# Remarks on the Integral Form of $\mathrm{D}=11$ Supergravity 

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#### Abstract

We make some considerations and remarks on $D=11$ supergravity and its integral form. We start from the geometrical formulation of supergravity and by means of the integral form technique we provide a superspace action that reproduces (at the quadratic level) the recent formulation of supergravity in pure spinor framework. We also make some remarks on Chevalley-Eilenberg cocycles and their Hodge duals.


[^0]
## 1 Introduction

The integral form of supergravity has been introduced in $[1,2,3]$ to provide an action principle for supersymmetric theories in the geometric formulation [10]. It has been shown that any superspace Lagrangian for the same theory can be reached by a suitable choice of the super embedding of the bosonic submanifold into a supermanifold or, in alternative way, the geometric formulation provides an interpolating model between all possible superspace realisations of the same theory. In the case of rigid supersymmetry, it has been shown in [4] how to choose the superembedding via the Picture Changing Operator to reproduce all possible superspace versions of the same theory. In the case of curved dynamical geometry, there are further complications that should be taken into account in order to provide an action principle (simple examples are discussed in [3, 5]). Besides the technical construction of different possible actions, the integral form of supergravity has the conceptual strength to clarify the difference between on-shell superspace, the absence of auxiliary fields, and the component formulation with dynamical equations of motion. A further aim is to translating the geometric understanding at the quantum level for the functional approach to quantum field theories and a solid variational principle is an important basic ingredient. The present note is not intended to solve these problems, but to collect some preliminary remarks and considerations regarding the application of the integral form of supergravity to the $\mathrm{D}=11$ model $[6,7,11]$. Some concerns regarding the action principle are given in sec. 1 , where it is pointed out that the equations of motion cannot emerge from a naive action principle, but they have to be supplemented by additional pieces of information since, in the geometric formulation, the geometric (a.k.a. rheonomic) Lagrangian is not closed. This is a narrow bottleneck that might prevent any simple solution to the problem. In sec. 2 we recall some basic formulas for the super volume and the geometry of curved supermanifolds. In sec. 3 the geometric formulation $[10,11,12]$ of $\mathrm{D}=11$ supergravity is recollected. In sec. 4, we provide a bridge between $\mathrm{D}=11$ supergravity obtained from pure spinor formulation of the super membrane $[15,16]$ and the integral form of supergravity. Notice that only by means of the action principle we can definitely compare the two frameworks. It has been noticed already in the case of $\mathrm{D}=10$ super Yang-Mills, see [17], the bridge between the two formalisms, but not in the context of dynamical supermanifolds. We hope that the present ideas might serve to understand better the recent developments [18] in the context of supergravity. Finally, in sec. 6, we describe an application of the Hodge duality on supermanifolds $[19,20]$ on the $\mathrm{D}=11$ cocycles which might provide a further tool to study the hidden symmetries as advocated in [11] and in the recent works [13, 14].

## 2 Few Remarks on Supergravity Action with PCO

Given the (11|0)-superform Lagrangian $\mathcal{L}^{(11 \mid 0)} \in \Omega^{(11 \mid 0)}\left(\mathcal{M}^{(11 \mid 32)}\right)$, the corresponding action on the entire supermanifold $\mathcal{M}^{(11 \mid 32)}$ is obtained in [10] by choosing an embedding
$i: \mathcal{M}^{(11)} \rightarrow \mathcal{M}^{(11 \mid 32)}$ and defining the integral

$$
\begin{equation*}
S=\int_{\mathcal{M}^{(11)} \hookrightarrow \mathcal{M}^{(11 \mid 32)}} i^{*} \mathcal{L}^{(11 \mid 0)} \tag{2.1}
\end{equation*}
$$

where $i^{*} \mathcal{L}^{(11 \mid 0)}$ is the pull-back on $\mathcal{M}^{(11)}$ of the full-superspace Lagrangian $\mathcal{L}^{(11 \mid 0)}$.
In the framework of integral forms we lift the Lagrangian to a top form $\mathcal{M}^{(11 \mid 32)}$ by means of a Picture Changing Operator (PCO) $\mathbb{Y}^{(0 \mid 32)}$ which is the Poincaré dual of the embedding and it can be realized as a multiplicative operator on the space of form $\Omega^{(p \mid q)}\left(\mathcal{M}^{(11 \mid 32)}\right)$. Using the usual technique in differential geometry we can rewrite $(2.1)$ as

$$
\begin{equation*}
S=\int_{\mathcal{M}^{(11 \mid 32)}} \mathcal{L}^{(11 \mid 0)} \wedge \mathbb{Y}^{(0 \mid 32)} \tag{2.2}
\end{equation*}
$$

where $\mathcal{L}^{(11 \mid 0)}$ is the geometric Lagrangian and $\mathbb{Y}^{(0 \mid 32)}$ depends upon the embedding. For example, associated to the trivial embedding, we have

$$
\begin{equation*}
\mathbb{Y}_{\text {s.t. }}^{(0 \mid 32)}=\theta^{1} \ldots \theta^{32} \delta\left(d \theta^{1}\right) \wedge \ldots \wedge \delta\left(d \theta^{32}\right) \tag{2.3}
\end{equation*}
$$

and the corresponding action $S=\int_{\mathcal{M}^{(11 \mid 32)}} \mathcal{L}^{(11 \mid 0)} \wedge \mathbb{Y}_{\text {s.t. }}^{(0 \mid 32)}=\int_{\mathcal{M}^{(11)}} \mathcal{L}^{\text {s.t. }}$ where $\mathcal{L}^{\text {s.t. }}$ is the space-time Lagrangian. $\mathbb{Y}^{(0 \mid 32)}$ is an element of the cohomology $H^{(0 \mid 32)}\left(\mathcal{M}^{(11 \mid 32)}, d\right)$. Changing the representative corresponds to the choice of different embeddings of the bosonic submanifold and it changes by $d$-exact terms: $\mathbb{Y}^{(0 \mid 16)} \mapsto \mathbb{Y}^{(0 \mid 32)}+d \Sigma^{(-1 \mid 32)}$ where we consider negative-degree integral forms (see, e.g., Appendix).

In general, the action will be independent of the choice of a representative if $\mathcal{L}^{(11 \mid 0)}$ is closed:

$$
\begin{align*}
S^{\prime} & =\int_{\mathcal{M}^{(11 \mid 32)}} \mathcal{L}^{(11 \mid 0)} \wedge \mathbb{Y}^{\prime(0 \mid 32)}=\int_{\mathcal{M}^{(11 \mid 32)}} \mathcal{L}^{(11 \mid 0)} \wedge\left(\mathbb{Y}^{(0 \mid 32)}+d \Sigma^{(-1 \mid 32)}\right) \\
& =\int_{\mathcal{M}^{(11 \mid 32)}} \mathcal{L}^{(11 \mid 0)} \wedge \mathbb{Y}^{(0 \mid 32)}-\int_{\mathcal{M}^{(11 \mid 32)}} d \mathcal{L}^{11 \mid 0)} \wedge \Sigma^{(-1 \mid 32)}+\text { b.t. } \\
& =S-\int_{\mathcal{M}^{11 \mid 32)}} d \mathcal{L}^{(11 \mid 0)} \wedge \Sigma^{(-1 \mid 32)}+\text { b.t. } \tag{2.4}
\end{align*}
$$

where with "b.t." we denote boundary terms. However, the closure of the Lagrangian is guaranteed only in a few known cases, in particular, when it is possible to add auxiliary fields that guarantee off-shell invariance of the Lagrangian. For $D=11$ supergravity, $d \mathcal{L}^{(11 \mid 0)} \neq 0$ and therefore we are not authorised to change the PCO $\mathbb{Y}^{(0 \mid 32)}$.

