,\bar r^{1/2}}\,\frac{r\bar r}{(r-\bar r)^2 + (z-\bar z)^2} \\ &= C\,\frac{\bar r^{1/2}}{r^{1/2}}\,\frac{|z-\bar z|}{(r-\bar r)^2 + (z-\bar z)^2} \leqslant \frac{C}{\big((r-\bar r)^2 + (z-\bar z)^2\big)^{1/2}} \, , \end{aligned}$$

because $\bar r/r \leqslant 2$ and $|z-\bar z| \leqslant \big((r-\bar r)^2 + (z-\bar z)^2\big)^{1/2}$.
b) If $\bar r > 2r$, observing that $\xi^3 F'(\xi^2)$ is bounded, we deduce

$$\begin{aligned} |G_r(r,z,\bar r,\bar z)| &\leqslant C\,\frac{|z-\bar z|}{r^{3/2}\,\bar r^{1/2}}\,\frac{r^{3/2}\,\bar r^{3/2}}{\big((r-\bar r)^2 + (z-\bar z)^2\big)^{3/2}} \\ &= C\,\frac{\bar r\,|z-\bar z|}{\big((r-\bar r)^2 + (z-\bar z)^2\big)^{3/2}} \leqslant \frac{C}{\big((r-\bar r)^2 + (z-\bar z)^2\big)^{1/2}} \, , \end{aligned}$$

because $\bar r < 2(\bar r - r) \leqslant 2\big((r-\bar r)^2 + (z-\bar z)^2\big)^{1/2}$. This proves estimate (2.11) for G_r.

Similar calculations give the same bound for the second component G_z too. Indeed, the first term in the right-hand side of (2.8) can be estimated exactly as above, using the obvious fact that $|r-\bar r| \leqslant \big((r-\bar r)^2 + (z-\bar z)^2\big)^{1/2}$. The second term involves the quantity $F(\xi^2) - 2\xi^2 F'(\xi^2)$, which is bounded by $C\xi^{-1}$ in case a) and by $C\xi^{-3}$ in case b). This concludes the proof of (2.11).

Now, since the integral kernel $G = (G_r, G_z)$ in (2.6) satisfies the same bound as the two-dimensional Biot-Savart kernel in $\mathbb{R}^2$, properties i) and ii) in Proposition 2.3 can be established exactly as in the 2D case. Estimate (2.9) thus follows from the Hardy-Littlewood-Sobolev inequality, and the bound (2.10) can be proved by splitting the integration domain and applying Hölder's inequality, see e.g. [8, Lemma 2.1]. □

The proof of Proposition 2.3 shows that the axisymmetric Biot-Savart law (2.6) has the same properties as the usual Biot-Savart law in the whole plane $\mathbb{R}^2$. In fact, it is possible to obtain in the axisymmetric situation weighted inequalities (involving powers of the distance r to the vertical axis) which have no analogue in the 2D case. As an example, we state here an interesting extension of estimate (2.9).

Proposition 2.4. — *Let $\alpha, \beta \in [0,2]$ be such that $0 \leqslant \beta - \alpha < 1$, and assume that $p, q \in (1,\infty)$ satisfy*

$$\frac{1}{q} = \frac{1}{p} - \frac{1+\alpha-\beta}{2} .$$

If $r^\beta \omega_\theta \in L^p(\Omega)$, then $r^\alpha u \in L^q(\Omega)^2$ and we have the bound

$$\|r^\alpha u\|_{L^q(\Omega)} \leqslant C\|r^\beta \omega_\theta\|_{L^p(\Omega)} . \tag{2.12}$$

Proof. — As in the proof of Proposition 2.3, all we need is to establish the pointwise estimate

$$\frac{r^\alpha}{\bar{r}^\beta}\Big(|G_r(r,z,\bar{r},\bar{z})| + |G_z(r,z,\bar{r},\bar{z})|\Big) \leqslant \frac{C}{\big((r-\bar{r})^2+(z-\bar{z})^2\big)^\lambda} , \tag{2.13}$$

for some $\lambda \in (0,1)$. Indeed, once (2.13) is known, the bound (2.12) follows immediately from the Hardy-Littlewood-Sobolev inequality if $q^{-1} = p^{-1} + \lambda - 1$. To prove (2.13), we proceed as before and distinguish three regions: a) $\bar{r}/2 \leqslant r \leqslant 2\bar{r}$; b) $r > 2\bar{r}$; c) $\bar{r} > 2r$. In each region, we bound the quantities $F'(\xi^2)$ and $F(\xi^2) - 2\xi^2 F'(\xi^2)$ appropriately using Remark 2.2. These calculations shows that inequality (2.13) holds with $\lambda = (1+\beta-\alpha)/2$ if and only if the exponents α, β satisfy $0 \leqslant \alpha \leqslant \beta \leqslant 2$. Since we need $\lambda < 1$ we assume in addition that $\beta - \alpha < 1$. The details are straightforward and can be left to the reader. □

Remarks 2.5. — **1.** Similarly, one can establish a weighted analogue of inequality (2.10).

2. If u is replaced by u_r, inequality (2.12) holds for all $\alpha, \beta \in [-1,2]$ such that $0 \leqslant \beta - \alpha < 1$.

3. If we take $\alpha = 1/q$ and $\beta = 1/p$, so that $\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$, inequality (2.12) is equivalent to

$$\|u\|_{L^q(\mathbb{R}^3)} \leqslant C\|\omega\|_{L^p(\mathbb{R}^3)} ,$$

which is well known for the Biot-Savart law in $\mathbb{R}^3$, see e.g. [10, Lemma 2.1].

Finally, we prove other weighted inequalities, which are similar to those considered in [5].

Proposition 2.6. — *The following estimates hold:*

$$\|u\|_{L^\infty(\Omega)} \leqslant C\|r\omega_\theta\|_{L^1(\Omega)}^{1/2}\,\|\omega_\theta/r\|_{L^\infty(\Omega)}^{1/2} , \tag{2.14}$$

$$\Big\|\frac{u_r}{r}\Big\|_{L^\infty(\Omega)} \leqslant C\|\omega_\theta\|_{L^1(\Omega)}^{1/3}\,\|\omega_\theta/r\|_{L^\infty(\Omega)}^{2/3} . \tag{2.15}$$

Proof. — Estimate (2.14) is stated in [5] in the slightly weaker form

$$\|u\|_{L^\infty(\Omega)} \leqslant C\|\omega_\theta\|_{L^1(\Omega)}^{1/4}\|r^2\omega_\theta\|_{L^1(\Omega)}^{1/4}\,\|\omega_\theta/r\|_{L^\infty(\Omega)}^{1/2} ,$$

but the proof given there actually yields the stronger bound (2.14), which can be compared with (2.10) in the case $p = 1$, $q = \infty$. To prove (2.15), we follow the same approach as in [5]. Using translation and scaling invariance, it is sufficient to show that the quantity

$$\frac{u_r}{r}\Big|_{r=1,z=0} = u_r(1,0) = \int_\Omega \frac{z}{\pi r^{1/2}}\,F'\left(\frac{(r-1)^2+z^2}{r}\right)\omega_\theta(r,z)\,\mathrm{d}r\,\mathrm{d}z \tag{2.16}$$

is bounded by $C\|\omega_\theta\|_{L^1(\Omega)}^{1/3}\,\|\omega_\theta/r\|_{L^\infty(\Omega)}^{2/3}$. We decompose $\Omega = I_1 \cup I_2$, where

$$I_1 = \Big\{(r,z) \in \Omega \,\Big|\, \frac{1}{2} \leqslant r \leqslant 2\,,\ -1 \leqslant z \leqslant 1\Big\} , \qquad I_2 = \Omega \setminus I_1 .$$

When integrating over the first region I_1, we use the fact that $|F'(s)| \leqslant Cs^{-1}$ and $r \approx 1$. We thus obtain

$$|u_r^{(1)}(1,0)| \leqslant C\int_{I_1} \frac{|z|}{(r-1)^2+z^2}\,|\omega_\theta(r,z)|\,\mathrm{d}r\,\mathrm{d}z \leqslant C\int_{I_1} \frac{|\omega_\theta(r,z)|}{\big((r-1)^2+z^2\big)^{1/2}}\,\mathrm{d}r\,\mathrm{d}z .$$

As in [5], or in part ii) or Proposition 2.3, we deduce that

$$|u_r^{(1)}(1,0)| \leqslant C\|\omega_\theta\|_{L^1(I_1)}^{1/2}\|\omega_\theta\|_{L^\infty(I_1)}^{1/2} \leqslant C\|\omega_\theta\|_{L^1(I_1)}^{1/3}\|\omega_\theta/r\|_{L^\infty(I_1)}^{2/3} . \tag{2.17}$$

In the complementary region, we use the optimal bound $|F'(s)| \leqslant Cs^{-5/2}$ and we observe that, if $(r,z) \in I_2$, then $(r-1)^2+z^2 \geqslant C(r^2+z^2)$ for some $C > 0$. We thus have

$$|u_r^{(2)}(1,0)| \leqslant C\int_{I_2} \frac{|z|\,r^2}{\bigl((r-1)^2+z^2\bigr)^{5/2}}\,|\omega_\theta(r,z)|\,\mathrm{d}r\,\mathrm{d}z \leqslant C\int_{I_2}\frac{|\omega_\theta(r,z)|}{r^2+z^2}\,\mathrm{d}r\,\mathrm{d}z .$$

Fix any $R > 0$ and denote $\Omega_R = \{(r,z) \in \Omega\,|\,\rho \leqslant R\}$, where $\rho = (r^2+z^2)^{1/2}$. Extending the integration domain from I_2 to $\Omega = \Omega_R \cup (\Omega\setminus\Omega_R)$, we compute

$$\begin{aligned}|u_r^{(2)}(1,0)| &\leqslant C\int_{\Omega_R}\frac{|\omega_\theta(r,z)|}{\rho^2}\,\mathrm{d}r\,\mathrm{d}z + C\int_{\Omega\setminus\Omega_R}\frac{|\omega_\theta(r,z)|}{\rho^2}\,\mathrm{d}r\,\mathrm{d}z\\ &\leqslant C\|\omega_\theta/r\|_{L^\infty(\Omega)}\int_{\Omega_R}\frac{1}{\rho}\,\mathrm{d}r\,\mathrm{d}z + CR^{-2}\int_{\Omega\setminus\Omega_R}|\omega_\theta(r,z)|\,\mathrm{d}r\,\mathrm{d}z\\ &\leqslant CR\|\omega_\theta/r\|_{L^\infty(\Omega)} + CR^{-2}\|\omega_\theta\|_{L^1(\Omega)} .\end{aligned}$$

Optimizing over $R > 0$ gives the bound $|u_r^{(2)}(1,0)| \leqslant C\|\omega_\theta\|_{L^1(\Omega)}^{1/3}\,\|\omega_\theta/r\|_{L^\infty(\Omega)}^{2/3}$, which together with (2.17) implies (2.15). ☐

Remark 2.7. — Estimate (2.15) can also be obtained by observing that

$$\Bigl\|\frac{u_r}{r}\Bigr\|_{L^\infty(\mathbb{R}^3)} \leqslant C\Bigl\|\frac{\omega_\theta}{r}\Bigr\|_{L^{3,1}(\mathbb{R}^3)} \leqslant C\Bigl\|\frac{\omega_\theta}{r}\Bigr\|_{L^1(\mathbb{R}^3)}^{1/3}\Bigl\|\frac{\omega_\theta}{r}\Bigr\|_{L^\infty(\mathbb{R}^3)}^{2/3} , \tag{2.18}$$

where $L^{3,1}(\mathbb{R}^3)$ denotes the Lorentz space. The first inequality in (2.18) is proved in [2, Proposition 4.1], and the second one follows by real interpolation.

