

REMARKS ON THE EQUILIBRIUM TURBULENT
BOUNDARY LAYER

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Two similarity laws are known for the mean-velocity profile in a turbulent boundary layer with constant pressure. These are Prandtl's law of the wall and von Kármán's momentum-defect law. Both concepts have recently been generalized empirically, the law of the wall to flows with arbitrary pressure gradient by Ludwig and Tillmann, and the defect law to a certain class of equilibrium flows by F. Clauser.

In the present paper it is shown that the pressure distribution corresponding to a given equilibrium flow can be computed precisely if it is assumed that a certain parameter $D = (\tau_w/q) dq/d\tau_w$ is constant, where q and τ_w are the dynamic pressure in the free stream and the shearing stress at the wall. The hypothesis $D = \text{constant}$ is suggested by a study of the integrated continuity equation and is supported by a rigorous analogy between the class of equilibrium flows defined by Clauser and the class of laminar flows studied by Falkner and Skan. The hypothesis $D = \text{constant}$ is also verified directly using experimental data for several equilibrium turbulent flows.

Two limiting cases of equilibrium flow are explored. The first limiting case is characterized by a completely logarithmic mean-velocity profile outside the sublayer and by a constant friction coefficient; this flow should appear in a wedge-shaped converging channel. The second limiting case is a continuously separating boundary layer which grows linearly; the dimensionless pressure gradient $(x/q) dq/dx$ is approximately twice that for the corresponding laminar flow. Typical shearing-stress profiles are computed for several equilibrium turbulent flows including the two limiting cases.

STATEMENT OF THE PROBLEM

1. Introduction. For fifty years the problem of turbulent shear flow has occupied a conspicuous place in the literature of fluid mechanics. During this time the principle of physical similarity has always been an important tool in the study of easily observable mean quantities like the mean-velocity distribution. For the turbulent boundary layer, in particular, great progress has been made in dealing with what I will call the direct problem of representing analytically the mean-velocity and shearing-stress profiles in flows which have been observed experimentally. However, no satisfactory solution has yet been found for the inverse problem of predicting the same quantities in flows which have not necessarily been observed but which develop in a specified environment.

For turbulent flow the primitive state of the art makes a solution of the direct problem very nearly a prerequisite for a solution of the inverse problem. This situation has no real analogy in laminar flow, if it is presumed that every fluid flow is ultimately determined by its environment through the agency of certain equations of motion and state together with certain initial and boundary conditions. The difference lies in the fact that the equations of motion for laminar flow are for practical purposes known, and any difficulties are therefore mathematical rather than physical in nature. On the other hand, the equations of mean motion for turbulent flow have yet to be formulated completely, and any discussion of turbulent shear flow must proceed in the vacuum created by imperfect understanding of transport processes in even the simplest turbulent motions.

2. The direct problem. Various theoretical and empirical attacks on the direct problem for the turbulent boundary layer have led long ago to the recognition of two similarity laws for the mean-velocity profile. These are Prandtl's law of the wall and von Kármán's momentum-defect law. Recently both concepts have been extended empirically, the law of the wall to flows with arbitrary pressure gradient by Ludwig and Tillmann⁽¹⁾, and the defect law to a special class of equilibrium flows by F. Clauser⁽²⁾.

Following these developments, I have proposed in another paper⁽³⁾ a formulation of the mean-velocity profile which incorporates the similarity laws in their extended form but which is not restricted to equilibrium flow. To be specific, consider turbulent flow of an incompressible fluid past a smooth plane surface, at which the relative velocity vanishes and the friction is Newtonian. For flows which are steady and two-dimensional in the mean, it is found empirically that the mean-velocity profile may be quite generally represented by a formula

$$\frac{u}{u_\tau} = f\left(\frac{y u_\tau}{\nu}\right) + \frac{\pi}{\kappa} w\left(\frac{y}{\delta}\right) \quad (1)$$

where $u_\tau^2 = \tau_w / \rho$. The quantity $\tau_w(x)$ is the wall shearing stress; $\delta(x)$ is a boundary-layer thickness which is uniquely defined; and κ is von Kármán's universal constant, taken here as 0.400. The function $f(z)$, called the law of the wall, has the properties that $f(z) \rightarrow z$ for $z \rightarrow 0$ ($z < 1$, say), and $f(z) \rightarrow (1/\kappa) \ln z + c$, where c is a second constant taken as 5.10, for $z \rightarrow \infty$ ($z > 50$, say). The function $w(\xi)$, which I have called the law of the wake, has by definition the properties that $w(0) = 0$, $w(1) = 2$, and $\int_0^2 \xi dw = 1$. The parameter

$\pi(x)$, which describes the relative amplitude of the wake and wall components, is related to the local friction coefficient $C_f = 2u_\tau^2/u_1^2$ by the expression

$$\kappa \frac{u_1}{u_\tau} = \ln \frac{\delta u_\tau}{\nu} + \kappa c + 2\pi \quad (2)$$

and to the displacement thickness $\delta^*(x)$ by the expression

$$\kappa \frac{\delta^* u_1}{\delta u_\tau} = 1 + \pi + \frac{\epsilon}{\delta u_\tau / \nu} \quad (3)$$

where $u_1(x)$ is the velocity in the external stream, ϵ is a constant*, and, by definition,

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{u_1}\right) dy \quad (4)$$

*The constant ϵ in Eq. (3) accounts for the departure of the flow in the sublayer from the logarithmic law of the wall. Using the notation $y u_\tau / \nu = \xi$ for convenience, ϵ is defined by

$$\epsilon = \kappa \int_0^\infty \xi \frac{d}{d\xi} \left[f(\xi) - \frac{1}{\kappa} \ln \xi - c \right] d\xi$$

and has a numerical value⁽⁴⁾ in the neighborhood of 27.

These relationships suggest that the development of a general turbulent boundary layer can be described in terms of two constants ρ and μ characterizing the fluid and four parameters u_1 , u_τ , δ , and π characterizing the state of the flow, the latter quantities being considered as functions of a single variable κ . Knowledge of these four parameters in any region determines not only the surface friction and the

rate of boundary-layer growth, but the complete mean-velocity profile and therefore, at least within the usual boundary-layer approximation, the shearing-stress field and the rate of energy transfer from the mean motion to the turbulent secondary motion.

I want to emphasize that the two functions called the law of the wall and the law of the wake are treated here as completely empirical functions established by direct observation of the mean-velocity profile. No attempt is made to discuss the problem of turbulence per se, and this omission is at the same time the greatest strength and the greatest weakness of the present development.

The important point is that Eq. (1) provides a complete and almost arbitrarily accurate analytic representation of the mean-velocity profile for a large class of flows as a linear combination of two supposedly universal functions $f(y\mu_T/\nu)$ and $w(y/\delta)$. Eq. (1) therefore constitutes a useful if tentative solution of the direct problem for the turbulent boundary layer.

3. Equilibrium flow. An equilibrium flow, as originally defined by Clauser, is one having a defect law of the form

$$\frac{u_1 - u}{u_T} = F\left(\frac{y}{\delta}\right) \quad (5)$$

outside the sublayer. An entirely equivalent statement, assuming the mean-velocity profile itself to be given by Eq. (1), is that the parameter Π is constant.

In the present paper I will be concerned almost entirely with equilibrium flows having the property (5). For reasons which will be-

come apparent later, the defect law is an essential element in the discussion, while the law of the wake is not. That is, the function $w(y/\delta)$ need not be the same for various equilibrium flows as long as Eq. (5) implies a relationship of the form of Eq. (1) and conversely. In what follows, however, I will retain the notation of Eq. (1) in order to show the dependence of various quantities on the single parameter, e.g. Π , which characterizes an equilibrium turbulent flow.

4. The inverse problem. Four parameters -- $u_1(x)$, $u_T(x)$, $\delta(x)$, and $\Pi(x)$ -- occur in ^{the mean-velocity profile.} Four independent relationships among these parameters are required in any formulation of the inverse problem. Two of these relationships are provided by the local friction law (2) and by the von Kármán momentum-integral equation which will be introduced in its proper place. A third relationship is ordinarily included in specified conditions for a particular flow; e.g. $\Pi = \text{constant}$ for an equilibrium flow, or $u_1 = u_1(x)$ for a prescribed ambient flow.

The assumption that a fourth equation can be found is not essentially different from the traditional single-parameter hypothesis. Both express the hope that the turbulent mixing process can somehow be represented by a single empirical relationship describing the response of the boundary layer to its environment. Numerous relationships, sometimes supported by physical arguments and sometimes not, have been proposed to serve this need in various engineering applications. However, the law of the wake and the concept of equilibrium flow are new elements in the problem which suggest that the office of fourth equation ought properly to be declared vacant, and it is my object in the present paper to propose a candidate for this office in the special case of equilibrium flow.