The Euler-Lagrange equations derived from the Lagrangian $\mathcal{L}^{(11 \mid 0)}$ do not coincide with the equations of motion coming from a variational principle of an action, as they have of take into account the PCO $\mathbb{Y}^{(0 \mid 32)}$. Varying with respect to any field $\phi$ of the theory we
get

$$
\begin{equation*}
\delta_{\phi} S=0 \Longrightarrow \int_{\mathcal{M}^{(11 \mid 32)}} \delta_{\phi} \mathcal{L}^{(11 \mid 0)}(\phi) \wedge \mathbb{Y}^{(0 \mid 32)}+\mathcal{L}^{(11 \mid 0)}(\phi) \wedge \delta_{\phi} \mathbb{Y}^{(0 \mid 32)}=0 \tag{2.5}
\end{equation*}
$$

Then, by using the requirement that $\mathbb{Y}^{(0 \mid 32)}$ is a representative of the cohomology and if the variation $\delta_{\phi} \mathbb{Y}^{(0 \mid 32)}$ can be expressed as a Lie derivative (for example by changing the embedding of the sub manifold into the supermanifold), it follows $\delta_{\phi} \mathbb{Y}{ }^{(0 \mid 32)}=d \mathbb{J}^{(-1 \mid 32)}$. Thus, by integration by parts and neglecting boundary terms, we get

$$
\begin{equation*}
\delta_{\phi} \mathcal{L}^{(11 \mid 0)}(\phi) \wedge \mathbb{Y}^{(0 \mid 32)}+d \mathcal{L}^{(11 \mid 0)} \wedge \mathbb{J}^{(-1 \mid 32)}=0 \tag{2.6}
\end{equation*}
$$

Different choices of PCO reflect different superspace Lagrangian with different amount of manifest isometries when $d \mathcal{L}^{(11 \mid 0)} \neq 0$ and the superspace Euler-Lagrangian equations $\delta_{\phi} \mathcal{L}^{(11 \mid 0)}=0$, as discussed in [10], can only be derived from $S$ if $d \mathcal{L}^{(11 \mid 0)}=0$ and if $\mathbb{Y}^{(0 \mid 32)}$ has no kernel. For a generic variation, we have to take into account also the variation of the PCO and the true equations of motion are

$$
\begin{equation*}
\delta_{\phi} \mathcal{L}^{(11 \mid 0)}(\phi) \wedge \mathbb{Y}^{(0 \mid 32)}+\mathcal{L}^{(11 \mid 0)}(\phi) \wedge \delta_{\phi} \mathbb{Y}^{(0 \mid 32)}=0 \tag{2.7}
\end{equation*}
$$

If the $\mathrm{PCO} \mathbb{Y}^{(0 \mid 32)}$ is independent from any supergravity field $\phi$, the second term drops and we are left with the equations

$$
\begin{equation*}
\delta_{\phi} \mathcal{L}^{(11 \mid 0)}(\phi) \wedge \mathbb{Y}^{(0 \mid 32)}=0 \tag{2.8}
\end{equation*}
$$

which are not equivalent to $\delta_{\phi} \mathcal{L}^{(11 \mid 0)}(\phi)=0$ unless $\mathbb{Y}{ }^{(0 \mid 32)}$ has no kernel. Let us consider the choice (2.3). It is evident that any form $\omega(\theta, d \theta)$ which is at least linear in $\theta$ and in $d \theta$ is in its kernel. However, there is another representative

$$
\begin{equation*}
\mathbb{Y}_{n o-\theta-k e r}^{(0 \mid 32)}=\left(1+\prod_{\alpha=1}^{32} \theta^{\alpha}\right) \prod_{\alpha=1}^{32} \delta\left(d \theta^{\alpha}\right)=\mathbb{Y}_{\text {s.t. }}^{(0 \mid 32)}+d\left(-\frac{1}{32} \theta^{\beta} \iota_{\beta} \prod_{\alpha=1}^{32} \delta\left(d \theta^{\alpha}\right)\right) \tag{2.9}
\end{equation*}
$$

where $\iota_{\alpha}=\partial / \partial_{d \theta^{\alpha}}$ is the contraction along the odd vector field $\partial_{\alpha} . \mathbb{Y}_{n o-\theta-k e r}^{(0 \mid 32)}$ has a smaller kernel since only linear functions in $d \theta$ are in its kernel. In order to provide a no-kernel PCO, we should add further contractions $\iota_{\alpha_{1}} \ldots \iota_{\alpha_{k}}$ along $k$ odd vector fields, but that decreases the form number to $-k$. That can be compensated by factors of forms $d x^{a_{1}} \wedge \cdots \wedge d x^{a_{k}}$ which have a non trivial kernel. Therefore, it appears rather difficult to have a fieldindependent PCO $\mathbb{Y}^{(0 \mid 32)}$ which has no kernel to justify the equations $\delta_{\phi} \mathcal{L}^{(11 \mid 0)}(\phi)=0$. Of course, the latter implies (2.8), but not vice-versa. The study of the kernel of $\mathbb{Y}(0 \mid 32)$ is anyway important to understand the supergravity equations of motion.

There is a further possibility $[33,25,26]$. The PCO (2.3) can be rewritten as (up to a
unessential coefficient \#)

$$
\begin{equation*}
\mathbb{Y}^{(0 \mid 32)}=\# \int\left[d^{32} w d^{32} p\right] e^{i p_{\alpha} \theta^{\alpha}+i w_{\alpha} d \theta^{\alpha}} \tag{2.10}
\end{equation*}
$$

where we add some auxiliary variables $w_{\alpha}$ and $p_{\alpha}$, which are commuting and anticommuting, respectively. Integrating on the latter we retrieve the original PCO (2.3), but before integrating this expression has no kernel and it might serve to define the superspace equations of motion. Checking that $d \mathbb{Y}^{(0 \mid 32)}=0$ is easy. However, the introduction of the new set of coordinates $w_{\alpha}$ and $p_{\alpha}$ might introduce new degrees of freedom since the fields $\phi$ and the Lagrangian could depend upon them. To avoid this, in [33, 25, 26] the differential is changed into $d \mapsto d+p_{\alpha} \partial_{w_{\alpha}}$ such that $d w_{\alpha}=p_{\alpha}$ and $d p_{\alpha}=0$, which is equivalent to say that the $d$-cohomology does not depend on them. Therefore, the Lagrangian would be independent of them if it would be an element of the cohomology, namely $d \mathcal{L}^{11 \mid 0}=0$ and $\mathcal{L}^{11 \mid 0} \neq d \Sigma$, but that it is not the case and therefore also this path is obstructed.