3. The semigroup associated with the linearized equation

This section is devoted to the study of the linearized vorticity equation (1.6):

$$\partial_t\omega_\theta = \Bigl(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Bigr)\omega_\theta , \tag{3.1}$$

which is considered in the half-plane $\Omega = \{(r,z)\,|\,r > 0,\ z \in \mathbb{R}\}$, with homogeneous Dirichlet condition at the boundary $r = 0$. Given initial data $\omega_0 \in L^1(\Omega)$, we denote by $\omega_\theta(t) = S(t)\omega_0$ the solution of (3.1) at time $t > 0$.

Lemma 3.1. — *For any $t > 0$, the evolution operator $S(t)$ associated with Eq. (3.1) is given by the explicit formula*

$$(S(t)\omega_0)(r,z) = \frac{1}{4\pi t}\int_\Omega \frac{\bar r^{1/2}}{r^{1/2}}\,H\Bigl(\frac{t}{r\bar r}\Bigr)\,e^{-\frac{(r-\bar r)^2+(z-\bar z)^2}{4t}}\,\omega_0(\bar r,\bar z)\,\mathrm{d}\bar r\,\mathrm{d}\bar z , \tag{3.2}$$

where the function $H : (0,\infty) \to \mathbb{R}$ is defined by

$$H(\tau) = \frac{1}{\sqrt{\pi\tau}}\int_{-\pi/2}^{\pi/2} e^{-\frac{\sin^2\phi}{\tau}}\cos(2\phi)\,\mathrm{d}\phi , \qquad \tau > 0 . \tag{3.3}$$

Proof. — If ω_θ is a solution of (3.1), we observe that the vector valued function $\omega = \omega_\theta e_\theta$ satisfies the usual heat equation $\partial_t\omega = \Delta\omega$ in the whole space $\mathbb{R}^3$. For any $t > 0$, we thus have the solution formula

$$\omega(x,t) = \frac{1}{(4\pi t)^{3/2}}\int_{\mathbb{R}^3} e^{-\frac{|x-\bar x|^2}{4t}}\,\omega(\bar x,0)\,\mathrm{d}\bar x , \qquad x \in \mathbb{R}^3 . \tag{3.4}$$

Denoting $x = (r\cos\theta, r\sin\theta, z)$ and $\bar x = (\bar r\cos\bar\theta, \bar r\sin\bar\theta, \bar z)$, we can write (3.4) in the form

$$\omega_\theta(r,z,t)\,e_\theta = \frac{1}{(4\pi t)^{3/2}}\int_0^\infty\int_{\mathbb{R}}\int_{-\pi}^{\pi} e^{-\frac{|x-\bar x|^2}{4t}}\,\omega_0(\bar r,\bar z)\,e_{\bar\theta}\,\bar r\,\mathrm{d}\bar\theta\,\mathrm{d}\bar z\,\mathrm{d}\bar r , \tag{3.5}$$

where e_θ is defined in (1.3) and

$$|x-\bar{x}|^2 \;=\; (r-\bar{r})^2+(z-\bar{z})^2+4r\bar{r}\sin^2\frac{\theta-\bar{\theta}}{2}\;.$$

Now, if we integrate over the angle $\bar{\theta}$ in (3.5) and use the definition (3.3) of H, we see that (3.5) is equivalent to $\omega_\theta(t)=S(t)\omega_0$ with $S(t)$ defined by (3.2). □

The function H cannot be expressed in terms of elementary functions, but all we need to know is the behavior of $H(\tau)$ as $\tau\to 0$ and $\tau\to\infty$.

Lemma 3.2. — *The function $H:(0,\infty)\to\mathbb{R}$ defined by* (3.3) *is smooth and satisfies the asymptotic expansions:*

i) $H(\tau)=1-\dfrac{3\tau}{4}+\mathcal{O}(\tau^2)$ *and* $H'(\tau)=-3/4+\mathcal{O}(\tau)$ *as* $\tau\to 0$;

ii) $H(\tau)=\dfrac{\pi^{1/2}}{4\tau^{3/2}}+\mathcal{O}\Big(\dfrac{1}{\tau^{5/2}}\Big)$ *and* $H'(\tau)=-\dfrac{3\pi^{1/2}}{8\tau^{5/2}}+\mathcal{O}\Big(\dfrac{1}{\tau^{7/2}}\Big)$ *as* $\tau\to\infty$.

Proof. — Expansion ii) follows immediately from (3.3) if we replace, in the expression $e^{-\frac{\sin^2\phi}{\tau}}$, the exponential function by its Taylor series at the origin. Expansion i) can be deduced from the formula

$$H(\tau)\;=\;\frac{1}{\sqrt{\pi}}\int_{-\frac{1}{\sqrt{\tau}}}^{\frac{1}{\sqrt{\tau}}} e^{-x^2}\,\frac{1-2\tau x^2}{\sqrt{1-\tau x^2}}\,\mathrm{d}x\;,$$

which is obtained by substituting $x=\frac{\sin\phi}{\sqrt{\tau}}$ in the integral (3.3). □

Remark 3.3. — We believe that the function H is decreasing, although we do not have a simple proof. In what follows, we only use the fact that the maps $\tau\mapsto\tau^\alpha H(\tau)$ and $\tau\mapsto\tau^\beta H'(\tau)$ are bounded if $0\leqslant\alpha\leqslant 3/2$ and $0\leqslant\beta\leqslant 5/2$.

Let $(S(t))_{t\geqslant 0}$ be the family of linear operators defined by (3.2) for $t>0$ and by $S(0)=\mathbf{1}$ (the identity operator). By construction, we have the semigroup property: $S(t_1+t_2)=S(t_1)S(t_2)$ for all $t_1,t_2\geqslant 0$. Further important properties are collected in the following proposition.

Proposition 3.4. — *The family $(S(t))_{t\geqslant 0}$ defined by* (3.2) *is a strongly continuous semigroup of bounded linear operators in $L^p(\Omega)$ for any $p\in[1,\infty)$. Moreover, if $1\leqslant p\leqslant q\leqslant\infty$, the following estimates hold:*
i) If $\omega_0\in L^p(\Omega)$, then

$$\|S(t)\omega_0\|_{L^q(\Omega)}\;\leqslant\;\frac{C}{t^{\frac{1}{p}-\frac{1}{q}}}\,\|\omega_0\|_{L^p(\Omega)}\;,\qquad t>0\;. \tag{3.6}$$

ii) If $f=(f_r,f_z)\in L^p(\Omega)^2$, then

$$\|S(t)\operatorname{div}_* f\|_{L^q(\Omega)}\;\leqslant\;\frac{C}{t^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}}\,\|f\|_{L^p(\Omega)}\;,\qquad t>0\;, \tag{3.7}$$

where $\operatorname{div}_* f=\partial_r f_r+\partial_z f_z$ denotes the two-dimensional divergence of f.

Proof. — We claim that

$$\frac{1}{4\pi t}\frac{\bar{r}^{1/2}}{r^{1/2}}H\Big(\frac{t}{r\bar{r}}\Big)\,e^{-\frac{(r-\bar{r})^2+(z-\bar{z})^2}{4t}}\;\leqslant\;\frac{C}{t}\,e^{-\frac{(r-\bar{r})^2+(z-\bar{z})^2}{5t}}\;, \tag{3.8}$$

for all $(r,z)\in\Omega$, all $(\bar{r},\bar{z})\in\Omega$, and all $t>0$. Indeed, since H is bounded, estimate (3.8) is obvious when $\bar{r}\leqslant 2r$. If $\bar{r}>2r$, we observe that $\tau^{1/2}H(\tau)$ is bounded, so that

$$\frac{\bar{r}^{1/2}}{r^{1/2}}H\Big(\frac{t}{r\bar{r}}\Big)\;\leqslant\;C\,\frac{\bar{r}^{1/2}}{r^{1/2}}\,\frac{r^{1/2}\,\bar{r}^{1/2}}{t^{1/2}}\;=\;C\,\frac{\bar{r}}{t^{1/2}}\;\leqslant\;C\,\frac{|r-\bar{r}|}{t^{1/2}}\;,$$

where in the last inequality we used the fact that $\bar{r}<2(\bar{r}-r)=2|r-\bar{r}|$. As $xe^{-x^2/4}\leqslant Ce^{-x^2/5}$ for all $x\geqslant 0$, we conclude that (3.8) holds in all cases. This provides for the integral kernel in (3.2) a pointwise upper bound in terms of the usual heat kernel in the whole plane $\mathbb{R}^2$, with a diffusion coefficient equal to 5/4 instead of 1. Thus estimate (3.6) follows immediately from Young's inequality, as in the 2D case.

To prove (3.7), we assume that $\omega_0 = \mathrm{div}_* f = \partial_r f_r + \partial_z f_z$, and we integrate by parts in (3.2) to obtain the identity

$$(S(t)\,\mathrm{div}_* f)(r,z) \;=\; \frac{1}{4\pi t}\int_\Omega \frac{\bar r^{1/2}}{r^{1/2}}\, e^{-\frac{(r-\bar r)^2+(z-\bar z)^2}{4t}}\,(A_r f_r + A_z f_z)\,\mathrm{d}\bar r\,\mathrm{d}\bar z\ ,$$

where

$$A_r \;=\; \frac{t}{r\bar r^2}\,H'\Bigl(\frac{t}{r\bar r}\Bigr) - \Bigl(\frac{1}{2\bar r} + \frac{r-\bar r}{2t}\Bigr)H\Bigl(\frac{t}{r\bar r}\Bigr)\ , \qquad A_z \;=\; -\frac{z-\bar z}{2t}\,H\Bigl(\frac{t}{r\bar r}\Bigr)\ .$$

Proceeding as above and using Remark 3.3, it is straightforward to verify that

$$\frac{1}{4\pi t}\frac{\bar r^{1/2}}{r^{1/2}}\, e^{-\frac{(r-\bar r)^2+(z-\bar z)^2}{4t}}\Bigl(|A_r| + |A_z|\Bigr) \;\leqslant\; \frac{C}{t^{3/2}}\, e^{-\frac{(r-\bar r)^2+(z-\bar z)^2}{5t}}\ , \tag{3.9}$$

for all $(r,z)\in\Omega$, all $(\bar r,\bar z)\in\Omega$, and all $t>0$. Thus estimate (3.7) follows again from Young's inequality, as in the 2D case.

Finally, we show that the semigroup $(S(t))_{t\geqslant 0}$ is strongly continuous in $L^p(\Omega)$ if $1\leqslant p<\infty$. All we need to verify is the continuity at the origin. Given $\omega_0\in L^p(\Omega)$, we denote by $\overline{\omega}_0 : \mathbb{R}^2\to\mathbb{R}$ the function obtained by extending ω_0 by zero outside Ω. Using the change of variables $\bar r = r + \sqrt{t}\rho$, $\bar z = z + \sqrt{t}\zeta$ in (3.2), we obtain the identity

$$\Bigl(S(t)\omega_0 - \omega_0\Bigr)(r,z) \;=\; \frac{1}{4\pi}\int_{\mathbb{R}^2} e^{-\frac{\rho^2+\zeta^2}{4}}\,\Psi(r,z,\rho,\zeta,t)\,\mathrm{d}\rho\,\mathrm{d}\zeta\ ,$$

for all $(r,z)\in\Omega$, where

$$\Psi(r,z,\rho,\zeta,t) \;=\; \Bigl(1+\frac{\sqrt{t}\rho}{r}\Bigr)^{1/2} H\Bigl(\frac{t}{r(r+\sqrt{t}\rho)}\Bigr)\,\overline{\omega}_0(r+\sqrt{t}\rho, z+\sqrt{t}\zeta) - \overline{\omega}_0(r,z)\ .$$

By Minkowski's integral inequality, we deduce that

$$\|S(t)\omega_0-\omega_0\|_{L^p(\Omega)} \;\leqslant\; \frac{1}{4\pi}\int_{\mathbb{R}^2} e^{-\frac{\rho^2+\zeta^2}{4}}\,\|\Psi(\cdot,\cdot,\rho,\zeta,t)\|_{L^p(\Omega)}\,\mathrm{d}\rho\,\mathrm{d}\zeta\ . \tag{3.10}$$

Now, the estimates we used in the proof of (3.8) show that

$$\Bigl(1+\frac{\sqrt{t}\rho}{r}\Bigr)^{1/2} H\Bigl(\frac{t}{r(r+\sqrt{t}\rho)}\Bigr) \;\leqslant\; C(1+|\rho|)\ , \tag{3.11}$$

whenever $r>0$ and $r+\sqrt{t}\rho>0$. This immediately implies that

$$\|\Psi(\cdot,\cdot,\rho,\zeta,t)\|_{L^p(\Omega)} \;\leqslant\; C(1+|\rho|)\|\omega_0\|_{L^p(\Omega)}\ ,$$

for all $(\rho,\zeta)\in\mathbb{R}^2$ and all $t>0$. Since the left-hand side of (3.11) converges to 1 as $t\to 0$, and since translations act continuously in $L^p(\mathbb{R}^2)$ for $p<\infty$, it also follows from estimate (3.11) and Lebesgue's dominated convergence theorem that

$$\|\Psi(\cdot,\cdot,\rho,\zeta,t)\|_{L^p(\Omega)} \;\longrightarrow\; 0 \quad \text{as } t\to 0\ ,$$

for all $(\rho,\zeta)\in\mathbb{R}^2$. Thus another application of Lebesgue's theorem implies that the right-hand side of (3.10) converges to zero as $t\to 0$, which is the desired result. □

As in the previous section, it is possible to derive weighted estimates for the linear semigroup (3.2). The proof of the following result is very similar to that of Propositions 2.4 and 3.4, and can thus be left to the reader.