II. KINEMATIC SIMILARITY

1. The continuity equation. The central idea in the discussion is the concept of kinematic similarity. This concept involves considerations sufficiently general so that it can only be introduced by making what may at first appear to be a digression of the wildest kind. For the sake of brevity I will put the matter in the form of a theorem, accepting the risk that this choice of terminology may approach the threshold of pain in persons accustomed to more approximate methods of dealing with turbulent boundary-layer flows.

THEOREM: Consider a shear flow with mean-velocity components $u(x, y, t)$ and $v(x, y, t)$ such that $\partial u / \partial x + \partial v / \partial y = 0$. Assume $u = v = 0$ at $y = 0$ and $\lim_{y \rightarrow 0} \mu \partial u / \partial y = \tau_w(x, t)$ where μ is a constant. Suppose that u is independent of y for y larger than some value $\delta(x, t)$. Define $u_\tau(x, t) = \sqrt{\tau_w / \rho}$ where ρ is a constant; $u_1(x, t) = u(x, \delta, t)$; $\delta^*(x, t) = \int_0^\delta (1 - u/u_1) dy$; $1/\lambda = -(1/u_\tau) \partial u_\tau / \partial x$; $v = \mu / \rho$; and

$$D(x, t) = \frac{\partial \ln u_1 / \partial x}{\partial \ln u_\tau / \partial x}$$

$$P(x, t) = \frac{\delta^*}{\delta} \frac{\partial \ln (\delta^* u_1 / v) / \partial x}{\partial \ln (u_1 / u_\tau) / \partial x}$$

Then the curve obtained by plotting $\lambda v / P \delta u$ against $y / P \delta$ for fixed x and t must leave the origin with slope unity and must coincide for $y > \delta$ with the straight line passing with slope D through the point (1,1).

To prove the theorem, note first that a velocity profile of the form $u/\mu_\tau = f(y\mu_\tau/\nu)$ automatically implies⁽⁴⁾ the relationship $v/\mu = y/\lambda$. The assumption of Newtonian friction at the wall, because it requires $u/\mu_\tau = y\mu_\tau/\nu +$ higher order terms in y , is therefore sufficient to establish the result of the theorem at the surface.

Outside the shear flow, on the other hand, $u(x, y, t) = u(x, \delta, t) = u_1(x, t)$, and $\partial v/\partial y = -\partial u_1(x, t)/\partial x$, so that $v(x, y, t) - v(x, \delta, t) = -(y - \delta)\partial u_1/\partial x$, and therefore

$$\frac{\lambda v}{\delta u} = \frac{\lambda v_1}{\delta u_1} + \frac{\mu_\tau}{u_1} \frac{\partial u_1/\partial x}{\partial \mu_\tau/\partial x} \left(\frac{y}{\delta} - 1 \right) \quad (6)$$

where $v_1 = v_1(x, t) = v(x, \delta, t)$. Now in the coordinate system $(\lambda v/\delta u, y/\delta)$ the straight line defined by the last equation intersects the straight line $\lambda v/\delta u = y/\delta$ at the point (P, P) , where

$$P = \frac{\frac{\lambda v_1}{\delta u_1} - D}{1 - D} \quad (7)$$

and

$$D(x, t) = \frac{\partial \ln u_1/\partial x}{\partial \ln \mu_\tau/\partial x} \quad (8)$$

Eliminating $\lambda v_1/\delta u_1$ between Eqs. (6) and (7), the integral of the continuity equation outside the shear flow becomes

$$\frac{1}{P} \frac{\lambda v}{\delta u} - 1 = D \left(\frac{1}{P} \frac{y}{\delta} - 1 \right) \quad (9)$$

Finally, from the definition (4) for displacement thickness δ^* it can be shown that

$$\frac{\partial \delta^*}{\partial x} = \frac{v_i}{u_i} + \frac{(\delta - \delta^*)}{u_i} \frac{\partial u_i}{\partial x}$$

Substituting for v_i/u_i in the expression (7) for P , there is obtained after a little manipulation

$$P(x, t) = \frac{\delta^*}{\delta} \frac{\partial \ln(\delta^* u_i / \nu) / \partial x}{\partial \ln(u_i / u_T) / \partial x} \quad (10)$$

and the result stated in the theorem follows.

2. Equilibrium turbulent flow. For an equilibrium turbulent flow the parameter Π is constant in the expression (2) for the velocity ratio u_i / u_T ,

$$\kappa \frac{u_i}{u_T} = \ln \frac{\delta u_T}{\nu} + \kappa C + 2\Pi$$

and in the expression (3) for the displacement thickness δ^* ,

$$\kappa \frac{\delta^* u_i}{\delta u_T} = 1 + \Pi + \frac{\epsilon}{\delta u_T / \nu}$$

When these relationships are used in the definition (10) for P , there is obtained immediately

$$P = 1 + \Pi = \text{constant} \quad (11)$$

At first or even second glance this result is astonishing. For consider that the parameters D and P have just been defined and connected by one of the most general theoretical relationships which might be developed for flows of boundary-layer type without being so general as to be useless. The relationships (2) and (3), on the other hand, are of the essence of contemporary empirical knowledge of phenomena in turbulent boundary layers; the emphasis is on the word empirical. That these apparently unrelated lines of investigation are found to converge in the simple equation (11) must be either a remarkable coincidence or a spontaneous manifestation of a fundamental order in the problem being studied here.

3. The laminar Falkner-Skan flows. Having an explicit formula for $P(x)$ in Eq. (10), it is natural to ask if any other boundary-layer flows are known for which P is independent of x . I will now show that the family of laminar flows first studied by Falkner and Skan⁽⁵⁾ has this property in common with the class of equilibrium turbulent flows defined by Clauser.

The Falkner-Skan flows are solutions of the laminar boundary-layer equations with the boundary conditions $u = v = 0$ at $y = 0, x > 0$; $u \rightarrow u_1(x)$ as $y \rightarrow \infty, x > 0$; and the special external condition

$$\frac{u_1}{u_0} = \left(\frac{x}{x_0} \right)^n \quad (12)$$

where $u_0, x_0 > 0$, and n are parameters. Taking a stream function of the form

$$\psi(x, y) = u_0 x_0 \sqrt{\left(\frac{2}{n+1}\right) \frac{v}{u_0 x_0}} \left(\frac{x}{x_0}\right)^{\frac{n+1}{2}} f(\eta, n) \quad (13)$$

with $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, and with

$$\eta(x, y) = \sqrt{\left(\frac{n+1}{2}\right) \frac{u_0 x_0}{v}} \left(\frac{x}{x_0}\right)^{\frac{n-1}{2}} \frac{y}{x_0} \quad (14)$$

there is obtained the non-linear ordinary differential equation

$$f''' + f f'' = \beta (f'^2 - 1) \quad (15)$$

where $\beta = 2n/(n+1)$ and the primes indicate differentiation with respect to η . The boundary conditions on $f(\eta, n)$ are $f = f' = 0$ at $\eta = 0$ and $f' \rightarrow 1$ as $\eta \rightarrow \infty$.

Hartree⁽⁶⁾ has tabulated $f'(\eta, n)$ together with $f''(0, n)$ for various values of β . These calculations have recently been repeated and the numerical results reported in greater detail by Smith⁽⁷⁾. No real solutions of Eq. (15) have been found for $\beta < -0.1988$; that is, for $-1 < n < -0.0904$, and the uniqueness of the solutions for negative β apparently requires an additional condition that $f' = u/u_1$ should approach unity from below as rapidly as possible for increasing η .

If the ratio v/u is computed for the Falkner-Skan flows using the expression (13) for the stream function, there is obtained

$$\eta_1 \frac{\lambda v}{\delta u} = 2 \left(\frac{1+n}{1-3n} \right) \left[\left(\frac{1-n}{1+n} \right) \eta - \frac{f(\eta)}{f'(\eta)} \right] \quad (16)$$

where $\eta_1 = \eta(x, \delta)$. The parameter λ is defined in the theorem

presented earlier. Now $f'(\eta)$ approaches unity for large η , and $f(\eta, n)$ therefore approaches $k(n) + \eta$, where $k(n)$ is a negative constant. Very far outside the boundary layer, it follows from Eq. (16) that

$$-\frac{\eta_1}{2k} \frac{\lambda v}{\delta \mu} - 1 = \frac{4n}{3n-1} \left(-\frac{\eta_1}{2k} \frac{y}{\delta} - 1 \right)$$

Comparing this expression with Eq. (9), it is seen that

$$D(n) = \frac{4n}{3n-1} \quad (17)$$

and

$$P(n) = -2 \frac{k(n)}{\eta_1(n)} \quad (18)$$

for the flows in question. Furthermore, it can be shown from the definition (4) for displacement thickness that $\delta^* = (y/\eta) \lim_{\eta \rightarrow \infty} [\eta - f(\eta)] = -k\delta/\eta_1$. Thus

$$P(n) = 2 \frac{\delta^*}{\delta} \quad (19)$$

for the Falkner-Skan flows.