## 3 Super-Vielbeins and the Super-Volume Form

We consider $D=11 N=1$ superspace parametrized by the coordinates $\left(x^{m}, \theta^{\mu}\right)$ with $m=0, \ldots, 10$ and $\mu=1, \ldots, 32$. We use the Majorana representation for the spinors $\theta^{\mu}$ using real coordinates. We use $a=0, \ldots, 10$ and $\alpha=1, \ldots, 32$, for flat indices of the target space with the supercharges defined as $Q_{\alpha}=\partial_{\alpha}+\left(\Gamma^{a} \theta\right)_{\alpha} \partial_{a}$ and $P_{a}=\partial_{a}$. The $\Gamma^{a}$ are 11-d Dirac matrices with $\left\{\Gamma^{a}, \Gamma^{b}\right\}=2 \eta^{a b} . \eta^{a b}$ is the flat metric on the tangent space, the signature is $(+,-, \ldots,-)$. The relevant Fierz identities are $\left(\bar{\psi} \Gamma^{a b} \psi\right)\left(\bar{\psi} \Gamma_{a} \psi\right)=0$ and the bi-spinor decomposition is

$$
\begin{equation*}
\psi_{\wedge} \bar{\psi}=\frac{1}{32}\left(\Gamma_{a}\left(\bar{\psi} \Gamma^{a} \psi\right)-\frac{1}{2} \Gamma_{a b}\left(\bar{\psi} \Gamma^{a b} \psi\right)+\frac{1}{5!} \Gamma_{a_{1} \ldots a_{5}}\left(\bar{\psi} \Gamma^{a_{1} \ldots a_{5}} \psi\right)\right) \tag{3.1}
\end{equation*}
$$

The supervielbeins $V^{a}$ and $\psi^{\alpha}$ are (1|0)-superforms with values in the tangent space and they can be decomposed on the basis $\left(d x^{m}, d \theta^{\alpha}\right)$ as

$$
\begin{equation*}
V^{a}=E_{m}^{a} d x^{m}+E_{\mu}^{a} d \theta^{\mu}, \quad \psi^{\alpha}=E_{m}^{\alpha} d x^{m}+E_{\mu}^{\alpha} d \theta^{\mu} \tag{3.2}
\end{equation*}
$$

We can write

$$
\begin{equation*}
T^{a}=\mathcal{D} V^{a}-\bar{\psi}^{\alpha} \Gamma_{\alpha \beta}^{a} \psi^{\beta}, \quad \rho^{\alpha}=\mathcal{D} \psi^{\alpha} . \tag{3.3}
\end{equation*}
$$

with $T^{a}, \rho^{\alpha}$ the vectorial and spinorial parts of the super torsion, respectively. The covariant derivatives are defined as $\mathcal{D} V^{a}=d V^{a}+\varpi^{a}{ }_{b} V^{b}$ and $\mathcal{D} \psi^{\alpha}=d \psi^{\alpha}+\frac{1}{4} \varpi_{a b} \Gamma_{\beta}^{a b, \alpha} \psi^{\beta}$ where $\varpi^{a b}$ is the spin connection. The equations $\varpi_{\beta}^{\alpha}=\frac{1}{4}\left(\Gamma^{a b}\right)^{\alpha}{ }_{\beta} \varpi_{a b}$ relate the spinorial representation with the vector representation. If we set $T^{a}=\rho^{\alpha}=0$ in order to fix the spin
connection $\varpi_{a b}$ in terms of $\left(V^{a}, \psi^{\alpha}\right)$. In general, for non-vanishing dynamical supergravity fields, we cannot also set $\rho^{\alpha}=0$. The supermatrix

$$
\mathbb{E}=\left(\begin{array}{ll}
E_{m}^{a} & E_{\mu}^{a}  \tag{3.4}\\
E_{m}^{\alpha} & E_{\mu}^{\alpha}
\end{array}\right)
$$

is the well-known supervielbein appearing in supegravity. In the case of flat space, it reads

$$
\begin{equation*}
\mathbb{E}_{\text {flat }} \equiv\left(V^{a}, \psi^{\alpha}\right)=\left(d x^{a}+\bar{\theta} \Gamma^{a} d \theta, d \theta^{\alpha}\right), \tag{3.5}
\end{equation*}
$$

and the fluctuations around $\mathbb{E}_{\text {flat }}$ in (3.5) are identified with the dynamical vielbein and gravitino. Given a supermatrix, we can define the invariant superfields

$$
\begin{equation*}
\operatorname{Ber}(\mathbb{E})=\frac{\operatorname{det}\left(E_{m}^{a}-E_{\mu}^{a}\left(E^{-1}\right)^{\mu}{ }_{\alpha} E_{m}^{\alpha}\right)}{\operatorname{det} E_{\mu}^{\alpha}}=\frac{\operatorname{det} E_{m}^{a}}{\operatorname{det}\left(E_{\mu}^{\alpha}-E_{m}^{\alpha}\left(E^{-1}\right)^{m}{ }_{a}^{a} E_{\mu}\right)}, \tag{3.6}
\end{equation*}
$$

which are the two equivalent expressions of the super determinant, well-defined when $\operatorname{det} E_{m}^{a} \neq 0$ and $\operatorname{det} E_{\mu}^{\alpha} \neq 0$.

The volume form is the integral form with form number 11 and picture number 32 (see the appendix for an introduction to integral forms and related notations):

$$
\begin{equation*}
\operatorname{Vol}^{(11 \mid 32)}=\epsilon^{a_{1} \ldots a_{11}} \delta\left(V^{a_{1}}\right)_{\wedge \ldots \wedge} \delta\left(V^{a_{11}}\right) \epsilon^{\alpha_{1} \ldots \alpha_{32}} \delta\left(\psi^{\alpha_{1}}\right) \wedge \cdots \wedge \delta\left(\psi^{\alpha_{32}}\right), \tag{3.7}
\end{equation*}
$$

which corresponds to a top form in supergeometry. It is closed, $d \mathrm{Vol}^{(11 \mid 32)}=\nabla \mathrm{Vol}^{(11 \mid 32)}=$ 0 , and the non-exactness depends on the (11|32) supermanifold on which it is defined (for example, if the supermanifold is compact, the top form is non-exact). The cohomological properties can be easily checked by applying the differential $d$, the Leibniz rule for $\nabla$ acting on the supervielbeins $V^{a}, \psi^{\alpha}$ and the distributional identity $\psi^{\alpha} \delta\left(\psi^{\alpha}\right)=0$.