Proposition 3.5. — *Let $1\leqslant p\leqslant q\leqslant\infty$ and $-1\leqslant\alpha\leqslant\beta\leqslant 2$. If $r^\beta\omega_0\in L^p(\Omega)$, then*

$$\|r^\alpha S(t)\omega_0\|_{L^q(\Omega)} \;\leqslant\; \frac{C}{t^{\frac{1}{p}-\frac{1}{q}+\frac{\beta-\alpha}{2}}}\,\|r^\beta\omega_0\|_{L^p(\Omega)}\ , \qquad t>0\ . \tag{3.12}$$

Moreover, if $-1 \leqslant \alpha \leqslant \beta \leqslant 1$ *and* $r^\beta f \in L^p(\Omega)^2$, *then*

$$\|r^\alpha S(t)\,\mathrm{div}_* f\|_{L^q(\Omega)} \leqslant \frac{C}{t^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}+\frac{\beta-\alpha}{2}}}\,\|r^\beta f\|_{L^p(\Omega)}\;, \qquad t > 0\;. \tag{3.13}$$

4. Local existence of solutions

Equipped with the results of the previous sections, we now take up the proof of Theorems 1.1 and 1.3. In view of the divergence-free condition in (1.5), the evolution equation (1.6) for the axisymmetric vorticity ω_θ can be written in the equivalent form

$$\partial_t\omega_\theta + \mathrm{div}_*(u\,\omega_\theta) \;=\; \Big(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big)\omega_\theta\;, \tag{4.1}$$

where $\mathrm{div}_*(u\,\omega_\theta) = \partial_r(u_r\omega_\theta) + \partial_z(u_z\omega_\theta)$. Given initial data ω_0, the integral equation associated with (4.1) is

$$\omega_\theta(t) \;=\; S(t)\omega_0 - \int_0^t S(t-s)\,\mathrm{div}_*(u(s)\omega_\theta(s))\,\mathrm{d}s\;, \qquad t > 0\;, \tag{4.2}$$

where $S(t)$ denotes the linear semigroup defined in (3.2). In this section, our goal is to prove local existence and uniqueness of solutions to (4.2) using a standard fixed point argument, in the spirit of Kato [13]. For the sake of clarity, we first treat the case where $\omega_0 \in L^1(\Omega)$, and then consider the more complicated situation where ω_0 is a finite measure with sufficiently small atomic part. This will establish the local well-posedness claims in Theorems 1.1 and 1.3, respectively.

4.1. **Local existence when the initial vorticity is integrable.**

Proposition 4.1. — *For any initial data* $\omega_0 \in L^1(\Omega)$, *there exists a positive time* $T = T(\omega_0)$ *such that the integral equation* (4.2) *has a unique solution*

$$\omega_\theta \in C^0([0,T], L^1(\Omega)) \cap C^0((0,T], L^\infty(\Omega))\;. \tag{4.3}$$

Moreover $\|\omega_\theta(t)\|_{L^1(\Omega)} \leqslant \|\omega_0\|_{L^1(\Omega)}$ *for all* $t \in [0,T]$, *and estimate* (1.10) *holds. Finally, if* $\|\omega_0\|_{L^1(\Omega)}$ *is small enough, the local existence time* $T > 0$ *can be taken arbitrarily large.*

Proof. — We follow the same approach as in the two-dimensional case, see e.g. [3, 7, 14]. Given $T > 0$ we introduce the function space

$$X_T \;=\; \Big\{\omega_\theta \in C^0((0,T], L^{4/3}(\Omega)) \,\Big|\, \|\omega_\theta\|_{X_T} < \infty\Big\}\;, \tag{4.4}$$

equipped with the norm

$$\|\omega_\theta\|_{X_T} \;=\; \sup_{0<t\leqslant T} t^{1/4}\|\omega_\theta(t)\|_{L^{4/3}(\Omega)}\;.$$

For any $t \geqslant 0$, we denote $\omega_{\mathrm{lin}}(t) = S(t)\omega_0$, where $S(t)$ is the linear semigroup (3.2). It then follows from Proposition 3.4 that $\omega_{\mathrm{lin}} \in X_T$ for any $T > 0$. For later use, we define

$$C_1(\omega_0,T) \;=\; \|\omega_{\mathrm{lin}}\|_{X_T} \;=\; \sup_{0<t\leqslant T} t^{1/4}\|S(t)\omega_0\|_{L^{4/3}(\Omega)}\;. \tag{4.5}$$

In view of (3.6), there exists a universal constant $C_2 > 0$ such that $C_1(\omega_0,T) \leqslant C_2\|\omega_0\|_{L^1(\Omega)}$ for any $T > 0$. Moreover, since $L^1(\Omega) \cap L^{4/3}(\Omega)$ is dense in $L^1(\Omega)$, it also follows from (3.6) that $C_1(\omega_0,T) \to 0$ as $T \to 0$, for any $\omega_0 \in L^1(\Omega)$.

Given $\omega_\theta \in X_T$ and $p \in [1,2)$, we define a map $\mathcal{F}\omega_\theta : (0,T] \to L^p(\Omega)$ in the following way:

$$(\mathcal{F}\omega_\theta)(t) \;=\; \int_0^t S(t-s)\,\mathrm{div}_*(u(s)\omega_\theta(s))\,\mathrm{d}s\;, \qquad 0 < t \leqslant T\;, \tag{4.6}$$

where it is understood that $u(s)$ is the velocity field obtained from $\omega_\theta(s)$ via the axisymmetric Biot-Savart law (2.6). Using estimate (3.7), Hölder's inequality, and the bound (2.9), we obtain for $t \in (0,T]$:

$$
\begin{aligned}
t^{1-\frac{1}{p}}\|(\mathcal{F}\omega_\theta)(t)\|_{L^p(\Omega)} &\leqslant t^{1-\frac{1}{p}} \int_0^t \frac{C}{(t-s)^{\frac{3}{2}-\frac{1}{p}}} \|u(s)\omega_\theta(s)\|_{L^1(\Omega)} \,\mathrm{d}s \\
&\leqslant t^{1-\frac{1}{p}} \int_0^t \frac{C}{(t-s)^{\frac{3}{2}-\frac{1}{p}}} \|u(s)\|_{L^4(\Omega)}\|\omega_\theta(s)\|_{L^{4/3}(\Omega)} \,\mathrm{d}s \\
&\leqslant t^{1-\frac{1}{p}} \int_0^t \frac{C}{(t-s)^{\frac{3}{2}-\frac{1}{p}}} \|\omega_\theta(s)\|_{L^{4/3}(\Omega)}^2 \,\mathrm{d}s \qquad (4.7)\\
&\leqslant t^{1-\frac{1}{p}} \int_0^t \frac{C}{(t-s)^{\frac{3}{2}-\frac{1}{p}}} \frac{\|\omega_\theta\|_{X_T}^2}{s^{\frac{1}{2}}} \,\mathrm{d}s \;\leqslant\; C\|\omega_\theta\|_{X_T}^2 \;.
\end{aligned}
$$

It is also straightforward to verify that the quantity $(\mathcal{F}\omega_\theta)(t)$ depends continuously on the time parameter $t \in (0,T]$ in the topology of $L^p(\Omega)$. Choosing $p = 4/3$, we deduce that $\mathcal{F}\omega_\theta \in X_T$ and $\|\mathcal{F}\omega_\theta\|_{X_T} \leqslant C_3\|\omega_\theta\|_{X_T}^2$ for some $C_3 > 0$. More generally, we have the Lipschitz estimate

$$
\|\mathcal{F}\omega_\theta - \mathcal{F}\tilde\omega_\theta\|_{X_T} \;\leqslant\; C_3\big(\|\omega_\theta\|_{X_T} + \|\tilde\omega_\theta\|_{X_T}\big)\|\omega_\theta - \tilde\omega_\theta\|_{X_T} \;, \qquad (4.8)
$$

for all $\omega_\theta, \tilde\omega_\theta \in X_T$.

Now we consider the map $\mathcal{G} : X_T \to X_T$ defined by $\mathcal{G}\omega_\theta = \omega_{\mathrm{lin}} - \mathcal{F}\omega_\theta$. We fix $R > 0$ such that $2C_3R < 1$, and denote by B_R the closed ball of radius R centered at the origin in X_T. If $C_1(\omega_0,T) \leqslant R/2$, the estimates above show that $\mathcal{G}$ maps B_R into B_R and is a strict contraction there, so that (by the Banach fixed point theorem) $\mathcal{G}$ has a unique fixed point ω_θ in B_R. By construction, ω_θ is a solution to the integral equation (4.2) in X_T. The condition $C_1(\omega_0,T) \leqslant R/2$ can be fulfilled in two different ways. If the initial data are small enough so that $C_2\|\omega_0\|_{L^1(\Omega)} \leqslant R/2$, the existence time $T > 0$ can be chosen arbitrarily, and the fixed point argument therefore establishes *global existence for small data in* $L^1(\Omega)$. On the other hand, for larger initial data ω_0, we can always choose $T > 0$ small enough so that $C_1(\omega_0,T) \leqslant R/2$, hence we also have *local existence for arbitrary data.*

Remarks 4.2. — **1.** For large data, the local existence time $T > 0$ given by the fixed point argument depends on the initial data, and it is not possible to bound T from below using the norm $\|\omega_0\|_{L^1(\Omega)}$ only. However, if $\omega_0 \in L^1(\Omega) \cap L^p(\Omega)$ for some $p > 1$, then (by Proposition 3.4) an upper bound on $\|\omega_0\|_{L^p(\Omega)}$ provides a lower bound on the local existence time T.

2. For later use, we note that the fixed point argument also proves that the solution ω_θ depends continuously on the initial data. More precisely, if $K \subset L^1(\Omega)$ is any compact set, or any sufficiently small neighborhood of a given point, we can take the same local existence time $T > 0$ for all initial data $\omega_0 \in K$, and there exists $C > 0$ such that

$$
\sup_{t\in[0,T]} \|\omega_\theta(t) - \tilde\omega_\theta(t)\|_{L^1(\Omega)} + \|\omega_\theta - \tilde\omega_\theta\|_{X_T} \;\leqslant\; C\|\omega_0 - \tilde\omega_0\|_{L^1(\Omega)} \;, \qquad (4.9)
$$

for all $\omega_0, \tilde\omega_0 \in K$, where $\omega_\theta, \tilde\omega_\theta \in X_T$ denote the solutions corresponding to the initial data $\omega_0, \tilde\omega_0$, respectively.

