Prog. Theor. Phys. Vol. 59 (1978), June

Remarks on the Indefinite-Metric Quantum Field Theory of General Relativity

Noboru NAKANISHI

Research Institute for Mathematical Science Kyoto University, Kyoto 606

February 18, 1978

Recently, the present author[1] has formulated the indefinite-metric quantum field theory of the gravitational field as an extension of the Kugo-Ojima formalism[2] of the Yang-Mills field. Our formalism is based on the BRS transformation $\boldsymbol{\delta}$ introduced in that work:[1] It is a nilpotent derivation defined on the basis of the infinitesimal general coordinate transformation $\Phi(x)\to\Phi'(x')$, where x_μ and x_μ' denote the coordinates of the *same* space-time point. Then $\boldsymbol{\delta}$ is not commutative with the differential operator ∂_μ, that is,

$$\boldsymbol{\delta}(\partial_\mu X)=\partial_\mu\boldsymbol{\delta}(X)+\kappa\partial_\mu c^\lambda\cdot\partial_\lambda X\,, \tag{1}$$

where c^λ denotes one of the Faddeev-Popov (F-P) ghosts and X is an arbitrary field polynomial obeying bose or fermi statistics, κ being the gravitational constant.

Very recently, Nishijima and Okawa[3] and Kugo and Ojima[4] have independently formulated the quantum field theory of the gravitational field on the basis of the BRS transformation introduced previously.[5] This operation, which we denote by $\boldsymbol{\delta}'$, is again a nilpotent derivation, but it is defined on the basis of the transformation $\Phi(x)\to\Phi'(x)$, so that it commutes with ∂_μ. The above authors assert that $\boldsymbol{\delta}'$ is more satisfactory than $\boldsymbol{\delta}$ because only the former corresponds to the quantum-theoretical generator of the BRS transformation, though their formalism is more complicated than ours.

Now, the purpose of this Letter is twofold. First, we show that our theory based on $\boldsymbol{\delta}$ and theirs based on $\boldsymbol{\delta}'$ are *mutually equivalent*, as far as the Landau-gauge case is concerned. Secondly, we extend our theory to the non-Landau-gauge case, which is much simpler than that in their formalism.

As is easily seen,[3),4),1)] $\boldsymbol{\delta}$ and $\boldsymbol{\delta}'$ are mutually related through

$$\boldsymbol{\delta}'(X)=\boldsymbol{\delta}(X)+\kappa c^\lambda\partial_\lambda X\,, \tag{2}$$

as far as X is a polynomial in ordinary tensors and c^ρ. The discrepancy between the $\boldsymbol{\delta}$ theory and the $\boldsymbol{\delta}'$ one arises when one introduces an auxiliary boson field b_ρ and another F-P ghost $\bar{c}_\rho$: In the former one assumes that

$$\boldsymbol{\delta}(b_\rho)=0,\ \boldsymbol{\delta}(\bar{c}_\rho)=ib_\rho\,, \tag{3}$$

while in the latter

$$\boldsymbol{\delta}'(b_\rho)=0,\ \boldsymbol{\delta}'(\bar{c}_\rho)=ib_\rho\,. \tag{4}$$

Evidently, (3) and (4) are different if (2) is applied also to b_ρ and $\bar{c}_\rho$. Correspondingly, the action integrals of both theories are different.

It should be noted, however, that *there is no reason for directly identifying* b_ρ *and* $\bar{c}_\rho$ *in the* $\boldsymbol{\delta}$ *theory with those in the* $\boldsymbol{\delta}'$ *theory.* That is, we should distinguish $(b_\rho, \bar{c}_\rho)$'s in both theories from each other. Hence we replace b_ρ and $\bar{c}_\rho$ in the $\boldsymbol{\delta}'$ theory by b_ρ' and $\bar{c}_\rho'$, respectively. We propose the *conversion formula*

$$b_\rho'=b_\rho-i\kappa c^\lambda\partial_\lambda\bar{c}_\rho,\ \bar{c}_\rho'=\bar{c}_\rho\,. \tag{5}$$

Then (3) is equivalent to

$$\boldsymbol{\delta}'(b_\rho')=0,\ \boldsymbol{\delta}'(\bar{c}_\rho')=ib_\rho'\,. \tag{6}$$

Indeed, with (3) and (2), (5) reduces to $b_\rho'=-i\boldsymbol{\delta}'(\bar{c}_\rho)$, i.e., (6); conversely, with (6) and (2), (5) reduces to $b_\rho=-i\boldsymbol{\delta}(\bar{c}_\rho')$, i.e., (3).