In addition, the volume form $\operatorname{Vol}^{(11 \mid 32)}$ is invariant with respect to Lorentz transformations

$$
\begin{equation*}
\delta V^{a}=\Lambda_{b}^{a} V^{b}, \quad \delta \psi^{\alpha}=\frac{1}{4} \Lambda_{a b}\left(\Gamma^{a b}\right)_{\beta}^{\alpha} \psi^{\beta} \tag{3.8}
\end{equation*}
$$

then, more precisely, it belongs to the equivariant cohomology.
Since the $V^{a}$ 's are anticommuting (bosonic 1-forms), we can replace the first delta's with their arguments

$$
\begin{equation*}
\operatorname{Vol}^{(11 \mid 32)}=\epsilon_{a_{1} \ldots a_{11}} V_{\wedge}^{a_{1}} \ldots \wedge V^{a_{11}} \epsilon^{\alpha_{1} \ldots \alpha_{32}} \delta\left(\psi^{\alpha_{1}}\right) \wedge \ldots \wedge \delta\left(\psi^{\alpha_{32}}\right) . \tag{3.9}
\end{equation*}
$$

The same cannot be done for $\psi^{\alpha}$ 's, since they are commuting. Notice that the $\delta\left(\psi^{\alpha}\right)$ does not transform as a tensor with respect to change of parametrization and therefore the index $\alpha$ is not a conventional covariant index summed with the Levi-Civita tensor
$\epsilon^{\alpha_{1} \ldots \alpha_{32}}$; the latter serves only to keep track of the order of the deltas, because of their anticommutation relations $\delta\left(\psi^{\alpha}\right)_{\wedge} \delta\left(\psi^{\beta}\right)=-\delta\left(\psi^{\beta}\right)_{\wedge} \delta\left(\psi^{\alpha}\right)$. Nonetheless, expression (3.9) is invariant under reparametrizations. One has to pay some attention to deal with those "covariant" expressions.

Inserting expressions (3.2) into (3.9), using the properties of the oriented delta's and of the 1 -forms, we get

$$
\begin{equation*}
\operatorname{Vol}^{(11 \mid 32)}=\operatorname{Ber}(\mathbb{E}) \epsilon_{m_{1} \ldots m_{11}} d x_{\wedge}^{m_{1}} \ldots \wedge d x^{m_{11}} \epsilon^{\mu_{1} \ldots \mu_{32}} \delta\left(d \theta^{\mu_{1}}\right)_{\wedge \ldots \wedge} \delta\left(d \theta^{\mu_{32}}\right), \tag{3.10}
\end{equation*}
$$

where the overall factor is the superdeterminant of $\mathbb{E}(3.7)$, as expected. Finally,

$$
\begin{equation*}
\int_{S M^{(11 \mid 32)}} \operatorname{Vol}^{(11 \mid 32)}=\int \operatorname{Ber}(\mathbb{E})\left[d^{11} x d^{32} \theta\right] \tag{3.11}
\end{equation*}
$$

gives the volume of the supermanifold $S M^{(11 \mid 32)}$.

## $4 \mathrm{D}=11$ Supergravity in the Geometric Framework

In order to deal with $\mathrm{D}=11$ supergravity, we refer to standard literature for the action and the details (see for example $[6,7,10]$ ), but we adopt the notations and the definitions given in [10]. The physical degrees of freedom are $V^{a}, \omega^{a b}, \psi^{\alpha}, A^{(3 \mid 0)}$ the first three fields are ( $1 \mid 0$ )-forms the last one is a $(3 \mid 0)$. In addition, we add also the $(6 \mid 0)$-form $B^{(6 \mid 0)}$ by consistency.

The corresponding curvature are given by

$$
\begin{align*}
R^{a b} & =d \omega^{a b}-\omega^{a c} \omega_{c}{ }^{b} \\
T^{a} & =\mathcal{D} V^{a}-\frac{i}{2} \bar{\psi} \Gamma^{a} \psi=d V^{a}-\omega^{a}{ }_{b} V^{b}-\frac{i}{2} \bar{\psi} \Gamma^{a} \psi \\
\rho & =\mathcal{D} \psi=d \psi-\frac{1}{4} \omega_{a b} \Gamma^{a b} \psi \\
F & =d A-\frac{1}{2} \bar{\psi} \Gamma_{a b} \psi V^{a} V^{b} \\
H & =d B-\frac{i}{2} \bar{\psi} \Gamma_{a_{1} \ldots a_{5}} \psi V^{a_{1}} \ldots V^{a 5}-\frac{15}{2} \bar{\psi} \Gamma_{a b} \psi V^{a} V^{b} \wedge A-15 F \wedge A \tag{4.1}
\end{align*}
$$

where $R^{a b}$ is the curvature of the connection, $T^{a}$ is the torsion, $\rho^{\alpha}$ is the torsion of the fermionic component of the supervielbein which is also identified with the curvature of the gravitino. $F$ is the 4 -form field strength of the 3 -form and $H$ is the field strength of the

6 -form $B$. In the flat space, namely, when all curvatures are zero, we see that

$$
\begin{align*}
\omega^{(4 \mid 0)} & =\frac{1}{2}\left(\bar{\psi} \Gamma_{a b} \psi\right) V^{a} V^{b}, \quad d \omega^{(4 \mid 0)}=0 . \\
d A^{(3 \mid 0)} & =\omega^{(4 \mid 0)}, \\
\omega^{(7 \mid 0)} & =\frac{i}{2}\left(\bar{\psi} \Gamma_{a_{1} \ldots a_{5}} \psi\right) V^{a_{1}} \ldots V^{a_{5}}, \quad d \omega^{(7 \mid 0)}=\omega^{(4 \mid 0)} \wedge \omega^{(4 \mid 0)} . \\
d B^{(6 \mid 0)} & =\omega^{(7 \mid 0)}+15 \omega^{(4 \mid 0)} \wedge A \tag{4.2}
\end{align*}
$$

where $\omega^{(4 \mid 0)}$ and $\omega^{(7 \mid 0)}+15 \omega^{(7 \mid 0)} \wedge A$ are two Chevalley-Eilenberg Cohomology class for the differential $d$ in the case of flat space and the field $A^{(3 \mid 0)}$ and $B^{(6 \mid 0)}$ are the potentials for the free differential algebra.

Acting with the differential on the curvatures, one obtains the Bianchi identities

$$
\begin{align*}
& \mathcal{D} R^{a b}=0,  \tag{4.3}\\
& \mathcal{D} T^{a}+R^{a}{ }_{b} V^{b}-i\left(\bar{\psi} \Gamma^{a} \rho\right)=0, \\
& \mathcal{D} \rho+\frac{1}{4} \Gamma_{a b} \psi R^{a b}=0, \\
& d F-\left(\bar{\psi} \Gamma_{a b} \rho\right) V^{a} V^{b}+\left(\bar{\psi} \Gamma_{a b} \psi\right) T^{a} V^{b}=0, \\
& d H-i\left(\bar{\psi} \Gamma_{a_{1} \ldots a_{5}} \rho\right) V^{a_{1}} \ldots V^{a_{5}}-\frac{5 i}{2}\left(\bar{\psi} \Gamma_{a_{1} \ldots a_{5}} \psi\right) T^{a_{1}} \ldots V^{a_{5}}-15\left(\bar{\psi} \Gamma_{a b} \psi\right) V^{a} V^{b} \wedge F-15 F \wedge F,
\end{align*}
$$

which relate the curvature and the supervielbeins $\left(V^{a}, \psi^{\alpha}\right)$.
Using the constraints

$$
\begin{align*}
T^{a} & =0 \\
F & =F_{a_{1} \ldots a_{4}} V^{a_{1}} \ldots V^{a_{4}}, \\
H & =H_{a_{1} \ldots a_{7}} V^{a_{1}} \ldots V^{a_{7}}, \\
\rho^{\alpha} & =\rho_{a b}^{\alpha} V^{a} V^{b}, \\
R^{a b} & =R_{c d}^{a b} V^{c} V^{d}+\bar{\Sigma}_{\alpha c}^{a b} \psi^{\alpha} V^{c}+\psi^{\alpha} K_{\alpha \beta}^{a b} \psi^{\beta} . \tag{4.4}
\end{align*}
$$

where $F_{a_{1} \ldots a_{4}}, H_{a_{1} \ldots a_{7}}, \rho_{a b}^{\alpha}, R_{c d}^{a b}, \Sigma_{\alpha c}^{a b}, K_{\alpha \beta}^{a b}$ are unconstrained superfields. These constraints are very strong and together with the Bianchi identities, they imply the equations of motion for the field components of (4.4). The superfields $\bar{\Sigma}_{\alpha c}^{a b}, K_{\alpha \beta}^{a b}$ are fixed in terms of the other superfields.