From Eq. (16) it is also seen that $\eta_1 \lambda v / \delta \mu$ is a function of $\eta_1 y / \delta = \eta$ alone, with n as parameter, and thus that $\lambda v / P \delta \mu = \lambda v / 2 \delta^* \mu$ as a function of $y / P \delta = y / 2 \delta^*$ is independent of Reynolds number. Several typical curves, computed from Smith's tables for various values of n , are shown in Fig. 1. These curves obviously do not depend on

the definition adopted for the boundary-layer thickness δ .

Eqs. (17) and (18) show that there is also a relationship between the two parameters D and P . This relationship, however, does depend on the definition of δ . If η_1 is arbitrarily taken as the value of η for which $u/u_1 = 0.99$, the curve plotted in Fig. 2 is obtained from the numerical solutions tabulated by Smith.

4. The limiting laminar sink flow. Although both Hartree and Smith have questioned the physical significance of solutions of the Falkner-Skan equation (15) for $\beta > 2$, Hartree made calculations for $\beta = 2$ ($n = \pm \infty$) and $\beta = 2.4$ without finding any anomalous behavior of $f(\eta, n)$. The situation appears to be that for sufficiently negative values of n , according to Eqs. (13) and (14), the reference velocity u_0 and therefore the free-stream velocity u_1 must be negative if ψ and η are to be real. Consequently, the external flow must be away from the origin in the range $-0.0904 < n < \infty$, or $-0.1988 < \beta < 2$, and toward the origin in the range $-\infty < n < -1$, or $2 < \beta < \infty$.

The limiting solution of Eq. (15) for $\beta \rightarrow +\infty$ ($n \rightarrow -1$ from below) is easily found either by a singular perturbation for large β , in which η is replaced by $\theta/\sqrt{\beta}$ and $f(\eta)$ is replaced by $h(\theta)/\sqrt{\beta}$; or by assuming a stream function of the form⁽³⁾

$$\psi = u_0 x_0 \sqrt{\frac{\nu}{|u_0 x_0|}} h(\theta)$$

with

$$\theta = \frac{y}{x} \sqrt{\frac{|u_0 x_0|}{\nu}}$$

and

$$u_1 x = u_0 x_0 = \text{constant}$$

In either event, taking the case of sink flow in the external stream, or negative u_0 and u_1 , the function $q(\theta) = h'(\theta) = u/u_1$ must satisfy the equation

$$q'' - q^2 + 1 = 0$$

where primes indicate differentiation with respect to θ and the boundary conditions are $q(0) = 0$, $q(\infty) = 1$. Multiplying by q' and integrating once, the surface shearing stress is found to be

$$\tau_w = \mu \frac{u_1}{x} \sqrt{\frac{4}{3}} \sqrt{\frac{|u_0 x_0|}{\nu}}$$

Integrating again, the velocity profile can be expressed in closed form* as

* These relationships are actually a boundary-layer approximation for large Reynolds number to the known exact solution of the Navier-Stokes equations for flow in a converging channel⁽⁸⁾. It is therefore not surprising to find that the present problem has no real solution if the flow in the external stream is away from rather than toward the origin (positive u_0 and u_1 , $\beta \rightarrow -\infty$, $n \rightarrow -1$ from above).

$$\theta(q) = \sqrt{\frac{3}{2}} \int_0^q \frac{dx}{(1-x)\sqrt{2+x}} = \frac{1}{\sqrt{2}} \ln \frac{(1 + \sqrt{\frac{2+q}{3}})(1 - \sqrt{\frac{2}{3}})}{(1 - \sqrt{\frac{2+q}{3}})(1 + \sqrt{\frac{2}{3}})}$$

so that

$$q(\theta) = \frac{u}{u_1} = 3 \tanh^2 \left(\frac{y}{x} \sqrt{\frac{|u_0 x_0|}{2\nu}} + \tanh^{-1} \sqrt{\frac{2}{3}} \right) - 2 \quad (20)$$

The most important property of the solution (20) of the Falkner-Skan equation for sink flow is that the streamlines throughout the flow are straight lines through the origin with $v/u = y/x$. It follows⁽⁴⁾ that the velocity profile can be written in the form $u/u_\tau = \psi(yu_\tau/\nu)$. Furthermore, the friction coefficient is constant, and therefore $D = d \ln u_1 / d \ln u_\tau = 1$.

5. The hypothesis $D = \text{constant}$. I have shown that the two parameters D and P occur naturally together in the integrated continuity equation; that D and P are separately constant for any one of the Falkner-Skan laminar flows; and that P is constant for an equilibrium turbulent flow. Little imagination is required to anticipate the specific hypothesis which will shortly be made for equilibrium turbulent flow, namely

$$D = \frac{d \ln u_1}{d \ln u_\tau} = \text{constant}$$

From the definition it is seen that the parameter D depends on the relative magnitude and rate of change of u_1 and u_τ , or alternatively of the free-stream dynamic pressure q and the surface shearing stress τ_w . The situation is therefore a happy one, in that the hypothesis $D = \text{constant}$ can be tested a priori.

Seven equilibrium or near-equilibrium flows are included in an extensive survey of the experimental literature reported elsewhere⁽³⁾. For six of these seven flows, Fig. 3 shows a plot in logarithmic coordinates of u_1/u_{1_0} against u_τ/u_{τ_0} , where u_{1_0} and u_{τ_0} are arbitrary constant reference velocities; u_τ is inferred from the law of the wall. The supposition that the data might define a straight line is borne out in each case, and the corresponding values of D and P are listed in the adjacent Table I and plotted in Fig. 4.

TABLE I
SUMMARY OF EXPERIMENTAL DATA ON D AND P IN EQUILIBRIUM
TURBULENT FLOW

Reference or Remarks	$P = 1 + \Pi$	$D = \frac{d \ln u_1}{d \ln u_\tau}$	D (revised)
Pure wall flow	1	1	
Ludwig and Tillmann ⁽¹⁾ , Channel VII	1.20 ± 0.02	1.22 ± 0.03	
Bauer ⁽⁹⁾ , 20° slope	1.22 ± 0.02	1.42 ± 0.03	
40° slope	1.24 ± 0.02	1.33 ± 0.03	
60° slope	1.23 ± 0.02	1.30 ± 0.03	
Wiegardt ⁽¹⁰⁾ , constant pressure	1.55 ± 0.01	0	
Clauser ⁽²⁾ , Series 1	2.54 ± 0.05	0.86 ± 0.05	0.795 ± 0.012
Series 2	4.93 ± 0.10	0.75 ± 0.05	0.863 ± 0.002
Pure wake flow	∞	1	

III. SOLUTION OF THE INVERSE PROBLEM

1. The von Kármán momentum-integral equation. Three of the four equations needed in the formulation of the inverse problem for equilibrium turbulent flow are the stipulation

$$\pi = \text{constant} \quad (21)$$

together with the local friction law (2)

$$\kappa \frac{u_1}{u_\tau} = \ln \frac{\delta u_\tau}{\nu} + \kappa c + 2\pi$$

and the momentum-integral equation of von Kármán, which for the conditions considered here is

$$\frac{\tau_w}{\rho} = u_\tau^2 = \frac{d}{dx} u_1^2 \theta + u_1 \delta^* \frac{du_1}{dx} \quad (22)$$

The new variables δ^* and θ in Eq. (22) are the boundary-layer displacement and momentum thicknesses respectively, defined by Eq. (4) and by

$$\theta = \int_0^\delta \frac{u}{u_1} \left(1 - \frac{u}{u_1}\right) dy \quad (23)$$