To conclude the proof of Proposition 4.1, we establish a few additional properties of the local solution $\omega_\theta \in X_T$. We first note that

$$
\lim_{t\to 0} t^{1/4}\|\omega_\theta(t)\|_{L^{4/3}(\Omega)} \;=\; \lim_{T\to 0} \|\omega_\theta\|_{X_T} \;=\; 0 \;, \qquad (4.10)
$$

because, when $T > 0$ is small, the fixed point argument holds in the ball B_R with $R = 2C_1(\omega_0,T)$. This proves (1.10) for $p = 4/3$. Next, using (4.7) with $p = 1$, we see that that the map $t \mapsto (\mathcal{F}\omega_\theta)(t)$ is continuous in the topology of $L^1(\Omega)$ and satisfies $\|(\mathcal{F}\omega_\theta)(t)\|_{L^1(\Omega)} \leqslant C\|\omega_\theta\|_{X_T}^2$ for all $t \in (0,T]$. In view of (4.10), this implies that the map $\omega_\theta - \omega_{\rm lin} \equiv -\mathcal{F}\omega_\theta$ belongs to $C^0([0,T],L^1(\Omega)$ and vanishes at $t=0$. In particular, using Proposition 3.4, we conclude that $\omega_\theta \in C^0([0,T],L^1(\Omega))$. That $\|\omega_\theta(t)\|_{L^1(\Omega)}$ is a non-increasing function of time is a well known fact, which will be discussed in Lemma 5.1 below. Finally, to prove that $\omega_\theta \in C^0((0,T],L^p(\Omega))$ for any $p \in (1,\infty]$ and that (1.10) holds, we use a standard bootstrap argument which we explain in some detail because it will be used again in Section 5. For $p \in (1,\infty]$ we define

$$M_p(T) \,=\, \sup_{0<t\leqslant T} t^{1-\frac{1}{p}}\|S(t)\omega_0\|_{L^p(\Omega)}\,, \quad N_p(T) \,=\, \sup_{0<t\leqslant T} t^{1-\frac{1}{p}}\|\omega_\theta(t)\|_{L^p(\Omega)}\,. \tag{4.11}$$

Then $M_p(T) \to 0$ as $T \to 0$, and we already know that $N_p(T) \to 0$ as $T \to 0$ for $p = 4/3$, hence for all $p \in (1,4/3]$ by interpolation. To prove the same result for $p > 4/3$, we split the integral in (4.2) in two parts and estimate the nonlinear term as follows:

$$\|u\,\omega_\theta\|_{L^r(\Omega)} \,\leqslant\, C\|u\|_{L^s(\Omega)}\|\omega_\theta\|_{L^{q_2}(\Omega)} \,\leqslant\, C\|\omega_\theta\|_{L^{q_1}(\Omega)}\|\omega_\theta\|_{L^{q_2}(\Omega)}\,, \quad s \,=\, \frac{2q_1}{2-q_1}\,,$$

where $\frac{4}{3} \leqslant q_1 < 2$, $\frac{4}{3} \leqslant q_2 \leqslant \infty$, and $\frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{2}$. We thus obtain

$$\begin{aligned}\|\omega_\theta(t)\|_{L^p(\Omega)} \,\leqslant\, \|S(t)\omega_0\|_{L^p(\Omega)} &+ C\int_0^{t/2} \frac{\|\omega_\theta(s)\|_{L^q(\Omega)}^2}{(t-s)^{\frac{2}{q}-\frac{1}{p}}}\,\mathrm{d}s \\ &+ C\int_{t/2}^{t} \frac{\|\omega_\theta(s)\|_{L^{q_1}(\Omega)}\|\omega_\theta(s)\|_{L^{q_2}(\Omega)}}{(t-s)^{\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{p}}}\,\mathrm{d}s\,,\end{aligned} \tag{4.12}$$

where the exponents $p \in [1,\infty]$, $q, q_1 \in [4/3,2)$ and $q_2 \in [4/3,\infty]$ are assumed to satisfy

$$\frac{1}{2} \,\leqslant\, \frac{2}{q} - \frac{1}{p}\,, \qquad \text{and} \quad \frac{1}{2} \,\leqslant\, \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{p} \,<\, 1\,. \tag{4.13}$$

Multiplying both sides of (4.12) by $t^{1-\frac{1}{p}}$ and taking the supremum over $t \in (0,T]$ we obtain the useful bound

$$N_p(T) \,\leqslant\, M_p(T) + C_{p,q}N_q(T)^2 + C_{p,q_1,q_2}N_{q_1}(T)N_{q_2}(T)\,, \tag{4.14}$$

where $C_{p,q}$ and C_{p,q_1,q_2} are positive constants. If we choose $q = q_1 = q_2 = 4/3$, we deduce from (4.14) that $N_p(T) \to 0$ as $T \to 0$ for any $p < 2$. Then, taking $q = 4/3$ and $q_1 = q_2$ sufficiently close to 2, we obtain the same result for any $p < \infty$. Finally, choosing $q = 4/3$, $q_1 = 3/2$, and $q_2 = 4$, we conclude from (4.14) that $N_\infty(T) \to 0$ as $T \to 0$, which proves (1.10).

Our final task is to discuss the uniqueness of the solution to (4.2). We first observe that, given $\omega_0 \in L^1(\Omega)$, the solution $\omega_\theta \in X_T$ given by the fixed point argument is, by construction, the only solution of (4.2) in X_T satisfying (4.10). In fact, using a nice argument due to Brezis [4], it is possible to prove uniqueness in a much larger class. Indeed, given any $T > 0$, assume that $\omega_\theta \in C^0([0,T],L^1(\Omega)) \cap C^0((0,T],L^\infty(\Omega))$ is a mild solution of (4.1) on $(0,T]$ in the sense that the integral equation

$$\omega_\theta(t) \,=\, S(t-t_0)\omega_\theta(t_0) - \int_{t_0}^{t} S(t-s)\,\mathrm{div}_*(u(s)\omega_\theta(s))\,\mathrm{d}s \tag{4.15}$$

is satisfied whenever $0 < t_0 \leqslant t \leqslant T$. The set $K = \{\omega_\theta(t)\,|\,t \in [0,T]\}$ is compact in $L^1(\Omega)$, hence the fixed point argument allows us to construct a local solution in $X_{\tilde{T}}$ for all initial data $\tilde{\omega}_0 \in K$, with a common existence time $\tilde{T} > 0$ (without loss of generality, we assume henceforth that $\tilde{T} \leqslant T/2$). That solution is denoted by

$\tilde{\omega}_\theta(t) = \Sigma(t)\tilde{\omega}_0$ for $t \in [0, \tilde{T}]$. By the observation above, for all $t_0 \in (0, \tilde{T}]$ we have the relation

$$\omega_\theta(t) \,=\, \Sigma(t-t_0)\omega_\theta(t_0)\ , \qquad t \in [t_0, t_0+\tilde{T}]\ , \tag{4.16}$$

because the left-hand side is a solution of (4.15) on the time interval $[t_0, t_0+\tilde{T}]$ and we obviously have $(t-t_0)^{1/4}\|\omega_\theta(t)\|_{L^{4/3}(\Omega)} \to 0$ as $t \to t_0$, which is the analogue of condition (4.10). Now, for any fixed $t \in (0, \tilde{T}]$, it follows from (4.9) that

$$\|\Sigma(t-t_0)\omega_\theta(t_0) - \Sigma(t-t_0)\omega_\theta(0)\|_{L^1(\Omega)} \,\leqslant\, C\|\omega_\theta(t_0) - \omega_\theta(0)\|_{L^1(\Omega)} \,\longrightarrow\, 0\ ,$$

as $t_0 \to 0$, and it is also clear that $\|\Sigma(t-t_0)\omega_\theta(0) - \Sigma(t)\omega_\theta(0)\|_{L^1(\Omega)} \to 0$ as $t_0 \to 0$. Thus taking the limit $t_0 \to 0$ in (4.16) we obtain the relation $\omega_\theta(t) = \Sigma(t)\omega_\theta(0)$ for $t \in [0, \tilde{T}]$, which means that the solution ω_θ we started with coincides on the time interval $[0, \tilde{T}]$ with the solution constructed from the initial data $\omega_\theta(0)$ by the fixed point argument. □

4.2. **The case where the initial vorticity is a finite measure.** We next consider the more general case where the initial vorticity ω_0 in (4.2) is a finite measure on Ω, which is no longer absolutely continuous with respect to the Lebesgue measure $\mathrm{d}r\,\mathrm{d}z$. For convenience we denote $\mu = \omega_0$, and we recall the canonical decomposition $\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$ where μ_{ac} is absolutely continuous with respect to Lebesgue's measure, μ_{pp} is a (countable) collection of point masses, and μ_{sc} is the "singularly continuous" part which has no atoms yet is supported on a set of zero Lebesgue measure. We have $\mu_{ac} \perp \mu_{sc} \perp \mu_{pp}$, which means that the three measures are mutually singular. In particular, the total variation norm of μ satisfies

$$\|\mu\|_{\mathrm{tv}} \,=\, \|\mu_{ac}\|_{\mathrm{tv}} + \|\mu_{sc}\|_{\mathrm{tv}} + \|\mu_{pp}\|_{\mathrm{tv}}\ .$$

The linear semigroup $S(t)$ acts on the measure μ by the formula

$$(S(t)\mu)(r,z) \,=\, \frac{1}{4\pi t}\int_\Omega \frac{\bar{r}^{1/2}}{r^{1/2}}\, H\Big(\frac{t}{r\bar{r}}\Big)\, e^{-\frac{(r-\bar{r})^2+(z-\bar{z})^2}{4t}}\,\mathrm{d}\mu(\bar{r},\bar{z})\ , \tag{4.17}$$

which generalizes (3.2), and we have the following estimates:

Proposition 4.3. — *Let μ be a finite measure on Ω. Then*

$$\sup_{t>0} t^{1-\frac{1}{p}}\,\|S(t)\mu\|_{L^p(\Omega)} \,\leqslant\, C\|\mu\|_{\mathrm{tv}}\ , \qquad 1 \leqslant p \leqslant \infty\ , \tag{4.18}$$

and

$$L_p(\mu) \,:=\, \limsup_{t\to 0}\, t^{1-\frac{1}{p}}\|S(t)\mu\|_{L^p(\Omega)} \,\leqslant\, C\|\mu_{pp}\|_{\mathrm{tv}}\ , \qquad 1 < p \leqslant \infty\ . \tag{4.19}$$

Proof. — Estimate (4.18) can be established as in Proposition 3.4, using the pointwise upper bound (3.8). To prove (4.19) we proceed as in the two-dimensional case [7, 11], with minor modifications. We know from (4.18) that $L_p(\mu) \leqslant C\|\mu\|_{\mathrm{tv}}$, hence using the canonical decomposition we find

$$L_p(\mu) \,\leqslant\, L_p(\mu_{ac}) + L_p(\mu_{sc}) + L_p(\mu_{pp}) \,\leqslant\, L_p(\mu_{ac}) + L_p(\mu_{sc}) + C\|\mu_{pp}\|_{\mathrm{tv}}\ .$$

Therefore, we only need to show that $L_p(\mu_{ac}) = L_p(\mu_{sc}) = 0$. In fact, it is sufficient to prove that for $p = \infty$, because the result then follows for $1 < p < \infty$ by interpolation.