In [10] the rheonomic Lagrangian is provided and it reads

$$
\begin{aligned}
\mathcal{L}^{(11 \mid 0)} & =\frac{1}{330} F_{a_{1} \ldots a_{4}} F^{a_{1} \ldots a_{4}} V^{c_{1}} \ldots \wedge V^{c_{11}} \epsilon_{c_{1} \ldots c_{11}} \\
& -\frac{1}{9} R^{a_{1} a_{2}} \wedge V^{a_{3}} \ldots \wedge V^{a_{11}} \epsilon_{a_{1} \ldots a_{11}} \\
& +2\left(\bar{\rho}_{\wedge} \Gamma_{c_{1} \ldots c_{8}} \psi\right)_{\wedge} V^{c_{1}} \ldots \wedge V^{c_{8}} \\
& +\left(\frac{1}{4}\left(\bar{\psi}_{\wedge} \Gamma^{a_{1} a_{2}} \psi\right)_{\wedge}\left(\bar{\psi}_{\wedge} \Gamma^{a_{3} a_{4}} \psi\right)+2 F F^{a_{1} \ldots a_{4}}\right) \wedge V^{a_{5}} \ldots \wedge V^{a_{11}} \epsilon_{a_{1} \ldots a_{11}} \\
& +\frac{7 i}{30} T^{a} \wedge V_{a \wedge}\left(\bar{\psi}_{\wedge} \Gamma^{b_{1} \ldots b_{5}} \psi\right)_{\wedge} V^{b_{6}} \ldots \wedge V^{b_{11}} \epsilon_{b_{1} \ldots b_{11}} \\
& -84 F_{\wedge} \omega^{7}+840 F_{\wedge} A_{\wedge} \omega^{4}-210 A_{\wedge} \omega^{4} \wedge \omega^{4}-840 F_{\wedge} F_{\wedge} A
\end{aligned}
$$

The fourth line will be absent if we set $T^{a}=0$; this single constraint is not sufficient to put the theory on-shell. Note that the Lagrangian being a superform can be expanded into $V^{a}$ and $\psi^{\alpha}$, but we can select those terms which are explicitly depending on $V^{a}$ ( $\rho, R^{a b}, F, H, T^{a}$ depend implicitly on $V$ 's) as follows

$$
\begin{equation*}
\mathcal{L}^{(11 \mid 0)}=\sum_{k=0}^{11} \mathcal{L}_{a_{1} \ldots a_{k}} V^{a_{1}} \ldots V^{a_{k}} \tag{4.5}
\end{equation*}
$$

where $\mathcal{L}_{a_{1} \ldots a_{k}} \neq 0$ if $k=0,2,4,5,6,7,8,9,11$. If the PCO has several factors of $V$ 's, acting multiplicatively $\mathcal{L}^{(11 \mid 0)} \wedge \mathbb{Y}^{(0 \mid 32)}$ it will kill several terms in the sum (4.5) simplifying the final superspace action. The choice of particular terms has to be motivated by symmetry requirements, for example manifest supersymmetry. In the following section, we provide an example of PCO which has been constructed in pure spinor supermembrane framework which select the term with $k=2$ which is a Chern-Simons-like term $\int A d A \wedge \omega^{4}$.

## 5 PCO, Membranes and the Integral Form of the Action

In the pioneering works [15] and later in [16] a supergravity action is built in the pure spinor formulation (later it has been studied from superparticle point of view in [21] and completed in [22, 23, 24]). Here would like to build a bridge between the pure spinor formulation and the geometric formulation and this can be done using the integral form formalism.

The action of supergravity obtained in [15] is built in terms of a pure spinor superfield $C^{(3)}=\lambda^{\alpha} \lambda^{\beta} \lambda^{\gamma} C_{\alpha \beta \gamma}(x, \theta)$ where $\lambda$ 's are the pure spinor (complex) coordinates that satisfy the quadratic constraints $\lambda \Gamma^{a} \lambda=0 .{ }^{1}$ The components $C_{\alpha \beta \gamma}(x, \theta)$ are defined up to a local

[^1]gauge transformation $\delta C_{\alpha \beta \gamma}(x, \theta)=\Gamma_{(\alpha \beta}^{a} \Sigma_{\beta) a}(x, \theta)$. The second crucial ingredient is the ghost-one BRST charge $Q=\int d \sigma \lambda^{\alpha} \nabla_{\alpha}$ which is nilpotent if $\lambda$ 's satisfy the pure spinor constraints. It has been verified that $Q C^{(3)}=0$ with the symmetries $\delta C^{(3)}=Q \Xi^{(2)}$ implies the linearised supergravity equations of motion. Then, the following action
\[

$$
\begin{equation*}
S_{\text {sugra }}=\int\left[d^{11} x d^{32} \theta D \lambda\right] C^{(3)} Q C^{(3)} \tag{5.1}
\end{equation*}
$$

\]

is a reasonable starting point for the complete supergravity action (see [16] for further developments) where the Lagrangian carries ghost number +7 . The integral is over the bosonic $D=11$ coordinates, the fermionic coordinates, and over the pure spinor coordinates $\lambda^{\alpha}$. However, for them, we need a special measure $[D \lambda]$ to be compatible with the pure spinor constraints and taking into account that the pure spinors are commuting variables. In pure spinor cohomology, there exists the following representative at ghost number +7

$$
\begin{equation*}
\Omega^{(7)}(\lambda, \theta)=\epsilon_{a_{1} \ldots a_{11}} \lambda \Gamma^{a_{1}} \theta \ldots \lambda \Gamma^{a_{7}} \theta \theta \Gamma^{a_{8} a_{9} a_{10} a_{11}} \theta . \tag{5.2}
\end{equation*}
$$

which is a Lorentz singlet and explicitly depends upon 9 ''s. Then, the measure $[D \lambda]=$ $d^{23} \lambda \mu(\lambda, \theta)$ is defined such that

$$
\begin{equation*}
\int\left[d^{32} \theta d^{23} \lambda \mu(\lambda, \theta)\right] \Omega^{(7)}(\lambda, \theta)=1 \tag{5.3}
\end{equation*}
$$

where $d^{32} \theta$ is the conventional Berezin integral and

$$
\begin{equation*}
\mu(\lambda, \theta)=\left(\theta^{23} \epsilon\right)_{\alpha_{1} \ldots \alpha_{9}} T^{\left[\alpha_{1} \ldots \alpha_{9}\right]\left(\beta_{1} \ldots \beta_{7}\right)} \frac{\partial}{\partial \lambda^{\beta_{1}}} \ldots \frac{\partial}{\partial \lambda^{\beta_{7}}} \delta^{23}(\lambda), \tag{5.4}
\end{equation*}
$$