These quantities are readily expressed in terms of u_1 , u_τ , δ , and π for the profile given by Eq. (1). Hereafter I propose to neglect the departure of the flow in the sublayer from the logarithmic law of the wall; then substitution of Eq. (1) in Eq. (4) yields Eq. (3) with $\epsilon = 0$,

$$\kappa \frac{\delta^* u_1}{\delta u_T} = \Omega_1(\pi) \quad (24)$$

and substitution in Eq. (23) yields

$$\frac{\kappa^2}{2} \left(\frac{\delta^* - \theta}{\delta} \right) \frac{u_1^2}{u_T^2} = \Omega_2(\pi) \quad (25)$$

where Ω_1 and Ω_2 are defined by

$$\Omega_1(\pi) = 1 + \pi \int_0^2 \frac{y}{\delta} dw \quad (26)$$

$$\Omega_2(\pi) = \Omega_1(\pi) + \pi \int_0^2 \frac{y}{\delta} \ln \frac{y}{\delta} dw + \pi^2 \int_0^2 \frac{y}{\delta} (2-w) dw \quad (27)$$

Taking the wake function $w(y/\delta)$ from Ref. 3, then

$$\Omega_1(\pi) = 1 + \pi$$

$$\Omega_2(\pi) = 1 + 1.600\pi + 0.761\pi^2$$

For future use it may be noted that the elimination of δ between Eqs. (24) and (25) provides a formula for the profile shape parameter δ^*/θ ,

$$\frac{\theta}{\delta^*} = 1 - \frac{2u_T}{\kappa u_1} \frac{\Omega_2}{\Omega_1} \quad (28)$$

In applying these relationships, a convenient first step is the use of Eqs. (28) and (24) successively in Eq. (22) to eliminate θ and δ^* in favor of u_1 , u_T , and δ . Remembering that π is constant,

the result is

$$\begin{aligned} \kappa^2 = & \frac{\delta}{u_1} \frac{du_1}{dx} \left(\kappa \frac{u_1}{u_T} \right)^2 \\ & - \frac{\delta}{u_T} \frac{du_T}{dx} \left[\left(\kappa \frac{u_1}{u_T} \right)^2 - \left(\kappa \frac{u_1}{u_T} \right) (3\Omega_1 - 1) + 2(2\Omega_2 - \Omega_1) \right] \\ & + \frac{d\delta}{dx} \left[\left(\kappa \frac{u_1}{u_T} \right) (\Omega_1 - 1) - 2(\Omega_2 - \Omega_1) \right] \end{aligned}$$

The second step is to suppress one of the three derivatives in the last equation with the aid of the local friction law (2) in differentiated form.

The natural choice for elimination is $d\delta/dx$, yielding

$$\begin{aligned} \kappa^2 = & \frac{\delta}{u_1} \frac{du_1}{dx} \left[\left(\kappa \frac{u_1}{u_T} \right)^2 \Omega_1 - 2 \left(\kappa \frac{u_1}{u_T} \right) (\Omega_2 - \Omega_1) \right] \quad (29) \\ & - \frac{\delta}{u_T} \frac{du_T}{dx} \left[\left(\kappa \frac{u_1}{u_T} \right)^2 \Omega_1 - 2 \left(\kappa \frac{u_1}{u_T} \right) \Omega_2 + 2\Omega_2 \right] \end{aligned}$$

The third step, suggested by experience with the special case $u_1 =$ constant, is the recognition of the quantity

$$z = \kappa \frac{u_1}{u_T}$$

as a fundamental independent variable. Differentiating the last expression, then

$$\frac{1}{z} \frac{dz}{dx} = \left(\frac{D-1}{D} \right) \frac{1}{u_1} \frac{du_1}{dx} = (D-1) \frac{1}{u_T} \frac{du_T}{dx}$$

where $D = d \ln u_1 / d \ln u_T$ by definition. Eq. (29) becomes finally

$$\kappa^2 \frac{dx}{\delta} = \frac{dz}{z} \left[z^2 \Omega_1 - 2z\Omega_2 - 2z\Omega_1 \left(\frac{D}{1-D} \right) + 2\Omega_2 \left(\frac{1}{1-D} \right) \right] \quad (30)$$

To recapitulate, Eq. (30) is the momentum-integral equation (22)

evaluated for the special mean-velocity profile (1) with constant π . The defect law (5) has been taken to apply throughout the flow, including the sublayer, so that Eqs. (29) and (30) are at least asymptotically valid for large Reynolds numbers. Because no assumption has yet been made concerning the parameter D , the effect of the manipulation just carried out has been to change the nature but not the number of the variables in the problem. It is conceivable that the form of Eq. (30) would eventually suggest the assumption $D = \text{constant}$ as a heuristic measure, even if attention had not been attracted to this hypothesis by consideration of the continuity equation.

2. Integration of the von Kármán equation. I will now assume that an equilibrium turbulent flow has the property

$$D = \frac{\mu_T}{\mu_1} \frac{d\mu_1}{d\mu_T} = \text{constant} \quad (31)$$

Eq. (31) evidently provides the fourth relationship needed in the formulation of the inverse problem, and is to be considered jointly with the momentum-integral equation (30), the local friction law (2), and the equilibrium condition (21). Now Eq. (31) is itself a differential equation which may be integrated immediately, with the result

$$\frac{\mu_1}{\mu_{1_0}} = \left(\frac{\mu_T}{\mu_{T_0}} \right)^D \quad (32)$$

where $\mu_{1_0} = \mu_1(x_0)$ and $\mu_{T_0} = \mu_T(x_0)$; x_0 is any convenient reference point. Returning to Eq. (30), the awkward factor δ on the left-hand side can be eliminated by observing, in view of Eqs. (2) and (32), that

$$\frac{\delta}{\delta_0} = \left(\frac{z}{z_0}\right)^{\frac{1}{1-D}} e^{z-z_0} \quad (33)$$

where $\delta_0 = \delta(x_0)$ and $z_0 = z(x_0)$. Substituting for δ in Eq. (30), there is obtained finally

$$\frac{x^2 z_0^{\frac{1}{1-D}} e^{z_0}}{\delta_0} dx = z^{\frac{1}{1-D}-1} e^z dz \left[z^2 \Omega_1 - 2z \Omega_2 - 2z \Omega_1 \left(\frac{D}{1-D}\right) + 2 \Omega_2 \left(\frac{1}{1-D}\right) \right] \quad (34)$$

The variables x and z are separated in Eq. (34), and integration can be carried out in closed form if $1/(1-D)$ is an integer or half-integer. For example, if $D = 4/3$,

$$\begin{aligned} x^2 \left(\frac{x-x_0}{\delta}\right) &= (5\Omega_1 - 2\Omega_2) z_0^3 e^{-z_0} [Ei(z) - Ei(z_0)] \\ &\quad - (5\Omega_1 - 2\Omega_2) z_0^2 \left[\frac{\delta}{\delta_0} \left(\frac{z}{z_0}\right)^2 - 1 \right] \\ &\quad - (4\Omega_1 - 2\Omega_2) z_0 \left[\frac{\delta}{\delta_0} \left(\frac{z}{z_0}\right) - 1 \right] \\ &\quad + 2\Omega_2 \left[\frac{\delta}{\delta_0} - 1 \right] \end{aligned} \quad (35)$$

where $Ei(z) = \int_{-\infty}^z \frac{e^x}{x} dx$ and

$$\frac{\delta}{\delta_0} = \left(\frac{z}{z_0}\right)^{-3} e^{z-z_0} \quad (36)$$

The quantities (x , δ , z) and (x_0 , δ_0 , z_0) in these expressions may obviously be considered as variables and parameters respectively or vice versa, depending on the application. Note that two independent constants of integration, δ_0 and z_0 , are encountered in integrating the system (31) and (34). Because $z = \kappa u_1 / u_T$ determines $\delta u_T / \nu$ for an equilibrium flow by virtue of the local friction law (2), this means that the two physical scales δ and ν / u_T may be specified independently at any one station.

I have evaluated Eqs. (35) and (36) numerically, taking $\Pi = 0.24$ and $D = 1.33$. These values are intended to represent the spillway flow with 40° slope studied by Bauer⁽⁹⁾ (Fig. 8 of Ref. 3). Calculations have also been made for $\Pi = 0.55$ and $D = 0$, corresponding to the flow with a constant external velocity of 33.0 meters per second studied by Wieghardt⁽¹⁰⁾ (Fig. 4 of Ref. 3), and for $\Pi = 1.54$ and $D = 0.795$, corres-

ponding to the flow with moderately rising pressure studied by Clauser⁽²⁾ (Fig. 15 of Ref. 3). Taking the constants δ_0 and z_0 in each case from the experimental data at a point well downstream, the calculated and measured values for $\delta^*(x)$, $u_1(x)$, and $z(x)$ are compared in Fig. 5.