From now on, we thus assume that μ is a non-atomic finite measure on Ω, and we denote by $|\mu|$ the positive measure which represents the total variation of μ. Given any point $\xi = (r,z) \in \mathbb{R}^2$ and any radius $\delta > 0$, we define

$$B(\xi,\delta) \,=\, \big\{\bar{\xi} \in \Omega \,\big|\, |\xi - \bar{\xi}| \leqslant \delta\big\}\ ,$$

where $|\xi - \bar{\xi}|$ is the Euclidean distance between ξ and $\bar{\xi}$. We claim that, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{\xi\in\Omega} |\mu|(B(\xi,\delta)) \,\leqslant\, \epsilon\ . \tag{4.20}$$

Indeed, if that property fails, there exist $\epsilon > 0$, a sequence (ξ_n) of points of Ω, and a sequence (δ_n) of positive real numbers such that $\delta_n \to 0$ as $n \to \infty$ and $|\mu|(B(\xi_n, \delta_n)) > \epsilon$ for all $n \in \mathbb{N}$. It is clear that the sequence (ξ_n) is bounded, because $|\mu|$ is a finite measure. Thus, after extracting a subsequence, we can assume that ξ_n converges as $n \to \infty$ to some point $\bar{\xi} = (\bar{r}, \bar{z}) \in \bar{\Omega}$. For any $\delta > 0$ we thus have $|\mu|(B(\bar{\xi}, \delta)) > \epsilon$, since $B(\bar{\xi}, \delta) \supset B(\xi_n, \delta_n)$ when n is sufficiently large. We conclude that

$$|\mu|\Big(\bigcap_{\delta>0} B(\bar{\xi}, \delta)\Big) \geqslant \epsilon > 0 .$$

But this is impossible, because the intersection above is empty if $\bar{\xi} \in \partial\Omega$ and equal to the singleton $\{\bar{\xi}\}$ if $\bar{\xi} \in \Omega$, and we assumed that the measure μ is non-atomic. Hence property (4.20), which can be interpreted as a weak form of absolute continuity with respect to Lebesgue's measure, must hold.

Now, for any given $t > 0$, there exists $\bar{\xi}(t) \in \Omega$ such that

$$|(S(t)\mu)(\bar{\xi}(t))| = \|S(t)\mu\|_{L^\infty(\Omega)} ,$$

because the map $\xi \mapsto (S(t)\mu)(\xi)$ is continuous and vanishes at infinity as well as on the boundary $\partial\Omega$. Using definition (4.17) and the pointwise estimate (3.8), we thus obtain

$$t\|S(t)\mu\|_{L^\infty(\Omega)} \leqslant C \int_{B(\bar{\xi}(t),\delta)} e^{-\frac{|\xi-\bar{\xi}(t)|^2}{5t}} \,\mathrm{d}|\mu|(\xi) + C \int_{\Omega \setminus B(\bar{\xi}(t),\delta)} e^{-\frac{|\xi-\bar{\xi}(t)|^2}{5t}} \,\mathrm{d}|\mu|(\xi) ,$$

where ϵ and δ are as in (4.20). The first integral is bounded by $C|\mu|(B(\bar{\xi}(t), \delta)) \leqslant C\epsilon$, and the second one by $Ce^{-\delta^2/(5t)}|\mu|(\Omega)$. It follows that

$$L_\infty(\mu) = \limsup_{t\to 0} t\|S(t)\mu\|_{L^\infty(\Omega)} \leqslant C\epsilon ,$$

and since $\epsilon > 0$ was arbitrary we conclude that $L_\infty(\mu) = 0$, which is the desired result. □

Proposition 4.4. — *There exist positive constants ϵ and C such that, for any initial data $\omega_0 \in \mathcal{M}(\Omega)$ with $\|(\omega_0)_{\mathrm{pp}}\|_{\mathrm{tv}} \leqslant \epsilon$, one can choose $T = T(\omega_0) > 0$ such that the integral equation (4.15) has a unique solution $\omega_\theta \in C^0((0,T], L^1(\Omega) \cap L^\infty(\Omega))$ satisfying (1.16) and such that $\omega_\theta(t) \rightharpoonup \omega_0$ as $t \to 0$. Moreover, if $\|\omega_0\|_{\mathrm{tv}}$ is small enough, the local existence time $T > 0$ can be taken arbitrarily large.*

Proof. — We briefly indicate how the proof of Proposition 4.1 has to be modified to handle the case where $\mu = \omega_0 \in \mathcal{M}(\Omega)$. We use exactly the same function space X_T defined in (4.4), and observe that the fixed point argument works in the ball $B_R \subset X_T$ provided $C_1(\mu, T) \leqslant R/2$, where $C_1(\mu, T)$ is defined as in (4.5) and $R > 0$ satisfies $2C_3R < 1$ with C_3 as in (4.8). From (4.18) we know that $C_1(\mu, T) \leqslant C_2\|\mu\|_{\mathrm{tv}}$ for any $T > 0$, hence we again obtain global existence and uniqueness in B_R if the initial vorticity is small enough so that $C_2\|\mu\|_{\mathrm{tv}} \leqslant R/2$. For larger data, we can use (4.19) which gives

$$\lim_{T\to 0} C_1(\mu, T) = L_{4/3}(\mu) \leqslant C_4\|\mu_{pp}\|_{\mathrm{tv}} ,$$

for some positive constant C_4. Thus, if the atomic part of the initial vorticity is small enough so that $C_4\|\mu_{pp}\|_{\mathrm{tv}} < R/2$, we can take $T > 0$ such that $C_1(\mu, T) \leqslant R/2$, and the fixed point argument proves local existence and uniqueness in B_R. In contrast, if $4C_3C_4\|\mu_{pp}\|_{\mathrm{tv}} \geqslant 1$, it is impossible to choose $R > 0$ and $T > 0$ so that the fixed point argument works in the ball $B_R \subset X_T$, and the method above completely fails.

Assuming that the fixed point argument works in the ball $B_R \subset X_T$, we can establish some additional properties of the solution $\omega_\theta \in B_R$ as in the case of integrable initial data. For instance, it is straightforward to verify that $\omega_\theta - \omega_{\mathrm{lin}} \in C^0((0,T), L^1(\Omega) \cap L^\infty(\Omega))$, where $\omega_{\mathrm{lin}}(t) = S(t)\mu$. However property (4.10) fails if

$\mu_{pp} \neq 0$, and we cannot argue as in Section 4.1 to show that $\|\omega_\theta(t) - \omega_{\rm lin}(t)\|_{L^1(\Omega)}$ converges to zero as $t \to 0$. To prove that, we first define

$$\delta \,=\, \limsup_{t\to 0} t^{1/4}\|\omega_\theta(t) - \omega_{\rm lin}(t)\|_{L^{4/3}(\Omega)} \,=\, \limsup_{T\to 0} \|\omega_\theta - \omega_{\rm lin}\|_{X_T}~.$$

Since $\omega_\theta - \omega_{\rm lin} = (\mathcal{F}\omega_{\rm lin} - \mathcal{F}\omega_\theta) - \mathcal{F}\omega_{\rm lin}$ and $\|\omega_{\rm lin}\|_{X_T} + \|\omega_\theta\|_{X_T} \leqslant 2R$, we can use (4.8) to obtain the estimate $\delta \leqslant 2C_3R\delta + \ell_{4/3}(\mu)$, where

$$\ell_p(\mu) \,=\, \limsup_{t\to 0} t^{1-\frac{1}{p}}\|\mathcal{F}\omega_{\rm lin}(t)\|_{L^p(\Omega)}~, \qquad 1 \leqslant p \leqslant \infty~.$$

As in the two-dimensional case [7, Section 2.3.4], a direct calculation, which exploits some cancellations in the nonlinear term $u_{\rm lin}(t)\cdot\nabla\omega_{\rm lin}(t)$ for small times, reveals that $\ell_p(\mu) = 0$ for any $p \in [1,\infty]$. This in turn implies that $\delta = 0$, since $2C_3R < 1$. Finally, using again the relation $\omega_\theta - \omega_{\rm lin} = (\mathcal{F}\omega_{\rm lin} - \mathcal{F}\omega_\theta) - \mathcal{F}\omega_{\rm lin}$ we conclude that

$$\limsup_{t\to 0} \|\omega_\theta(t) - \omega_{\rm lin}(t)\|_{L^1(\Omega)} \,\leqslant\, CR\delta + \ell_1(\mu) \,=\, 0~. \tag{4.21}$$

It follows in particular from (4.21) that $\omega_\theta(t) \rightharpoonup \mu$ as $t \to 0$, because we can use the explicit formula (4.17) to verify that $\omega_{\rm lin}(t) \rightharpoonup \mu$ as $t \to 0$. By construction ω_θ is a solution of (4.2), hence of (4.15), and both inequalities in (1.16) hold.

Finally, if $\omega_\theta \in C^0((0,T], L^1(\Omega)\cap L^\infty(\Omega))$ is a mild solution on $(0,T)$ satisfying (1.16) and such that $\omega_\theta(t) \rightharpoonup \omega_0$ as $t \to 0$, we can take the limit $t_0 \to 0$ in (4.15) and conclude that ω_θ satisfies (4.2), so that ω_θ coincides with the solution constructed by the fixed point argument. □

Remark 4.5. — In Proposition 4.4 we only claim uniqueness of solutions under assumption (1.16), which means (after restricting the existence time) that ω_θ belongs to the ball $B_R \subset X_T$ where the fixed point argument works. As in the two-dimensional case, one may conjecture that uniqueness holds among all solutions $\omega_\theta \in C^0((0,T), L^1(\Omega)\cap L^\infty(\Omega))$ such that $\|\omega_\theta(t)\|_{L^1(\Omega)}$ is uniformly bounded and $\omega_\theta(t) \rightharpoonup \mu$ as $t \to 0$. We hope to come back to that interesting question in a future work.

Remark 4.6. — The solutions constructed in Propositions 4.1 and 4.4 are in fact smooth for positive times, and satisfy the axisymmetric vorticity equation (4.1) in the classical sense. This can be proved using standard smoothing properties of the Navier-Stokes equations that are not specific to the axisymmetric case, see e.g. [15] and Proposition 5.5 below.

5. A priori estimates and global existence

We continue the proof of Theorems 1.1 and 1.3 by showing that the local solutions constructed in Sections 4.1 and 4.2 can be extended to global solutions for positive times. Since we are not interested in the behavior for small times, we can assume without loss of generality that the initial vorticity is integrable. Let thus $\omega_\theta \in C^0([0,T], L^1(\Omega)) \cap C^0((0,T], L^\infty(\Omega))$ be a solution of the integral equation (4.2), hence also of the differential equation (4.1), with initial data $\omega_0 \in L^1(\Omega)$. Our goal here is to derive a priori estimates on various norms of ω_θ.