where $T^{\left[\alpha_{1} \ldots \alpha_{9}\right]\left(\beta_{1} \ldots \beta_{7}\right)}=\epsilon^{a_{1} \ldots a_{11}} \Gamma_{a_{1}}^{\alpha_{1} \beta_{1}} \ldots \Gamma_{a_{7}}^{\alpha_{7} \beta_{7}} \Gamma_{a_{8} \ldots a_{11}}^{\alpha_{8} \alpha_{9}}$. Notice that it depends upon the complementary $23 \theta^{\prime}$ 's of $\Omega^{7}$ such that the Berezin integral gives exactly one, the derivatives $\partial / \partial \lambda^{\beta}$ act by integration-by-parts on $\Omega^{(7)}$ and finally the integration $\delta^{23}(\lambda) d^{23} \lambda$ gives one by the conventional definition of Dirac delta distributions. Finally, applying the formula (5.3) to the Lagrangian in (5.1), one selects several pieces reproducing the quadratic part of the supergravity action.

Still working on flat space, we define the new PCO

$$
\begin{equation*}
\mathbb{Y}^{(0 \mid 32)}=\epsilon_{\alpha_{1} \ldots \alpha_{32}} \theta^{\alpha_{1}} \ldots \theta^{\alpha_{23}}\left(V_{a_{1}} \Gamma^{a_{1}} \iota\right)^{\alpha_{24}} \ldots\left(V_{a_{9}} \Gamma^{a_{9}} \iota\right)^{\alpha_{32}} \delta^{32}(\psi) \tag{5.5}
\end{equation*}
$$

which is equivalent to the $\mathbb{Y}_{\text {s.t. }}^{(0 \mid 32)}$ since in the flat space $V^{a}=d x^{a}+\bar{\theta} \Gamma^{a} \psi$ and by integration-by-part, we see that differs from $\mathbb{Y}_{\text {s.t. }}^{(0 \mid 32)}$ by exact terms. Note that the form degree is zero since the form degree of $V$ 's compensate the form degree of $\iota_{\alpha}$. Now, we observe that if we replace $\lambda$ with $\psi$ 's in (5.2), namely $\Omega^{(7)}(\lambda, \theta) \mapsto \Omega^{(7 \mid 0)}(\psi, \theta)$ we get a 7 -form, but clearly it will be not closed since the $\psi$ are not pure spinors. Nonetheless, given the Chevalley-

Eilenberg cohomology $\omega^{(4 \mid 0)}$ discussed in the previous section we have

$$
\begin{equation*}
\mathbb{Y}^{(0 \mid 32)} \wedge \omega^{(4 \mid 0)}=\mu^{(-7 \mid 0)} \operatorname{Vol}^{(11 \mid 32)} \tag{5.6}
\end{equation*}
$$

where $\operatorname{Vol}^{(11 \mid 32)}$ is the super volume of the manifold and $\mu^{(-7 \mid 0)}$ is given by

$$
\begin{equation*}
\mu^{(-7 \mid 0)}=\left(\theta^{23} \epsilon\right)_{\alpha_{1} \ldots \alpha_{9}} T^{\left[\alpha_{1} \ldots \alpha_{9}\right]\left(\beta_{1} \ldots \beta_{7}\right)} \iota_{\beta_{1}} \ldots \iota_{\beta_{7}} \tag{5.7}
\end{equation*}
$$

where we replaced the derivatives w.r.t. $\lambda$ with the contraction along the odd vector fields $\nabla_{\alpha}$. The structure is exactly the same as constructed in (5.4) and finally inserting the PCO $\mathbb{Y}^{(0 \mid 32)}$ in the action, it selects the Lagrangian which is dual to $\Omega^{(7 \mid 0)}(\psi, \theta)$ as the pure spinor measure $\mu(\lambda, \theta)$ is dual to the cohomology class $\Omega^{(7)}(\lambda, \theta)$. Therefore, instead of using the pure spinor cohomology class, which we do not have in our framework, we use the Chevalley-Eilenberg cohomology $\omega^{(4 \mid 0)}$ which is well-defined and it plays a crucial role in the $\mathrm{D}=11$ supergravity construction.

Then, plugging the PCO $\mathbb{Y}^{(0 \mid 32)}$ we select only two terms. The other drop out because of the number of $V$ 's at the first order.
$\int_{\mathcal{S M}^{(11 \mid 32)}} \mathcal{L}^{(11 \mid 0)} \wedge \mathbb{Y}^{(0 \mid 32)} \mapsto \int_{\mathcal{S M}^{(11 \mid 32)}}\left(F \wedge A_{\wedge}\left(\bar{\psi}_{\wedge} \Gamma_{a b} \psi\right)_{\wedge} V^{a}{ }_{\wedge} V^{b}-F_{\wedge} F_{\wedge} A\right) \wedge \mathbb{Y}^{(0 \mid 32)}(5.8)$
Notice that the result is not the complete answer, but it gives only an indication that the supergravity action in the geometric formulation in (4.5) contains a superspace action similar to pure spinor formulation (5.1). In order to get the full result one needs to convert $\mathbb{Y}^{(0 \mid 32)}$ to a curved one where the $V^{a}$ and $\psi^{\alpha}$ in (5.5) are the dynamical fields. This program will be tackled in subsequent publications and here we only discuss a first step toward the complete construction.

A curved PCO is constructed as follows (see [32]). We introduce the super-Euler vector as follows

$$
\begin{align*}
& \mathcal{E}=\theta^{\mu} \partial_{\mu}+f^{m}(x, \theta) \partial_{m}=X^{a} \nabla_{a}+\Theta^{\alpha} \nabla_{\alpha}, \\
& \Theta^{\alpha}=\theta^{\mu} E_{\mu}^{\alpha}+f^{m}(x, \theta) E_{m}^{\alpha}, \quad X^{a}=\theta^{\mu} E_{\mu}^{a}+f^{m}(x, \theta) E_{m}^{a}, \tag{5.9}
\end{align*}
$$

where the combinations $\left(X^{a}, \Theta^{\alpha}\right)$ are the new curved coordinates with flat indices (see [9]). We can set $X^{a}=0$ by choosing the function $f^{m}(x, \theta)=E_{\alpha}^{m} \Theta^{\alpha}$ yielding

$$
\begin{equation*}
\iota_{\mathcal{E}} \psi^{\alpha}=\Theta^{\alpha}, \quad \iota_{\mathcal{E}} V^{a}=0 . \tag{5.10}
\end{equation*}
$$