The value $D = 0.795$ attributed to Clauser's flow is different from the experimental value $D = 0.86$ listed in Table I. The need for some revision in the original value of D can be argued from the momentum-integral equation in the differentiated form (29). Let this equation be rewritten as

$$\kappa^2 \frac{\lambda}{\delta} = z^2 \Omega_1 (1-D) - 2z \left[\Omega_1 D + \Omega_2 (1-D) \right] + 2 \Omega_2$$

Now the quantity $\kappa^2 \lambda / \delta$ can also be evaluated by differentiating the local friction law (2) to obtain

$$\kappa^2 \frac{\lambda}{\delta} = \frac{\kappa^2}{d\delta/dx} \left[z(1-D) + 1 \right]$$

The first of these two equations assumes two-dimensional momentum balance as well as similarity in the mean-velocity profile in the sense required by the law of the wall and the defect law. The second equation assumes similarity only. Taking $\Pi = 1.54$ and $D = 0.86$ for Clauser's first series of experiments, together with a typical value $z = 13$, it is found that the computed values of λ/δ and $d\delta/dx$ are negative. I have therefore preferred to reverse the calculation, estimating $d\delta/dx$ from the experiments and calculating D instead. The revised values*

* That the revisions should be in opposite directions for Clauser's two flows is suggested by slight discrepancies in momentum balance reported elsewhere (Cf. the functions $\bar{\Phi}(x)$ in Figs. 15 and 16 of Ref. 3).

for D are listed in Table I, together with an estimate of probable error corresponding to an uncertainty of ± 20 percent in $d\delta/dx$.

3. The shearing-stress profile. The distribution of shearing stress within an equilibrium turbulent boundary layer may be found by integrating the boundary-layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (37)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_1 \frac{du_1}{dx} + \frac{1}{\rho} \frac{\partial \tau}{\partial y} \quad (38)$$

for the profile of Eq. (1) with constant π . Noting that $\tau_w = \rho u_\tau^2$ by definition, the result of a tedious lot of algebra is

$$\begin{aligned} \kappa^2 \frac{\lambda}{\delta} \left(1 - \frac{\tau}{\tau_w}\right) &= \frac{y}{\delta} \left[\zeta^2 - D z^2 + (1-D) \zeta z (\omega_1 - 1) \right. \\ &\quad \left. - 2(1-D) z (\omega_2 - \omega_1) - 2\zeta \omega_1 + 2\omega_2 \right] \end{aligned} \quad (39)$$

where $z = \kappa u_1 / u_\tau$, $\zeta = \kappa u / u_\tau$, and ω_1 and ω_2 are incomplete integrals corresponding to Ω_1 and Ω_2 in Eqs. (26) and (27). The functions

$\omega_1(\pi, y/\delta)$ and $\omega_2(\pi, y/\delta)$ are defined and tabulated in Ref. 3.

Taking $\delta u_\tau / \nu = 5000$ for four equilibrium flows which have been observed experimentally -- first, $P = 1.24$, $D = 1.33$; second $P = 1.55$, $D = 0$; third, $P = 2.54$, $D = 0.80$; and fourth, $P = 4.93$, $D = 0.86$ -- the mean-velocity profile according to Eq. (1) and the total shearing-stress profile according to Eq. (39) are shown in Fig. 6. Also shown as a cross-hatched region is the velocity defect in the equivalent wake⁽³⁾.

4. Uniqueness. One important consequence of the hypothesis $D = \text{constant}$ for an equilibrium turbulent boundary layer has to do with the matter of uniqueness. Given a value for D , an integral of Eq. (34) can presumably be found in the form $(x-x_0)/\delta_0 = E(z, z_0, \pi, D)$. Eliminating the thickness δ_0 in favor of δ with the aid of Eq. (33), this integral can be written $(x-x_0)/\delta = F(z, z_0, \pi, D)$. Now if π (or P) and D are separately constant for an equilibrium flow, it is reasonable to suppose that these quantities are related by some function $D(P)$ like the one for laminar flow shown in Fig. 2. If so, then the dependence of the flow on the parameter D need not be stated explicitly; thus $(x-x_0)/\delta = G(z, z_0, \pi)$. Finally, it is possible to specify the origin or initial point x_0, δ_0, z_0 , etc. in such a way that z_0 depends on π alone. For example, assume that the momentum thickness θ vanishes for $x = x_0 = 0$, and note that Eq. (28) then requires $\kappa u_\tau / \mu_\tau = z = z_0 = 2\Omega_2(\pi) / \Omega_1(\pi)$. The integral of Eq. (34) under these conditions can therefore be expressed ultimately as

$$\frac{x}{\delta} = H(z, \pi)$$

But if $\delta u_\tau / \nu$ and δ/x for constant π are functions of $z = \kappa u_\tau / \mu_\tau$ alone, then so are $u_\tau x / \nu$ and u_τ / ν . So also are $\delta^*/x, \delta^*/\theta, u_\tau \theta / \nu$, and similar quantities, by virtue of various relationships derived earlier for equilibrium turbulent flow.

Given $D = \text{constant}$, therefore, the conclusion is that quantities like $\delta^*/\theta, C_f = 2u_\tau^2/\mu_\tau^2$, and $R_\theta = u_\tau \theta / \nu$ can be expressed as one-parameter functions of a uniquely defined streamwise Reynolds number $R = u_\tau x / \nu$ for the class of equilibrium turbulent flows even when u_τ depends on x . This conclusion does not require the assumption

that the wake function $w(y/\delta)$ in Eq. (1) is universal, because the mean-velocity profile in an equilibrium flow is adequately expressed for the purpose of this argument by the defect law (5). Neither would the conclusion stated here be changed if the exact mean-velocity profile in the sublayer had been considered in Eqs. (24) and (25), as the quantities $\Omega_1(\pi)$ and $\Omega_2(\pi)$ could then be replaced by $\Omega_1(\pi, z)$ and $\Omega_2(\pi, z)$.

IV. THE FUNCTION $D(P)$

1. The pure wall flow. In this and the next section I will attempt to treat the limiting cases $\Pi = 0$ and $\Pi = \infty$ by arguments which amount to extrapolations based on the idea of kinematic similarity. In the temporary absence of experimental evidence these arguments may be accepted or rejected on their merits without seriously prejudicing any of the earlier discussion.

Consider first the limiting case $\Pi = 0$ in Eq. (1). The mean-velocity profile is given by the law of the wall,

$$\frac{u}{u_\tau} = f\left(\frac{y u_\tau}{\nu}\right)$$

and it follows⁽⁴⁾ that u/u_τ and $y u_\tau/\nu$ are constant on mean streamlines and that

$$\frac{\lambda \nu}{\delta u} = \frac{y}{\delta} \tag{40}$$

where $1/\lambda = -(1/u_\tau) du_\tau/dx$ as before. The shearing-stress profile is most readily obtained by putting $\omega_1 = \omega_2 = 1$ in Eq. (39);

$$x^2 \frac{\lambda}{\delta} \left(1 - \frac{\tau}{\tau_w}\right) = \frac{y}{\delta} \left[-D \left(x \frac{u_1}{u_\tau}\right)^2 + \left(x \frac{u}{u_\tau}\right)^2 - 2 \left(x \frac{u}{u_\tau}\right) + 2 \right] \tag{41}$$

At $y = \delta$, therefore, where $u = u_1$, and $\tau = 0$,

$$x^2 \frac{\lambda}{\delta} = \left(x \frac{u_1}{u_\tau}\right)^2 (1 - D) - 2 \left(x \frac{u_1}{u_\tau}\right) + 2 \tag{42}$$

Several arguments can be found, all of them unfortunately somewhat porous, for supposing that if the parameter D is constant for the

pure wall flow it ought to have the value unity. Certainly the point $P = D = 1$ is favorably located in Fig. 4 with respect to the experimentally determined points in accelerating flow, if the hypothetical function $D(P)$ for equilibrium turbulent flow is to resemble the one in Fig. 2 for laminar flow. The statement $D = 1$ can also be argued from the physical premise that the mean-velocity profile for pure wall flow has only one characteristic length, ν/μ_τ , so that $\delta\mu_\tau/\nu$ ought to be constant.

Alternatively, suppose for the sake of regularity that ν/μ in Eq. (40) and τ/τ_w in Eq. (41) are functions of y/δ alone. Then λ/δ is constant from (40), and μ_1/μ_τ is constant and thus $D = 1$ from (42). Moreover, $\delta\mu_\tau/\nu$ is constant from (2), so that $d\delta/dx$ is also constant and δ varies linearly with x .