Lemma 5.1. — *The solution of* (4.2) *satisfies* $\|\omega_\theta(t)\|_{L^1(\Omega)} \leqslant \|\omega_0\|_{L^1(\Omega)}$ *for all* $t \in [0,T]$. *Moreover, if* $\omega_0 \not\equiv 0$, *the map* $t \mapsto \|\omega_\theta(t)\|_{L^1(\Omega)}$ *is strictly decreasing.*

Proof. — We first assume that $\omega_0 \geqslant 0$ and $\omega_0 \not\equiv 0$. By the strong maximum principle, the solution $\omega_\theta(t)$ of (4.1) is strictly positive for $t \in (0,T]$. Integrating by parts and using the fact that $\omega_\theta(t)$ satisfies the homogeneous Dirichlet boundary condition on $\partial\Omega$, we easily find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \,=\, -2\int_{\mathbb{R}} \partial_r\omega_\theta(0,z,t)\,\mathrm{d}z \,<\, 0~, \tag{5.1}$$

where the last inequality follows from Hopf's lemma. This proves the claim for positive solutions. Note that, in deriving (5.1), we did not use the precise expression of the velocity field u in (4.1). In the general case where ω_0 can change sign, we decompose $\omega_\theta(t) = \omega_\theta^+(t) - \omega_\theta^-(t)$, where $\omega_\theta^\pm(t)$ are defined as the solutions of the linear equations

$$\partial_t \omega_\theta^\pm + \mathrm{div}_*(u\,\omega_\theta^\pm) \;=\; \Bigl(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Bigr)\omega_\theta^\pm\;, \tag{5.2}$$

with initial data $\omega_\theta^\pm(0) = \max(\pm\omega_0, 0) \geqslant 0$. Both equations in (5.2) involve the same velocity field u, which is associated to the full solution ω_θ via the axisymmetric Biot-Savart law (2.6). The analogue of (5.1) holds for the solutions $\omega_\theta^\pm(t)$ of (5.2), hence

$$\begin{aligned}\|\omega_\theta(t)\|_{L^1(\Omega)} \;&\leqslant\; \int_\Omega \Bigl(\omega_\theta^+(r,z,t) + \omega_\theta^-(r,z,t)\Bigr)\,\mathrm{d}r\,\mathrm{d}z \\ &\leqslant\; \int_\Omega \Bigl(\omega_\theta^+(r,z,0) + \omega_\theta^-(r,z,0)\Bigr)\,\mathrm{d}r\,\mathrm{d}z \;=\; \|\omega_0\|_{L^1(\Omega)}\;,\end{aligned}$$

for $0 \leqslant t \leqslant T$. When $t > 0$, the second inequality is strict if either $\omega_\theta^+(0)$ or $\omega_\theta^-(0)$ is nonzero, and if both quantities are nonzero the first inequality is also strict (by the strong maximum principle). If $\omega_0 \not\equiv 0$, this proves that the L^1 norm of the solution $\omega_\theta(t)$ is strictly decreasing at initial time, and a similar argument shows that it is strictly decreasing over the whole interval $[0,T]$. □

Higher L^p norms of the vorticity ω_θ are more difficult to control, because the velocity field in (4.1) does not satisfy $\mathrm{div}_*\, u = 0$. As in [17, 23], we thus consider the related quantity $\eta = \omega_\theta/r$, which satisfies Eq. (1.8) with initial data $\eta_0 = \omega_0/r \in L^1(\mathbb{R}^3)$. Using the existence result in Proposition 4.1 and the weighted estimates on the linear semigroup given in Proposition 3.5, it is easy to verify that $\eta \in C^0([0,T], L^1(\mathbb{R}^3)) \cap C^0((0,T], L^\infty(\mathbb{R}^3))$. Moreover, by Lemma 5.1, the map $t \mapsto \|\eta(t)\|_{L^1(\mathbb{R}^3)} \equiv \|\omega_\theta(t)\|_{L^1(\Omega)}$ is decreasing for nonzero solutions. Since the advection field $u - (2/r)e_r$ in (1.8) satisfies

$$\mathrm{div}\Bigl(u - \frac{2e_r}{r}\Bigr) \;=\; -2\,\mathrm{div}\,\frac{e_r}{r} \;=\; -4\pi\delta_{r=0} \;\leqslant\; 0\;,$$

a classical method due to Nash [20] gives the following a priori estimate:

Lemma 5.2. — [5, Lemma 3.8] *For any initial data $\eta_0 \in L^1(\mathbb{R}^3)$, the solution of (1.8) satisfies, for $1 \leqslant p \leqslant \infty$,*

$$\|\eta(t)\|_{L^p(\mathbb{R}^3)} \;\leqslant\; \frac{C}{t^{\frac{3}{2}(1-\frac{1}{p})}}\,\|\eta_0\|_{L^1(\mathbb{R}^3)}\;, \qquad 0 < t \leqslant T\;. \tag{5.3}$$

Equivalently, the axisymmetric vorticity ω_θ satisfies, for $p \in [1,\infty]$, the weighted estimate

$$\|r^{\frac{1}{p}-1}\omega_\theta(t)\|_{L^p(\Omega)} \;\leqslant\; \frac{C}{t^{\frac{3}{2}(1-\frac{1}{p})}}\,\|\omega_0\|_{L^1(\Omega)}\;, \qquad 0 < t \leqslant T\;. \tag{5.4}$$

Using (5.4), we now establish our main a priori estimate on the solutions of (1.6).

Proposition 5.3. — *Any solution $\omega_\theta \in C^0([0,T], L^1(\Omega)) \cap C^0((0,T], L^\infty(\Omega))$ of (4.2) with initial data $\omega_0 \in L^1(\Omega)$ satisfies, for all $p \in [1,\infty]$,*

$$\|\omega_\theta(t)\|_{L^p(\Omega)} \;\leqslant\; \frac{C_p(\|\omega_0\|_{L^1(\Omega)})}{t^{1-\frac{1}{p}}}\;, \qquad 0 < t \leqslant T\;, \tag{5.5}$$

where $C_p(s) = \mathcal{O}(s)$ as $s \to 0$.

Proof. — We can assume without loss of generality that $M := \|\omega_0\|_{L^1(\Omega)} > 0$. We know from Lemma 5.1 that $\|\omega_\theta(t)\|_{L^1(\Omega)} \leqslant M$ for $t \in [0,T]$, hence (5.5) holds for $p = 1$. To prove (5.5) for $p = 2$, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \omega_\theta^2\,\mathrm{d}r\,\mathrm{d}z \;=\; -2\int_\Omega |\nabla\omega_\theta|^2\,\mathrm{d}r\,\mathrm{d}z + \int_\Omega\Bigl(\frac{u_r}{r} - \frac{1}{r^2}\Bigr)\omega_\theta^2\,\mathrm{d}r\,\mathrm{d}z\;. \tag{5.6}$$

The celebrated Nash inequality [20] asserts that

$$\begin{aligned}\int_\Omega \omega_\theta^2 \,\mathrm{d}r\,\mathrm{d}z \;&\leqslant\; C\Bigl(\int_\Omega |\omega_\theta|\,\mathrm{d}r\,\mathrm{d}z\Bigr)\Bigl(\int_\Omega |\nabla\omega_\theta|^2\,\mathrm{d}r\,\mathrm{d}z\Bigr)^{1/2}\\ &\leqslant\; CM\Bigl(\int_\Omega |\nabla\omega_\theta|^2\,\mathrm{d}r\,\mathrm{d}z\Bigr)^{1/2}~.\end{aligned}$$

On the other hand, using estimate (5.4) with $p=\infty$ and Proposition 2.6, we obtain

$$\Bigl\|\frac{u_r(t)}{r}\Bigr\|_{L^\infty(\Omega)} \;\leqslant\; C\,\|\omega_\theta(t)\|_{L^1(\Omega)}^{1/3}\Bigl\|\frac{\omega_\theta(t)}{r}\Bigr\|_{L^\infty(\Omega)}^{2/3} \;\leqslant\; \frac{CM}{t}~.$$

Thus, if we define

$$f(t) \;=\; \int_\Omega \omega_\theta(r,z,t)^2\,\mathrm{d}r\,\mathrm{d}z~, \qquad 0\leqslant t\leqslant T~,$$

we deduce from (5.6) that $f:[0,T]\to\mathbb{R}$ satisfies the differential inequality

$$f'(t) \;\leqslant\; -\frac{K_1}{M^2}\,f(t)^2 + \frac{K_2M}{t}\,f(t)~, \qquad 0<t\leqslant T~, \tag{5.7}$$

where K_1, K_2 are positive constants. If we set $f(t)=t^\alpha g(t)$ with $\alpha=K_2M$, we see that (5.7) reduces to the simpler differential inequality $g'(t)\leqslant -K_1M^{-2}t^\alpha g(t)^2$, which can be integrated of the time interval $[t_0,t]\subset(0,T]$ to give the bound

$$\begin{aligned}\frac{1}{g(t)} \;&\geqslant\; \frac{1}{g(t_0)} + \frac{K_1}{M^2}\,\frac{1}{\alpha+1}\Bigl(t^{\alpha+1}-t_0^{\alpha+1}\Bigr)\\ &\geqslant\; \frac{K_1}{M^2}\,\frac{1}{\alpha+1}\Bigl(t^{\alpha+1}-t_0^{\alpha+1}\Bigr) \;\xrightarrow[t_0\to 0]{}\; \frac{K_1}{M^2}\,\frac{t^{\alpha+1}}{\alpha+1}~.\end{aligned}$$

We conclude that

$$\|\omega_\theta(t)\|_{L^2(\Omega)}^2 \;=\; f(t) \;=\; t^\alpha g(t) \;\leqslant\; \frac{\alpha+1}{K_1}\,\frac{M^2}{t} \;=\; \frac{K_2M+1}{K_1}\,\frac{M^2}{t}~,$$

for $0<t\leqslant T$, which proves (5.5) for $p=2$ (hence for $1\leqslant p\leqslant 2$ by interpolation). To reach the same conclusion for higher values of p, we proceed exactly as in the proof of Proposition 4.1. Using the notations (4.11), we know from Proposition 3.4 that $M_p(T)\leqslant CM$ for any $p\in[1,\infty]$, and from the argument above that $N_p(T)\leqslant C(M)$ for $p\in[1,2]$. The relation (4.14) then shows that $N_p(T)\leqslant C(M)$ for all $p>2$, and a second iteration gives the desired result for $p=\infty$ too. □

Remark 5.4. — Proposition 5.3 shows in particular that the L^p norms of the vorticity $\omega_\theta(t)$ cannot blow up in finite time. In view of Remark 4.2, this implies that all solutions constructed in Sections 4.1 and 4.2 are global for positive times, and that the conclusions of Lemma 5.1 and Proposition 5.3 hold for all $t>0$.

With Proposition 5.3 at hand, it is straightforward to show that the solutions of the vorticity equation (1.6) are smooth for positive times, and that estimates similar to (5.5) hold for the derivatives too. For later use, we state the following result.

Proposition 5.5. — *Under the assumptions of Proposition 5.3, we have for all* $p\in[1,\infty]$ *:*

$$\|\nabla\omega_\theta(t)\|_{L^p(\Omega)} \;\leqslant\; \frac{C_p(\|\omega_0\|_{L^1(\Omega)})}{t^{\frac{3}{2}-\frac{1}{p}}}~, \qquad t>0~, \tag{5.8}$$

where $C_p(s)=\mathcal{O}(s)$ *as* $s\to 0$.

Proof. — It is possible to prove estimate (5.8) by working directly on the integral representation (4.2), but we find it easier to deduce it from Proposition 5.3 using general smoothing properties of the Navier-Stokes equations. We know from (2.10) and (5.5) that the velocity field associated with the solution ω_θ of (4.1) satisfies, for any $t_0 > 0$,

$$\|u(t_0)\|_{L^\infty(\Omega)} \leqslant C\|\omega_\theta(t_0)\|_{L^1(\Omega)}^{1/2}\,\|\omega_\theta(t_0)\|_{L^\infty(\Omega)}^{1/2} \leqslant \frac{C(M)}{\sqrt{t_0}}\,, \tag{5.9}$$

where $M = \|\omega_0\|_{L^1(\Omega)}$ and $C(s) = \mathcal{O}(s)$ as $s \to 0$. On the other hand, there exist positive constants a and A such that the solution $u(t)$ of the Navier-Stokes equations (1.1) in $\mathbb{R}^3$ with data $u(t_0) \in L^\infty(\mathbb{R}^3)$ at time t_0 satisfies

$$(t - t_0)\|\nabla^2 u(t)\|_{L^\infty(\mathbb{R}^3)} \leqslant A\|u(t_0)\|_{L^\infty(\mathbb{R}^3)}\,, \tag{5.10}$$

whenever $t_0 < t < t_0 + a\|u(t_0)\|_{L^\infty(\mathbb{R}^3)}^{-2}$, see [15, Proposition 4.1]. Here $\nabla^2 u$ denotes the collection of all second-order derivatives of u. Since $\|u(t_0)\|_{L^\infty(\Omega)} \equiv \|u(t_0)\|_{L^\infty(\mathbb{R}^3)}$, we can combine estimates (5.9), (5.10) by fixing $t > 0$ and choosing, for instance,

$$t_0 = \frac{t}{2}\,\frac{2C(M)^2 + a}{C(M)^2 + a}\,,$$

so that $t_0 < t < t_0 + at_0 C(M)^{-2} \leqslant t_0 + a\|u(t_0)\|_{L^\infty(\mathbb{R}^3)}^{-2}$. We thus obtain

$$\|\nabla^2 u(t)\|_{L^\infty(\mathbb{R}^3)} \leqslant \frac{A}{t - t_0}\,\frac{C(M)}{\sqrt{t_0}} = \frac{\tilde{C}(M)}{t^{3/2}}\,, \tag{5.11}$$

where $\tilde{C}(s) = \mathcal{O}(s)$ as $s \to 0$. Using the pointwise estimate

$$|\nabla\omega_\theta| \leqslant |\partial_r\omega_\theta| + |\partial_z\omega_\theta| \leqslant C\Big(|\nabla^2 u| + \frac{1}{r}\,|\partial_r u_z|\Big)\,,$$

and the fact that $\partial_r u_z$ vanishes at $r = 0$, we see that (5.11) implies (5.8) with $p = \infty$. The case $p < \infty$ easily follows by interpolation, in view of (5.5). ☐

6. Long-time behavior

In this final section, we study the long-time behavior of the solutions of the axisymmetric vorticity equation (1.6) constructed in Sections 4 and 5. In particular we prove estimate (1.11), and we obtain the asymptotic formula (1.13) in the particular case where the initial vorticity has a definite sign and a finite impulse in the sense of (1.12). This will conclude the proof of Theorems 1.1 and 1.3.