Applying the covariant differential $\nabla$ on $\Theta^{\alpha}$, we get

$$
\begin{equation*}
\nabla \Theta^{\alpha}=\nabla \iota_{\mathcal{E}} \psi^{\alpha}=\psi^{\alpha}-\iota_{\mathcal{E}} \rho^{\alpha}+\Omega^{\alpha}{ }_{\beta} \psi^{\beta} \quad \Longrightarrow \quad \psi^{\alpha}=\left[(1+\Omega)^{-1}\right]_{\beta}^{\alpha}\left(\nabla \Theta^{\beta}+\iota_{\mathcal{E}} \rho^{\beta}\right), \tag{5.11}
\end{equation*}
$$

where $\rho^{\alpha}$ is the spinorial component of the supertorsion (field strength of the gravitino)
and $\Omega^{\alpha}{ }_{\beta}=\iota_{\mathcal{E}} \varpi_{a b}\left(\Gamma^{a b}\right)^{\alpha}{ }_{\beta}$ is a gauge parameter built in terms of the spin connection $\varpi$. In general, $\iota_{\mathcal{E}} \rho^{\alpha}$ does not vanish for dynamical supergravity fields and $\Omega^{\alpha}{ }_{\beta}=\varpi_{a b, \rho} \iota_{\mathcal{E}} E^{\rho}\left(\Gamma^{a b}\right)^{\alpha}{ }_{\beta}$. Notice that $\Omega^{\alpha}{ }_{\beta}$ is proportional to $\Theta$, hence it could be dropped in the expression of the curved PCO because of the product of the four $\Theta$ 's in front of the delta's. By using the expression in (5.11), the curved PCO reads

$$
\begin{equation*}
\mathbb{Y}_{\text {curved }}^{(0 \mid 32)}=\prod_{\alpha=1}^{32} \Theta^{\alpha} \delta\left(\nabla \Theta^{\alpha}\right)=\prod_{\alpha=1}^{32} \iota_{\mathcal{E}} \psi^{\alpha} \delta\left((1+\Omega)^{\alpha}{ }_{\beta} \psi^{\beta}-\iota_{\mathcal{E}} T^{\alpha}\right) ; \tag{5.12}
\end{equation*}
$$

it is closed because $\nabla^{2} \Theta^{\alpha}=R_{a b}\left(\Gamma^{a b}\right)^{\alpha} \Theta^{\beta}$, since the indices on the $\Theta$ 's are the flat Lorentz indices and because of the product of all $\Theta^{\alpha}$ in front of the delta's. Moreover, since $\iota_{\mathcal{E}} T^{\alpha}=\Theta^{\beta} E^{b} T_{\beta b}{ }^{\alpha}$, i.e., it is proportional to $\Theta, \mathbb{Y}^{(0 \mid 11)}$ can be reduced to

$$
\begin{equation*}
\mathbb{Y}^{(0 \mid 32)}=\prod_{\alpha=1}^{32} \Theta^{\alpha} \delta\left(\nabla \Theta^{\alpha}\right)=\prod_{\alpha=1}^{32} \Theta^{\alpha} \delta\left(E^{\alpha}\right) . \tag{5.13}
\end{equation*}
$$

As a last remark, the PCO (5.12) is not manifestly supersymmetric. As for the flat case, we can build the new PCO by introducing some $V$ 's in the game as follows

$$
\begin{equation*}
\mathbb{Y}^{(0 \mid 32)}=\epsilon_{\alpha_{1} \ldots \alpha_{32}} \Theta^{\alpha_{1}} \ldots \Theta^{\alpha_{23}}\left(V_{a_{1}} \Gamma^{a_{1}} \iota\right)^{\alpha_{24}} \ldots\left(V_{a_{9}} \Gamma^{a_{9}} \iota\right)^{\alpha_{32}} \delta^{32}(\psi) \tag{5.14}
\end{equation*}
$$

with the constraint that $T^{a}=0$ must be imposed from the beginning or as a condition for the closure of the PCO (see also [3] for a complete discussion in $\mathrm{D}=3 \mathrm{~N}=1$ supergravity).

## $6 \quad \mathrm{D}=11$ Cocycles and Hodge Duality

In this last section, we make some considerations on the Chevalley-Eilenberg cocycles $\omega^{(4 \mid 0)}$ and $\omega^{(7 \mid 0)}$ written in terms of the gravitinos $\psi^{\alpha}$ and vielbeins $V^{a}$. In particular we construct the Laplace-Beltrami operator and we act on those cocycles to check the Hodge theory (a complete discussion will be provided in a separate publication [30], a discussion in the context of superLie algebras can also be found in [31]).

The cocycles $\omega^{(4 \mid 0)}$ and $\omega^{(7 \mid 0)}$ satisfy the following equations

$$
\begin{equation*}
d \omega_{4}=0, \quad d \omega_{7}=-\frac{1}{2} \omega_{4} \wedge \omega_{4} \tag{6.1}
\end{equation*}
$$

The second equation is a consequence of the Fierz identities

$$
\bar{\psi} \Gamma_{a_{1} \ldots a_{5}} \psi \bar{\psi} \Gamma^{a_{5}} \psi=\bar{\psi} \Gamma_{\left[a_{1} a_{2}\right.} \psi \bar{\psi} \Gamma_{\left.a_{3} a_{4}\right]} \psi
$$

Let us consider the super Hodge dual (defined in [19, 20]) of those superforms

$$
\begin{align*}
& \star \omega_{4}=V^{a_{1}} \ldots V^{a_{9}} \epsilon_{a_{1} \ldots a_{9} b_{1} b_{2}} \bar{\iota} \Gamma^{b_{1} b_{2}}\left\langle\delta^{32}(\psi),\right. \\
& \star \omega_{7}=V^{a_{1}} \ldots V^{a_{6}} \epsilon_{a_{a_{1} \ldots a_{6} b_{1} b_{5}}} \overline{\Gamma^{b_{1} \ldots b_{5}} \iota \delta^{32}(\psi),} \tag{6.2}
\end{align*}
$$

where $\bar{\tau} \Gamma^{b_{1} b_{2}} \iota=\frac{\delta}{\delta \psi} \Gamma^{b_{1} b_{2}} \frac{\delta}{\delta \psi}$ are the derivatives with respect to the argument of the delta functions. Therefore they act by integration by parts. In particular, if we compute the wedge product of $\omega_{4}$ with $\star \omega_{4}$ (and analogously for $\omega_{7}$ ) we get the volume form

$$
\begin{equation*}
\omega_{4} \wedge \star \omega_{4}=V_{1} \ldots V_{11} \delta\left(\psi_{1}\right) \ldots \delta\left(\psi_{32}\right), \quad \omega_{7} \wedge \star \omega_{7}=V_{1} \ldots V_{11} \delta\left(\psi_{1}\right) \ldots \delta\left(\psi_{32}\right) . \tag{6.3}
\end{equation*}
$$