If $D = 1$ there is no entrainment of fluid in the boundary layer considered here, because $y = \delta$ is a mean streamline. This and other points of resemblance between the pure wall flow and the limiting Falkner-Skan flow for $n = -1$ or $\beta = \infty$ suggest that the flow here is actually a sink flow moving toward the origin. This view is supported by the following argument; if u , y , δ , μ_1 , and μ_τ are positive and $D = 1$, then Eq. (42) requires λ/δ and λ to be negative, at least for large Reynolds numbers. Eq. (40) then requires ν/μ and ν to be also negative. Finally, $d\mu_\tau/dx$ and $d\mu_1/dx$ are both positive from the definitions of λ and D . These statements can only be reconciled with the statement that $d\delta/dx$ is a (negative) constant if the flow is proceeding toward the origin $x = 0$ through negative values of x .

A comparison of the laminar and turbulent sink flows emphasizes the fact that any velocity profile which could be written in the form of the law of the wall with constant μ_1/μ_τ would be a possible profile in

sink flow. Only the function of Eq. (20) has the additional property that the boundary-layer momentum equation is satisfied at the same time that $\tau = \mu \partial u / \partial y$. For turbulent flow, on the other hand, the boundary-layer momentum equation is automatically satisfied because u is given and τ is computed therefrom. In either case the friction coefficient is constant but its value can apparently be chosen arbitrarily.

The logarithmic mean-velocity profile and the corresponding shearing-stress profile* in the pure wall flow are shown in Fig. 6 for $\delta u_\tau / \nu = 5000$ and $D = 1$.

* I have also computed the function τ / τ_w for $\delta u_\tau / \nu = 10^3, 10^6, \text{ and } 10^9$, and have found that the various curves can really not be distinguished in the figure. That is, τ / τ_w as a function of y / δ is for practical purposes independent of the Reynolds number $\delta u_\tau / \nu$. These calculations presumably refer to physically different boundary layers, not to different stations in the same boundary layer.

2. The pure wake flow. The profile parameter Π in Eqs. (2) and (3) is a measure of the relative magnitude of the wake and wall components in the mean-velocity profile. According to Eq. (3), Π becomes indefinitely large when τ_w approaches zero. However, when Eq. (1) is multiplied by u_τ / u_1 and u_τ is put equal to zero, Π having first been eliminated by Eq. (3), the mean-velocity profile becomes

$$\frac{u}{u_1} = \frac{\delta^*}{\delta} w \left(\frac{y}{\delta} \right) = \frac{1}{2} w \left(\frac{y}{\delta} \right) \quad (43)$$

since $w = 2$ when $u = u_1$ by definition. This is the profile at a point of separation or reattachment. The pure wake flow is obtained on assuming

that this same profile holds for all values of x . In the present instance, δ^*/δ , θ/δ , and δ^*/θ are constants having the values 0.500, 0.120, and 4.18 respectively. (3)

To begin with, an important property of the pure wake flow follows directly from the momentum-integral equation of von Kármán for two-dimensional flow. Taking $\theta/\delta = \text{constant}$ and $\tau_w = 0$, Eq. (22) becomes

$$\frac{1}{\delta} \frac{d\delta}{dx} + \left(2 + \frac{\delta^*}{\theta}\right) \frac{1}{u_i} \frac{du_i}{dx} = 0 \quad (44)$$

and therefore

$$\delta u_i^{2 + \delta^*/\theta} = \text{constant} \quad (45)$$

Furthermore, the shearing-stress profile is readily obtained by integrating the boundary-layer system (37) and (38) directly for the mean-velocity distribution (43). The result is

$$\frac{v/u}{d\delta/dx} = \frac{y}{\delta} - \left(\frac{1 + \delta^*/\theta}{2 + \delta^*/\theta}\right) \frac{1}{W} \int_0^{y/\delta} W d\zeta \quad (46)$$

$$\frac{\tau/\rho u_i^2}{d\delta/dx} = \left(\frac{1}{2 + \delta^*/\theta}\right) \frac{y}{\delta} - \left(\frac{1 + \delta^*/\theta}{2 + \delta^*/\theta}\right) \frac{W}{4} \int_0^{y/\delta} W d\zeta + \left(\frac{\delta^*/\theta}{2 + \delta^*/\theta}\right) \frac{1}{4} \int_0^{y/\delta} W^2 d\zeta \quad (47)$$

Now suppose that either v/u or τ/ρ for this particular flow is a function of y/δ alone. Eq. (46) or (47) then implies

$$\frac{d\delta}{dx} = - \left(2 + \frac{\delta^*}{\theta}\right) \frac{\delta}{u_i} \frac{du_i}{dx} = \text{constant}$$

so that δ varies linearly with x ; and this property is obviously interchangeable with the original condition on v/μ or τ/ρ .

Finally, if $d\delta/dx$ is in fact constant for the turbulent boundary-layer flow with $\tau_w = 0$, then Eq. (45) requires, on taking for convenience $\delta = 0$ at $x = 0$,

$$u, x^{1/(2+\delta^*/\theta)} = u, x^{0.162} = \text{constant}$$

The corresponding Falkner-Skan flow with $\tau_w = 0$ was characterized by

$$u, x^{0.0904} = \text{constant}$$

and the present result is at least consistent with the empirical observation that turbulent flow will in general support a more rapid pressure rise.

Clauser's data in Fig. 4 suggest that the value of D which is appropriate for the pure wake flow is $D = 1$. Consider also that Eq. (46) evaluated at $y = \delta$, in conjunction with Eq. (44), requires

$$\frac{\lambda v_i}{\delta u_i} = D \left(\frac{3 + \delta^*/\theta}{2} \right)$$

This expression substituted in the defining equation (7) for P yields*

$$P = \frac{D}{(1-D)} \left(\frac{1 + \delta^*/\theta}{2} \right)$$

But $P = 1 + \pi$ is infinite for the pure wake flow, and therefore D must be equal to unity.

This expression for P is valid for laminar flow as long as μ/μ_i depends only on y/δ and τ_w is zero. Knowing that $P = 2\delta^/\delta$ and $D = 4n/(3n-1)$, according to Eqs. (17) and (19), an estimate of the limiting value of n in

the separating Falkner-Skan flow is readily obtained. Taking $\delta^*/\delta = 1/2$ and $\delta^*/\theta = 4$ as reasonable values, then $D = 2/7$ or $n = -1/11 = -0.0909$. As quoted earlier, the exact value for n is -0.0904 .

The mean-velocity distribution (43) and the shearing-stress distribution (47) are plotted in Fig. 6 for the wake function $w(y/\delta)$ of Ref. 3. The shearing stress being computed as τ/τ_{max} , it is not necessary to specify the value of $d\delta/dx$ for the hypothetical pure wall flow considered here.

The present formulation does not in fact yield a value for the derivative $d\delta/dx$, and it would be surprising if it did. However, an estimate for $d\delta/dx$ can be based on the supposition that the pure wake flow studied here corresponds in some sense to the half-wake studied experimentally by Liepmann and Laufer⁽¹¹⁾. The two flows differ in the presence or absence of a streamwise pressure gradient and in the constraint at the boundary $y = 0$. Keeping in mind the observed insensitivity of the wake component, i.e. the defect law, to wall conditions such as roughness in equilibrium flows with finite τ_w , and reserving the question of the finite normal velocity in the free shear layer at the point corresponding to the wall, the two mixing processes might be expected to be similar at least near the free boundary at $y = \delta$. If so, a tentative estimate⁽³⁾ for $d\delta/dx$ in the separating equilibrium flow is $d\delta/dx = \delta/x = 0.252$.

3. An interpolation formula. In any practical application of the concept of equilibrium flow, for example to diffuser design, some interpolation method is needed to supplement the experimental values of $D(P)$ in Table I. The method proposed here depends on the development of two

quantities which, unlike the parameters D and P , remain finite as π increases from zero to infinity. One such quantity is the strength of the equivalent wake, $2\pi\mu_T/\kappa\mu_1 = 2\pi/z$. Another is the rate of mass entrainment in the boundary layer. Defining

$$\frac{s}{\mu_1} = \frac{d\delta}{dx} - \frac{v_1}{\mu_1} = \frac{1}{\rho\mu_1} \frac{d}{dx} [\rho\mu_1(\delta - \delta^*)] \quad (48)$$

then the quantity s is seen to be the velocity of propagation of the boundary $y = \delta$ with respect to the free stream. Now the ratio s/μ_1 may be expressed in terms of the local friction coefficient and the parameter π with the aid of Eqs. (2) and (3);

$$\frac{s}{\mu_1} = \frac{\delta}{z} \frac{dz}{dx} (z - \pi)$$

where $z = \kappa\mu_1/\mu_T$. But $(\delta/z) dz/dx$ is a known function of z , π , and D from Eq. (30); thus

$$\frac{1}{\kappa^2} \frac{s}{\mu_1} = \frac{(z - \pi)(1 - D)}{(1 - D)(z^2\Omega_1 - 2z\Omega_2) - 2Dz\Omega_1 + 2\Omega_2} \quad (49)$$

The case of pure wake flow may be treated separately to obtain $2\pi/z = 1$, and, from Eqs.(48) and (46),

$$\frac{1}{\kappa^2} \frac{s}{\mu_1} = \frac{1}{2\kappa^2} \frac{d\delta}{dx} \left(\frac{1 + \delta^*/\theta}{2 + \delta^*/\theta} \right)$$

For given values of π and D and for a specified value of $\delta\mu_T/\nu$, the quantities z , $2\pi/z$, and $s/\kappa^2\mu_1$ may be computed from Eqs. (2) and (49). Plotting $s/\kappa^2\mu_1$ against $2\pi/z$ for the six flows of Fig. 6 and using

essentially straight line interpolation except near $\Pi = 0$, the inverse calculation* for $D(P)$ leads for $\delta u_T/\nu = 5000$ to the curve shown in

*For Clauser's second flow it appears that the parameter D ought to be taken as 0.866 if a smooth curve is to be obtained in the coordinates $(2\Pi/z, s/x^2 u_1)$.