6.1. **Convergence to zero in scale invariant norms.** Let us assume that $\omega_\theta \in C^0([0,\infty), L^1(\Omega)) \cap C^0((0,\infty), L^\infty(\Omega))$ is a global solution of the vorticity equation (4.1), with initial data $\omega_0 \in L^1(\Omega)$. We know from Lemma 5.1 that the L^1 norm $\|\omega_\theta(t)\|_{L^1(\Omega)}$ is a decreasing function of time, and our goal here is to prove that this quantity actually converges to zero as $t \to \infty$. We first show that $\omega_\theta(r, z, t)$ is essentially confined, for large times, in a ball of radius $\mathcal{O}(\sqrt{t})$ in Ω.

Proposition 6.1. — *Let $\omega_0 \in L^1(\Omega)$ and $M = \|\omega_0\|_{L^1}$. For any $\epsilon > 0$, there exist positive constants $K_3(\epsilon, \omega_0)$ and $K_4(\epsilon, M)$ such that the solution of* (4.1) *with initial data ω_0 satisfies, for all $t \geqslant 0$,*

$$\int_{\Omega(t)} |\omega_\theta(r, z, t)|\,\mathrm{d}r\,\mathrm{d}z \leqslant \epsilon\,, \tag{6.1}$$

where $\Omega(t) = \big\{(r, z) \in \Omega\,\big|\,\sqrt{r^2 + z^2} \geqslant K_3 + K_4\sqrt{t}\big\}$.

Proof. — By the maximum principle, it is sufficient to establish (6.1) for positive solutions of (4.1) (in the general case, the result follows by decomposing ω_θ as in the proof of Lemma 5.1). We thus assume that $\omega_0 \geqslant 0$ and that $M = \int_\Omega \omega_0\,\mathrm{d}r\,\mathrm{d}z > 0$. The only property of the velocity field that will be used to obtain (6.1) is the a priori estimate (5.9).

We first prove confinement in the radial direction. For $R \geqslant 0$ and $t \geqslant 0$, we define

$$f(R,t) \;=\; \int_R^\infty \left\{ \int_{\mathbb{R}} \omega_\theta(r,z,t)\,\mathrm{d}z \right\} \mathrm{d}r\ .$$

Then $f(R,t)$ is a non-increasing function of R such that $f(0,t) = \|\omega_\theta(t)\|_{L^1(\Omega)}$ and $f(R,t) \to 0$ as $R \to \infty$. Moreover, using (4.1), it is easy to verify that f satisfies the evolution equation

$$\partial_t f(R,t) \;=\; \partial_R^2 f(R,t) + \frac{1}{R}\,\partial_R f(R,t) + \int_{\mathbb{R}} u_r(R,z,t)\omega_\theta(R,z,t)\,\mathrm{d}z\ , \tag{6.2}$$

for any $R > 0$. In view of (5.9), we have the estimate

$$\begin{aligned} \int_{\mathbb{R}} u_r(R,z,t)\omega_\theta(R,z,t)\,\mathrm{d}z \;&\leqslant\; \frac{C(M)}{\sqrt{t}} \int_{\mathbb{R}} \omega_\theta(R,z,t)\,\mathrm{d}z \\ &=\; -\frac{C(M)}{\sqrt{t}}\,\partial_R f(R,t)\ , \end{aligned} \tag{6.3}$$

for some positive constant $C(M)$. Since $\partial_R f \leqslant 0$, we deduce from (6.2), (6.3) that

$$\partial_t f(R,t) \;\leqslant\; \partial_R^2 f(R,t) - \frac{C(M)}{\sqrt{t}}\,\partial_R f(R,t)\ , \qquad R > 0\ . \tag{6.4}$$

Solving the differential inequality (6.4), with homogeneous Neumann boundary condition at $R = 0$, is a straightforward task. For instance, if we extend $f(\cdot,t)$ to the whole real line by setting $\overline{f}(R,t) = f(0,t)$ for $R \leqslant 0$, the extension satisfies inequality (6.4) for all $R \in \mathbb{R}$. We deduce that

$$f(R,t) \;\leqslant\; g(R - 2C(M)\sqrt{t}, t)\ , \qquad R \geqslant 0\ , \quad t \geqslant 0\ ,$$

where g is the solution of the heat equation $\partial_t g = \partial_R^2 g$ on $\mathbb{R}$ with initial data $g(R,0) = \overline{f}(R,0)$. Given any $\epsilon > 0$, we choose $R_0 > 0$ large enough so that $f(R_0,0) \leqslant \epsilon$. If $R > R_0$ we estimate

$$\begin{aligned} g(R,t) \;&=\; \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{R_0} e^{-\frac{(R-r)^2}{4t}}\,\overline{f}(r,0)\,\mathrm{d}r + \frac{1}{\sqrt{4\pi t}} \int_{R_0}^\infty e^{-\frac{(R-r)^2}{4t}} f(r,0)\,\mathrm{d}r \\ &\leqslant\; \frac{e^{-\frac{(R-R_0)^2}{4t}}}{\sqrt{4\pi t}} \int_{-\infty}^{R_0} e^{-\frac{(R_0-r)^2}{4t}} M\,\mathrm{d}r + \frac{\epsilon}{\sqrt{4\pi t}} \int_{R_0}^\infty e^{-\frac{(R-r)^2}{4t}}\,\mathrm{d}r \\ &\leqslant\; M\,e^{-\frac{(R-R_0)^2}{4t}} + \epsilon\ . \end{aligned}$$

The right-hand side is smaller than 2ϵ if $R \geqslant R_0 + 2\sqrt{t}\log(M/\epsilon)^{1/2}$. Summarizing, we have shown that $f(R,t) \leqslant 2\epsilon$ provided $R \geqslant R_0 + K\sqrt{t}$, with $K = 2C(M) + 2\log(M/\epsilon)^{1/2}$.

The argument is similar for the confinement in the vertical direction. For $Z \in \mathbb{R}$ and $t \geqslant 0$, we define

$$\begin{aligned} h_+(Z,t) \;&=\; \int_Z^\infty \left\{ \int_0^\infty \omega_\theta(r,z,t)\,\mathrm{d}r \right\} \mathrm{d}z\ , \\ h_-(Z,t) \;&=\; \int_{-\infty}^Z \left\{ \int_0^\infty \omega_\theta(r,z,t)\,\mathrm{d}r \right\} \mathrm{d}z\ . \end{aligned}$$

Then $Z \mapsto h_+(Z,t)$ is non-increasing, $Z \mapsto h_-(Z,t)$ is non-decreasing, and we have the differential inequalities

$$\partial_t h_\pm(Z,t) \;\leqslant\; \partial_Z^2 h_\pm(Z,t) \mp \frac{C(M)}{\sqrt{t}}\,\partial_Z h_\pm(Z,t)\ ,$$

which allow us to compare $h_\pm(Z,t)$ with suitably translated solutions of the one-dimensional heat equation. Proceeding as above, we find that, for any $\epsilon > 0$, there exist positive constants Z_0 and K such that $h_+(Z,t) \leqslant 2\epsilon$ if $Z \geqslant Z_0 + K\sqrt{t}$, and $h_-(Z,t) \leqslant 2\epsilon$ if $Z \leqslant -Z_0 - K\sqrt{t}$. It follows that

$$\int_0^\infty \int_{|z|\geqslant Z_0+K\sqrt{t}} \omega_\theta(r,z,t)\,\mathrm{d}z\,\mathrm{d}z \;\leqslant\; 4\epsilon\;, \qquad t \geqslant 0\;.$$

Combining this result with the previous estimate on $f(R,t)$, we obtain (6.1). □

Proposition 6.2. — *For any initial data $\omega_0 \in L^1(\Omega)$, the solution of the axisymmetric vorticity equation* (4.1) *satisfies $\|\omega_\theta(t)\|_{L^1(\Omega)} \to 0$ as $t \to \infty$.*

Proof. — We know from Lemma 5.1 that $\|\omega_\theta(t)\|_{L^1(\Omega)}$ converges to some limit ℓ as $t \to \infty$. To prove that $\ell = 0$, we use a standard rescaling argument. Let $(\lambda_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers such that $\lambda_n \to \infty$ as $n \to \infty$. We define for all $n \in \mathbb{N}$:

$$w_n(r,z,t) \;=\; \lambda_n^2\,\omega_\theta(\lambda_n r, \lambda_n z, \lambda_n^2(1+t))\;, \qquad (r,z) \in \Omega\;, \quad t \geqslant 0\;,$$

Denoting $w_n(t) = w_n(\cdot,\cdot,t)$, we claim that the sequence $(w_n(0))_{n\in\mathbb{N}}$ is relatively compact in $L^1(\Omega)$. Indeed, by Proposition 6.1, for all $\epsilon > 0$ there exists a compact set $\Omega_0 \subset \Omega$ such that

$$\sup_{n\in\mathbb{N}} \int_{\Omega\setminus\Omega_0} |w_n(r,z,0)|\,\mathrm{d}r\,\mathrm{d}z \;\leqslant\; \epsilon\;. \tag{6.5}$$

In addition, Proposition 5.5 asserts that

$$\sup_{n\in\mathbb{N}} \|\nabla w_n(0)\|_{L^\infty(\Omega)} \;=\; \sup_{n\in\mathbb{N}} \lambda_n^3 \|\nabla\omega_\theta(\lambda_n^2)\|_{L^\infty(\Omega)} \;<\; \infty\;. \tag{6.6}$$

Combining (6.5), (6.6) and using the Riesz criterion [21, Theorem XIII.66], we obtain the desired compactness. Thus, after extracting a subsequence, we can assume that $w_n(0)$ converges in $L^1(\Omega)$ to some limit $\overline{w}$, which satisfies $\|\overline{w}\|_{L^1(\Omega)} = \ell$ because $\|w_n(0)\|_{L^1(\Omega)} = \|\omega_\theta(\lambda_n^2)\|_{L^1(\Omega)} \to \ell$ as $n \to \infty$.