In the $\boldsymbol{\delta}$ theory, the gauge-fixing Lagrangian density $\mathcal{L}_{\rm GF}$ and the F-P ghost one $\mathcal{L}_{\rm FP}$ are given by[1]

$$\mathcal{L}_{\rm GF}=-(2\kappa)^{-1}\sqrt{-g}g^{\mu\nu}(\partial_\mu b_\nu+\partial_\nu b_\mu), \tag{7}$$

$$\mathcal{L}_{\rm FP}=\tfrac{1}{2}\,i\sqrt{-g}g^{\mu\nu}(\partial_\mu\bar{c}_\rho\cdot\partial_\nu c^\rho+\partial_\nu\bar{c}_\rho\cdot\partial_\mu c^\rho), \tag{8}$$

respectively. Hence in terms of b_ρ' and $\bar{c}_\rho'$, we have

$$\begin{aligned}\mathcal{L}_{\rm GF}+\mathcal{L}_{\rm FP}=&-(2\kappa)^{-1}\tilde{g}^{\mu\nu}(\partial_\mu b_\nu'+\partial_\nu b_\mu')\\&+(2\kappa)^{-1}i\boldsymbol{\delta}'(\tilde{g}^{\mu\nu})(\partial_\mu\bar{c}_\nu'+\partial_\nu\bar{c}_\mu')\\&-i\partial_\lambda(\tilde{g}^{\mu\nu}c^\lambda\partial_\mu\bar{c}_\nu')\end{aligned} \tag{9}$$

with $\tilde{g}^{\mu\nu}\equiv\sqrt{-g}g^{\mu\nu}$.*) We thus see that the Lagrangian density $\mathcal{L}$ of the $\boldsymbol{\delta}$ theory is equivalent to that of the $\boldsymbol{\delta}'$ theory,[3),4)] as far as the Landau-gauge case is concerned.

Next, we consider an extension of the $\boldsymbol{\delta}$ theory to the non-Landau-gauge case. If we give up the general linear invariance as in Refs. 3) and 4), then we may add

$$-(\alpha/\kappa)\sqrt{-g}b^\rho b_\rho \tag{10}$$

to $\mathcal{L}$, where $b^\rho\equiv\eta^{\rho\sigma}b_\sigma$ with $\eta^{\rho\sigma}$ being the Minkowski metric. Of course, (10) is *not* equivalent to const $b'^\rho b_\rho'$ of Refs. 3) and 4).

The field equations then become

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R-B^{\mu\nu}+\alpha b^\rho b_\rho g^{\mu\nu}=\kappa T^{\mu\nu}, \tag{11}$$

$$\partial_\mu(\sqrt{-g}g^{\mu\nu})-2\alpha\sqrt{-g}b^\nu=0\,, \tag{12}$$

$$g^{\mu\nu}\partial_\mu\partial_\nu c^\rho+2\alpha b^\nu\partial_\nu c^\rho=0\,, \tag{13}$$

$$g^{\mu\nu}\partial_\mu\partial_\nu\bar{c}_\rho+2\alpha b^\nu\partial_\nu\bar{c}_\rho=0 \tag{14}$$

in the same notation as in Ref. 1). The covariant derivative of (11) becomes

$$g^{\mu\nu}\partial_\mu\partial_\nu b_\rho+2\alpha b^\nu\partial_\nu b_\rho=0\,. \tag{15}$$

The BRS current $J_b{}^\mu$ and the F-P ghost one $J_c{}^\mu$ remain *unchanged.* It is natural that they are independent of α.

The asymptotic-field Lagrangian density $\mathcal{L}^{\rm asym}$ acquires a term $-\alpha\beta^\rho\beta_\rho$, and the asymptotic-field equations become

$$\Box\varphi_{\mu\nu}-(1-\alpha)(\partial_\mu\beta_\nu+\partial_\nu\beta_\mu)=0\,, \tag{16}$$

$$\partial^\mu\varphi_{\mu\nu}-\tfrac{1}{2}\partial_\nu\varphi_\mu{}^\mu+\alpha\beta_\nu=0\,, \tag{17}$$

$$\Box\beta_\rho=0,\ \Box\gamma^\rho=0,\ \Box\bar{\gamma}_\rho=0\,. \tag{18}$$

The four-dimensional commutation relations are simply obtained from those in the Landau gauge[1] by replacing $E(x-y)$ by $(1-\alpha)E(x-y)$,[6] just as in quantum electrodynamics. Then the proof of the unitarity of the physical S-matrix remains unchanged.

Thus our formalism based on $\boldsymbol{\delta}$ is much simpler than the one based on $\boldsymbol{\delta}'$.

It was quite beneficial to the present

*) $\boldsymbol{\delta}'(\tilde{g}^{\mu\nu})=-\kappa[\partial_\lambda c^\mu\cdot\tilde{g}^{\lambda\nu}+\partial_\lambda c^\nu\cdot\tilde{g}^{\mu\lambda}-\partial_\lambda(c^\lambda\tilde{g}^{\mu\nu})]$.

author that he could communicate with the authors of Refs. 3) and 4) prior to making their preprints.

1) N. Nakanishi, Prog. Theor. Phys. **59** (1978), 972. See also N. Nakanishi, preprint RIMS-240.
2) T. Kugo and I. Ojima, preprint KUNS-420.
3) K. Nishijima and M. Okawa, preprint (Tokyo Univ.) UT-301.
4) T. Kugo and I. Ojima, in preparation.
5) R. Delbourgo and M. R. Medrano, Nucl. Phys. **B110** (1976), 476.
K. S. Stelle, Phys. Rev. **D16** (1977), 953.
6) T. Kimura, Prog. Theor. Phys. **55** (1976), 1259.