Fig. 4. This calculation, as might be expected, is not at all sensitive to the value chosen for $\delta u_T/\nu$.

4. The hypothetical function $D(P)$. Perhaps the most instructive physical interpretation of the hypothesis $D = \text{constant}$ comes from the fact that the mean streamlines must intersect at a common origin for any region in which v/u for fixed x is a linear function of y . One such region is the one near the wall, including the sublayer in the case of turbulent flow; here $v/u = y/\lambda$. In the absence of a boundary layer there is a corresponding relationship $v/u = D y/\lambda$ for the non-viscous ambient flow. Thus the parameter D describes the way in which any divergence or convergence of the external flow, which is to say any pressure gradient, affects the shear flow in the neighborhood of the wall. These remarks apply equally for laminar and turbulent boundary layers. Furthermore, for turbulent flow the interpretation just given, like the defect law itself, does not involve the viscosity of the fluid explicitly -- hence the term kinematic similarity.

What is ultimately needed, however, is not a physical interpretation but a physical principle, from which might be deduced not only the existence of a function $D(P)$ for equilibrium turbulent flow but the

form of this function. Although Fig. 3 clearly justifies the assumption $D = \text{constant}$ as an interpolation device for the particular equilibrium flows in question, other reasons must be found for making this assumption in the general case. The theorem presented earlier, in which the two parameters D and P first occur, amounts at best only to circumstantial evidence. So does the parallel treatment given here to laminar and turbulent equilibrium flows. In the absence of a physical principle, therefore, any discussion of a function $D(P)$ requires an act of faith in that neither of the two statements $D = \text{constant}$ or $P = \text{constant}$ can be said to imply the other. The problem having been stated in these terms, it follows from experience with the special but by no means trivial case $\mu_1 = \text{constant}$ or $D = 0$ that a serious attempt should be made to account directly and specifically for the concept of a defect law.

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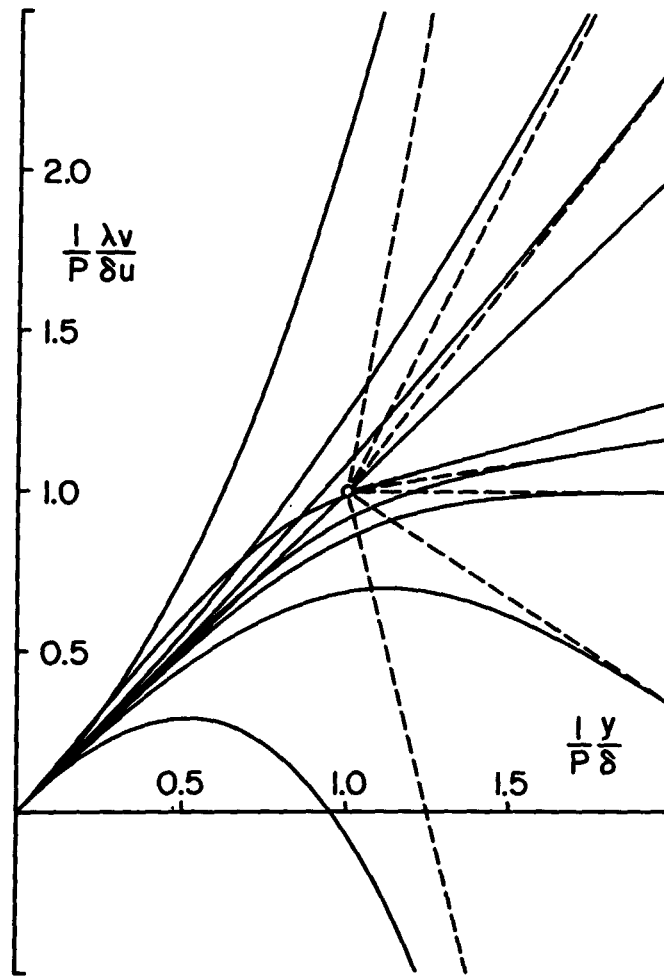


Fig. 1. Kinematic Similarity for the Laminar Falkner-Skan Flows

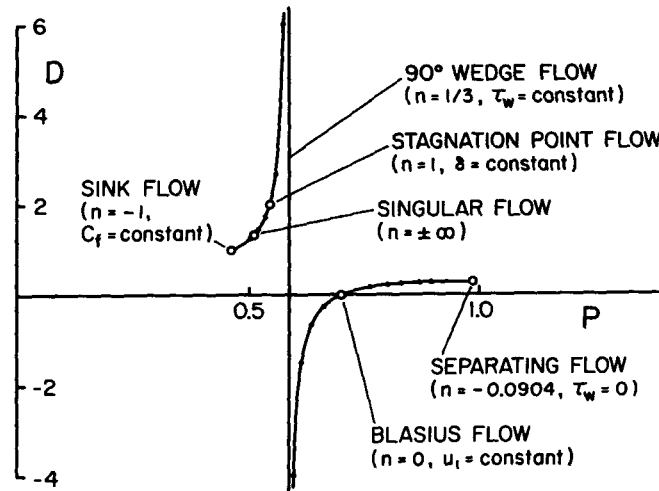


Fig. 2. The Theoretical Function $D(P)$ for the Laminar Falkner-Skan Flows

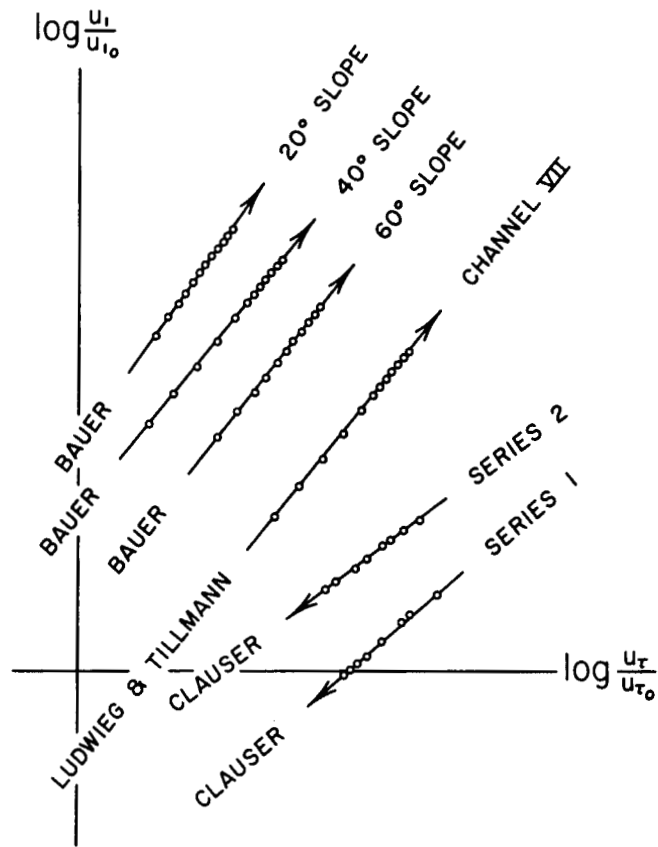


Fig. 3. Test of the Hypothesis $D = d \ln u_1 / d \ln u_\tau = \text{constant}$ for Equilibrium Turbulent Flow