Now, we fix $t > 0$, and repeating the procedure above we extract yet another subsequence so that $w_n(t)$ converges in $L^1(\Omega)$ to some limit $\overline{w}(t)$. By construction, for each $n \in \mathbb{N}$, the function $w_n(r,z,t)$ solves the vorticity equation (4.1) with initial data $w_n(r,z,0)$. As in the proof of Proposition 4.1, we thus denote $w_n(t) = \Sigma(t)w_n(0)$. Taking the limit $n \to \infty$ and using the fact that the solutions of (4.1) depend continuously on the initial data in $L^1(\Omega)$, see Remark 4.2.2, we deduce that $\overline{w}(t) = \Sigma(t)\overline{w}$. But since $\|\overline{w}(t)\|_{L^1(\Omega)} = \|\overline{w}\|_{L^1(\Omega)} = \ell$, we get a contradiction with the strict decay of the L^1 norm established in Lemma 5.1, unless $\ell = 0$. □

Corollary 6.3. — *Under the assumptions of Proposition 6.2, we have*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \|\omega_\theta(t)\|_{L^p(\Omega)} \;=\; 0\;, \qquad 1 \leqslant p \leqslant \infty\;. \tag{6.7}$$

Proof. — Given any $t_0 > 0$, we can apply Proposition 5.3 to the solution of (4.1) restricted to the time interval $[t_0,\infty)$. For any $p \in [1,\infty]$, we thus find

$$\|\omega_\theta(t)\|_{L^p(\Omega)} \;\leqslant\; \frac{C_p(\|\omega(t_0)\|_{L^1(\Omega)})}{(t-t_0)^{1-\frac{1}{p}}}\;, \qquad t > t_0\;,$$

hence

$$\limsup_{t\to\infty} t^{1-\frac{1}{p}} \|\omega_\theta(t)\|_{L^p(\Omega)} \;\leqslant\; C_p(\|\omega(t_0)\|_{L^1(\Omega)})\;.$$

Taking now the limit $t_0 \to \infty$ and using Proposition 6.2 together with the fact that $C_p(s) = \mathcal{O}(s)$ as $s \to 0$, we obtain (6.7). □

6.2. **Asympotic behavior of positive solutions with finite impulse.** We now consider the particular situation where the initial vorticity $\omega_0 \in L^1(\Omega)$ is non-negative and has a finite impulse. We denote

$$M = \int_\Omega \omega_0(r,z)\,\mathrm{d}r\,\mathrm{d}z\;, \qquad \mathcal{I} = \int_\Omega r^2\omega_0(r,z)\,\mathrm{d}r\,\mathrm{d}z\;. \tag{6.8}$$

As is well-known, the impulse $\mathcal{I}$ is conserved for solutions of (4.1).

Lemma 6.4. — *For any non-negative solution of* (4.1) *in* $L^1(\Omega)$ *with finite impulse, we have*

$$\int_\Omega r^2\omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z = \int_\Omega r^2\omega_0(r,z)\,\mathrm{d}r\,\mathrm{d}z\;, \qquad t \geqslant 0\;.$$

Proof. — Using (4.1) we find by a direct calculation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega r^2\omega_\theta\,\mathrm{d}r\,\mathrm{d}z &= \int_\Omega r^2\Bigl(\partial_r^2+\partial_z^2+\frac{1}{r}\partial_r-\frac{1}{r^2}\Bigr)\omega_\theta\,\mathrm{d}r\,\mathrm{d}z - \int_\Omega r^2\,\mathrm{div}_*(u\omega_\theta)\,\mathrm{d}r\,\mathrm{d}z\\ &= 2\int_\Omega r u_r\omega_\theta\,\mathrm{d}r\,\mathrm{d}z\;.\end{aligned}$$

The last integral actually vanishes. Indeed, using the explicit formulas (2.6) and (2.7), we obtain

$$\int_\Omega r u_r\omega_\theta\,\mathrm{d}r\,\mathrm{d}z = -\frac{1}{\pi}\int_\Omega\int_\Omega \frac{z-\bar z}{r^{1/2}\,\bar r^{1/2}}\,F'(\xi^2)\omega_\theta(\bar r,\bar z)\omega_\theta(r,z)\,\mathrm{d}\bar r\,\mathrm{d}\bar z\,\mathrm{d}r\,\mathrm{d}z = 0\;,$$

because the integrand is odd with respect to the permutation $(r,z)\leftrightarrow(\bar r,\bar z)$. This gives the desired result. □

Under the assumption that $\mathcal{I} < \infty$, one can obtain precise information on the long-time behavior of the axisymmetric vorticity. We begin with the linear case:

Lemma 6.5. — *If* $\omega_0 \in L^1(\Omega)$ *is non-negative and* $\mathcal{I} < \infty$*, one has*

$$\lim_{t\to\infty} t^2(S(t)\omega_0)(r\sqrt{t},z\sqrt{t}) = \frac{\mathcal{I}}{16\sqrt{\pi}}\,r\,e^{-\frac{r^2+z^2}{4}}\;, \qquad (r,z)\in\Omega\;, \tag{6.9}$$

where convergence holds in $L^p(\Omega)$ *for* $1 \leqslant p \leqslant \infty$.

Proof. — In view of the explicit formula (3.2), we have for all $(r,z)\in\Omega$:

$$\begin{aligned}&t^2(S(t)\omega_0)(r\sqrt{t},z\sqrt{t})\\ &\quad= \frac{r}{4\pi}\int_\Omega K\Bigl(\frac{\sqrt{t}}{r\bar r}\Bigr)\,\exp\Bigl(-\frac{1}{4}\Bigl[\Bigl(r-\frac{\bar r}{\sqrt{t}}\Bigr)^2+\Bigl(z-\frac{\bar z}{\sqrt{t}}\Bigr)^2\Bigr]\Bigr)\,\bar r^2\omega_0(\bar r,\bar z)\,\mathrm{d}\bar r\,\mathrm{d}\bar z\;,\end{aligned}$$

where $K(\tau) = \tau^{3/2}H(\tau)$ is uniformly bounded and converges to $\sqrt{\pi}/4$ as $\tau\to+\infty$. The pointwise result (6.9) thus follows from Lebesgue's dominated convergence theorem. Convergence in L^p norms is easy to prove if ω_0 has compact support in Ω, and can be established in the general case by an approximation argument. □

It follows in particular from (6.9) that $\|S(t)\omega_0\|_{L^p(\Omega)} = \mathcal{O}(t^{-2+\frac{1}{p}})$ as $t\to\infty$. Our last result is the extension of Lemma 6.5 to the nonlinear case.

Proposition 6.6. — *If* $\omega_0 \in L^1(\Omega)$ *is non-negative and* $\mathcal{I} < \infty$*, the solution of* (4.1) *with initial data* ω_0 *satisfies*

$$\lim_{t\to\infty} t^2\omega_\theta(r\sqrt{t},z\sqrt{t},t) = \frac{\mathcal{I}}{16\sqrt{\pi}}\,r\,e^{-\frac{r^2+z^2}{4}}\;, \qquad (r,z)\in\Omega\;, \tag{6.10}$$

where convergence holds in $L^p(\Omega)$ *for all* $p\in[1,\infty]$*. Thus* $\|\omega_\theta(t)\|_{L^p(\Omega)} = \mathcal{O}(t^{-2+\frac{1}{p}})$ *as* $t\to\infty$.

Proof. — We estimate the integral term in (4.2) in the following way. First, using (3.13) with $\alpha = 0$ and $\beta = 1$, we write

$$\Big\|\int_0^{t/2} S(t-s)\,\mathrm{div}_*(u(s)\omega_\theta(s))\,\mathrm{d}s\Big\|_{L^1(\Omega)} \leqslant \int_0^{t/2} \frac{C}{t-s}\,\|u(s)\|_{L^\infty(\Omega)}\|r\omega_\theta(s)\|_{L^1(\Omega)}\,\mathrm{d}s\,.$$

From (2.14) we have

$$\|u\|_{L^\infty}\|r\omega_\theta\|_{L^1} \leqslant C\|r\omega_\theta\|_{L^1}^{3/2}\|\omega_\theta/r\|_{L^\infty}^{1/2} \leqslant C\|\omega_\theta\|_{L^1}^{3/4}\|r^2\omega_\theta\|_{L^1}^{3/4}\|\omega_\theta/r\|_{L^\infty}^{1/2}\,,$$

hence using Lemmas 5.1 and 5.2 we obtain

$$\|u(s)\|_{L^\infty(\Omega)}\|r\omega_\theta(s)\|_{L^1(\Omega)} \leqslant C\,\frac{M^{1/2}\mathcal{I}^{3/4}}{s^{3/4}}\,\|\omega_\theta(s)\|_{L^1}^{3/4}\,,$$

where $M, \mathcal{I}$ are defined in (6.8). Similarly, we find

$$\Big\|\int_{t/2}^{t} S(t-s)\,\mathrm{div}_*(u(s)\omega_\theta(s))\,\mathrm{d}s\Big\|_{L^1(\Omega)} \leqslant \int_{t/2}^{t} \frac{C}{(t-s)^{\frac12}}\,\|u(s)\|_{L^\infty(\Omega)}\|\omega_\theta(s)\|_{L^1(\Omega)}\,\mathrm{d}s\,,$$

where

$$\|u(s)\|_{L^\infty(\Omega)}\|\omega_\theta(s)\|_{L^1(\Omega)} \leqslant C\,\frac{M^{1/2}\mathcal{I}^{1/4}}{s^{3/4}}\,\|\omega_\theta(s)\|_{L^1}^{5/4}\,.$$

Thus it follows from (4.2) that

$$\begin{aligned}\|\omega_\theta(t)\|_{L^1(\Omega)} \leqslant \|S(t)\omega_0\|_{L^1(\Omega)} &+ CM^{1/2}\mathcal{I}^{3/4}\int_0^{t/2}\frac{\|\omega_\theta(s)\|_{L^1}^{3/4}}{(t-s)\,s^{3/4}}\,\mathrm{d}s\\ &+ CM^{1/2}\mathcal{I}^{1/4}\int_{t/2}^{t}\frac{\|\omega_\theta(s)\|_{L^1}^{5/4}}{(t-s)^{1/2}\,s^{3/4}}\,\mathrm{d}s\,. \qquad (6.11)\end{aligned}$$

Since $\|S(t)\omega_0\|_{L^1(\Omega)} = \mathcal{O}(t^{-1})$ as $t \to \infty$ by Lemma 6.5, the integral inequality (6.11) implies (by a straightforward bootstrap argument) that $\|\omega_\theta(t)\|_{L^1(\Omega)} = \mathcal{O}(t^{-1})$ as $t \to \infty$. In a similar way, one can show that $\|\omega_\theta(t)\|_{L^p(\Omega)} = \mathcal{O}(t^{-2+1/p})$ as $t \to \infty$ for all $p \in [1, \infty]$.

To prove (6.10), we fix $t_0 \geqslant 1$ and consider the integral equation (4.15) for $t \geqslant t_0$. If we bound the nonlinear term exactly as in (6.11), using the additional information that $\|\omega_\theta(t)\|_{L^1(\Omega)} = \mathcal{O}(t^{-1})$, we obtain the estimate

$$t\|\omega_\theta(t) - S(t-t_0)\omega_\theta(t_0)\|_{L^1(\Omega)} \leqslant \frac{C(M,\mathcal{I})}{\sqrt{t_0}}\,, \qquad (6.12)$$

for some positive constant $C(M, \mathcal{I})$. We now rescale the solution $\omega_\theta(r, z, t)$ as in (6.10) and take the limit as $t \to \infty$. Using (6.12) and Lemma 6.5, we obtain

$$\limsup_{t\to\infty}\|t^2\omega_\theta(\cdot\sqrt{t}, \cdot\sqrt{t}, t) - \Phi\|_{L^1(\Omega)} \leqslant \frac{C(M,\mathcal{I})}{\sqrt{t_0}}\,, \qquad (6.13)$$

where $\Phi : \Omega \to \mathbb{R}$ denotes the function defined by the expression in the right-hand side of (6.10). If we take the limit $t_0 \to \infty$ in (6.13), we see that (6.10) holds if convergence is understood in $L^1(\Omega)$. A similar argument shows that convergence also holds $L^p(\Omega)$ for all $p \in [1, \infty]$. □

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Thierry GALLAY
Institut Fourier; UMR CNRS 5582; Université de Grenoble I; BP 74; 38402
Saint-Martin-d'Hères, France

Thierry.Gallay@ujf-grenoble.fr (Corresponding author)

Vladimír ŠVERÁK
School of Mathematics; University of Minnesota; 127 Vincent Hall; 206 Church St. SE;
Minneapolis, MN 55455, USA
sverak@math.umn.edu