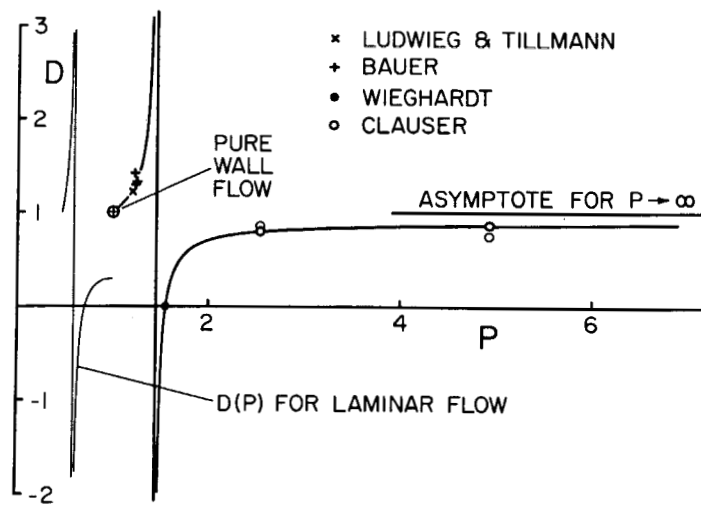


Fig. 4. The Experimental Function $D(P)$ for Equilibrium Turbulent Flow

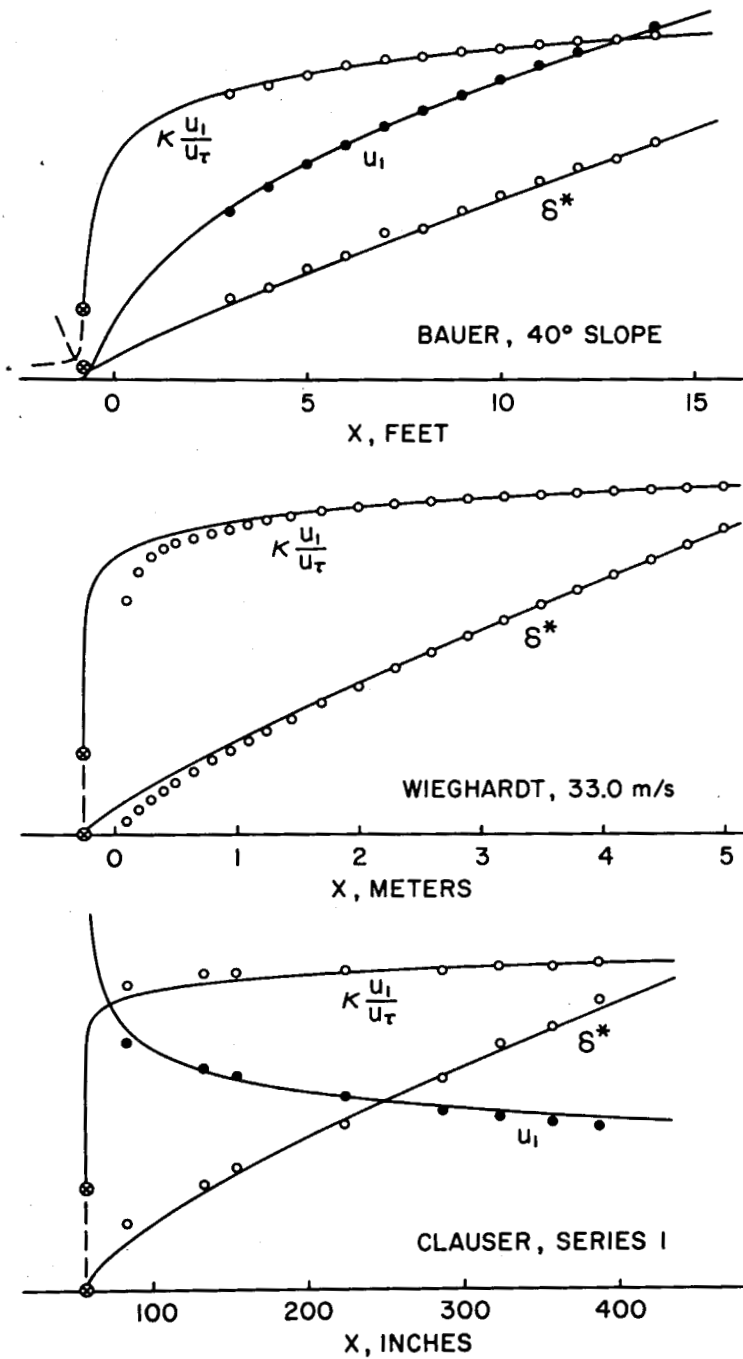


Fig. 5. Comparison of Calculated and Observed Development of Three Equilibrium Turbulent Flows

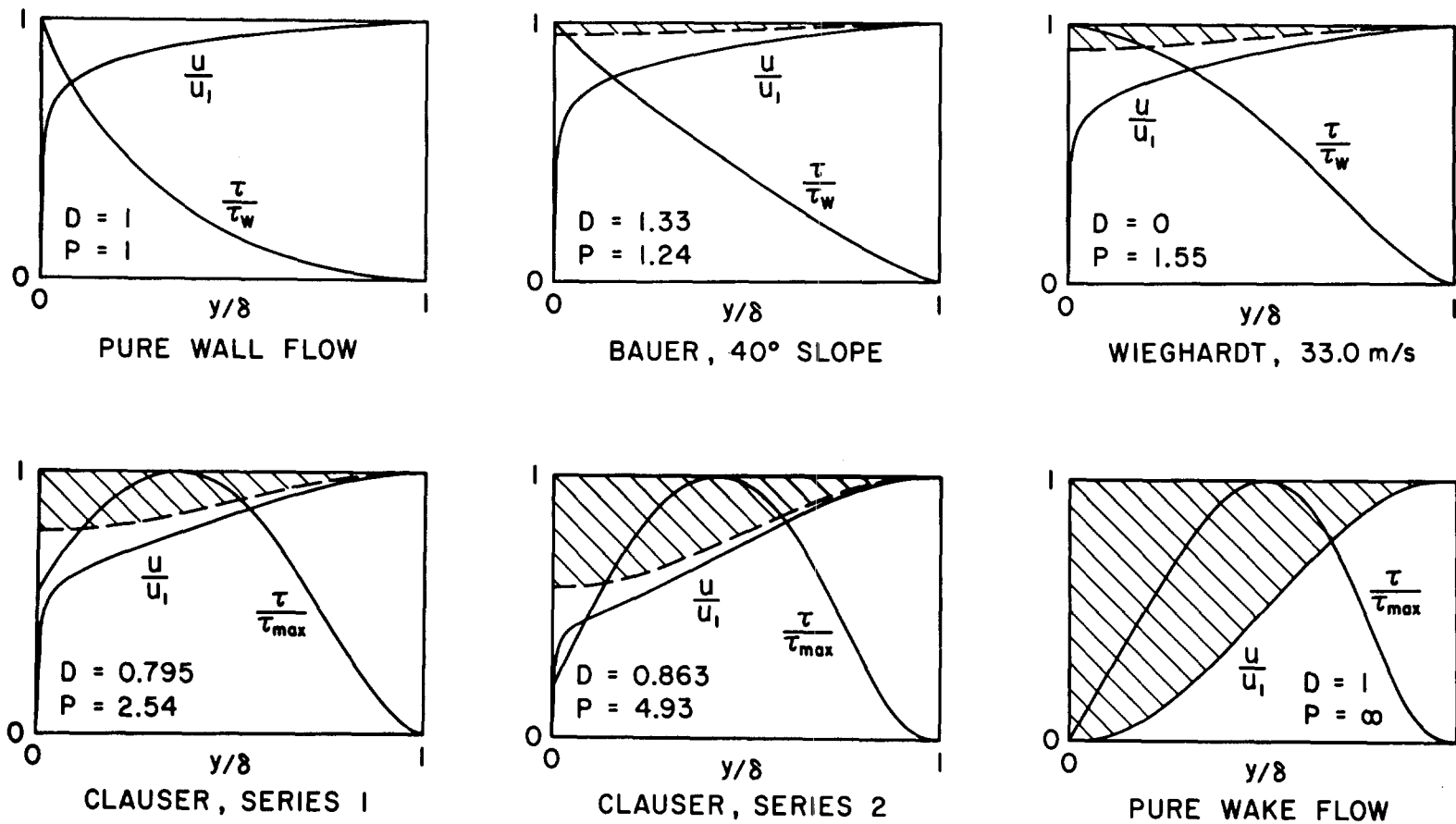


Fig. 6. Typical Mean-velocity and Total Shearing-stress Profiles in Several Equilibrium Turbulent Flows at the Same Local Reynolds Number $\delta u_{\tau}/\nu = 5000$