Notice that the first one has degrees (7|32) (due to the presence of 9 vielbeins and two derivatives), while the second one has degree (4|32). Notice that both are closed

$$
\begin{align*}
d \star \omega_{4} & =9\left(\bar{\psi} \Gamma^{a_{1}} \psi\right) V^{a_{2}} \ldots V^{a_{9}} \epsilon_{a_{1} \ldots a_{9} b_{1} b_{2}} \Gamma^{b_{1} b_{2}} \iota \delta^{32}(\psi) \\
& =9 \operatorname{tr}\left(\Gamma^{a_{1}} \Gamma^{b_{1} b_{2}}\right) V^{a_{2}} \ldots V^{a_{9}} \epsilon_{a_{1} \ldots a_{9} b_{1} b_{2}} \delta^{32}(\psi)=0, \\
d \star \omega_{7} & =6 V^{a_{2}} \ldots V^{a_{6}} \epsilon_{a_{1} \ldots a_{6} b_{1} \ldots b_{5}}^{\bar{\iota}} \Gamma^{b_{1} \ldots b_{5}} \iota \delta^{32}(\psi) \\
& =6 \operatorname{tr}\left(\Gamma^{a_{1}} \Gamma^{b_{1} \ldots b_{5}}\right) V^{a_{2}} \ldots V^{a_{9}} \epsilon_{a_{1} \ldots a_{6} b_{1} \ldots b_{5}}^{32} \delta^{32}(\psi)=0 \tag{6.4}
\end{align*}
$$

they vanish because of the trace between the gamma matrices. On the other hand, if we compute the Hodge dual of $d \omega_{7}$, we get

$$
\begin{equation*}
\star d \omega_{7}=-\frac{1}{4} V^{a_{1}} \ldots V^{a_{7}} \epsilon_{a_{1} \ldots a_{7} b_{1} \ldots b_{4}} \bar{\iota} \Gamma^{b_{1} b_{2}} \iota \bar{\iota} \Gamma^{b_{3} b_{4}} \iota \delta^{32}(\psi) \tag{6.5}
\end{equation*}
$$

Using again the Fierz identities, we can recast the derivatives as follows

$$
\begin{equation*}
\star d \omega_{7}=-\frac{1}{4} V^{a_{1}} \ldots V^{a_{7}} \epsilon_{a_{1} \ldots a_{7} b_{1} \ldots b_{4}} \bar{\iota} \Gamma^{b_{1} b_{2} b_{3} b_{4} b_{5}} \iota \bar{\iota} \Gamma_{b_{5}} \iota \delta^{32}(\psi) \tag{6.6}
\end{equation*}
$$

and then we can compute the differential

$$
\begin{equation*}
d \star d \omega_{7}=-\frac{7}{4} \bar{\psi} \Gamma^{a_{1}} \psi \ldots V^{a_{7}} \epsilon_{a_{1} \ldots a_{7} b_{1} \ldots b_{4}} \bar{\Gamma} \Gamma^{b_{1} b_{2} b_{3} b_{4} b_{5}} \iota \iota \Gamma_{b_{5}} \iota \delta^{32}(\psi) \tag{6.7}
\end{equation*}
$$

The integration by parts of produces two different structures, one vanishes because of the usual trace of gamma matrices, but the second structure gives the expression

$$
\begin{equation*}
d \star d \omega_{7}=-\frac{7}{2} \star \omega_{7} \quad \Longrightarrow \quad \star d \star d \omega_{7}=-\frac{7}{2} \omega_{7} \tag{6.8}
\end{equation*}
$$

then finally it leads (together the vanishing of $d \star \omega_{7}=0$ ), to the Laplace-Beltrami differ-
ential $\Delta=d \star d \star+\star d \star d$ acting on those cocycles and it yields

$$
\begin{equation*}
\Delta \omega_{7}=-\frac{7}{2} \omega_{7}, \quad \Delta \omega_{4}=0 \tag{6.9}
\end{equation*}
$$

The second equation follows from $d \omega_{4}=0$. The second equation is one side of the Hodge theorem, since $\omega^{(4)}$ is a Chevalier-Eilenberg cocycle, $\Delta \omega_{4}=0$ implies that it is also the harmonic representative. Vice-versa, $\omega^{(7)}$ is not a cohomology class and $\Delta \omega_{7} \neq 0$.

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## Appendix: A Brief Review on Integral Forms

In this appendix, we want to recall the main definitions and computation techniques for integral forms. For a more exhaustive review or for a more rigorous approach to integral forms we suggest [27, 28, 29].

We consider a supermanifold $\mathcal{S} \mathcal{M}^{(n \mid m)}$ with $n$ bosonic and $m$ fermionic dimensions. We denote the local coordinates in an open set as $\left(x^{a}, \theta^{\alpha}\right), a=1, \ldots, n, \alpha=1, \ldots, m$. A generic ( $p \mid 0$ )-form, i.e., a superform, has the following local expression

$$
\begin{equation*}
\omega^{(p \mid 0)}=\omega_{\left[i_{1} \ldots i_{r}\right]\left(\alpha_{1} \ldots \alpha_{s}\right)}(x, \theta) d x^{i_{1}} \wedge \ldots \wedge d x^{i_{r}} \wedge d \theta^{\alpha_{1}} \wedge \ldots \wedge d \theta^{\alpha_{s}}, p=r+s \tag{6.10}
\end{equation*}
$$

The coefficients $\omega_{\left[i_{1} \ldots i_{r}\right]\left(\alpha_{1} \ldots \alpha_{s}\right)}(x, \theta)$ are a set of superfields and the indices $a_{1} \ldots a_{r}, \alpha_{1} \ldots \alpha_{s}$ are anti-symmetrized and symmetrised, respectively, because of the rules (we omit the " $\wedge$ " symbol)

$$
\begin{equation*}
d x^{i} d x^{j}=-d x^{j} d x^{i}, d \theta^{\alpha} d \theta^{\beta}=d \theta^{\beta} d \theta^{\alpha}, d x^{i} d \theta^{\alpha}=d \theta^{\alpha} d x^{i} \tag{6.11}
\end{equation*}
$$

Namely, we assign parity 1 to odd forms and 0 to even forms:

$$
\begin{equation*}
|d x|=1,|d \theta|=0 . \tag{6.12}
\end{equation*}
$$

Since superforms are generated both by commuting and anti-commuting forms, we immediately see that there is no top form. In other words, if one looks for the analogous of the determinant bundle on a supermanifold, one has to consider a different space of forms,
namely the integral forms. A generic integral form locally reads

$$
\begin{equation*}
\omega^{(p \mid m)}=\omega_{\left[i_{1} \ldots i_{r}\right]}^{\left(\alpha_{1} \ldots \alpha_{s}\right)}(x, \theta) d x^{i_{1}} \wedge \ldots \wedge d x^{i_{r}} \wedge \iota_{\alpha_{1}} \ldots \iota_{\alpha_{s}} \delta\left(d \theta^{1}\right) \wedge \ldots \wedge \delta\left(d \theta^{m}\right) \tag{6.13}
\end{equation*}
$$

where $\delta(d \theta)$ is a (formal) Dirac delta function and $\iota_{\alpha}$ denotes the interior product. The integration on $d \theta$ 's is defined algebraically by setting

$$
\begin{equation*}
\int_{d \theta} \delta(d \theta)=1, \quad \int_{d \theta} f(d \theta) \delta(d \theta)=f(0) \tag{6.14}
\end{equation*}
$$

for a generic test function $f(d \theta)$. The symbol $\delta(d \theta)$ satisfies the usual distributional equations

$$
\begin{equation*}
d \theta \delta(d \theta)=0, \delta(\lambda d \theta)=\frac{1}{\lambda} \delta(d \theta), d \theta \delta^{(1)}(d \theta)=-\delta(d \theta), d \theta \delta^{(p)}(d \theta)=-p \delta^{(p-1)}(d \theta), \tag{6.15}
\end{equation*}
$$

We sometimes denote by $\iota_{\alpha} \delta\left(d \theta^{\alpha}\right) \equiv \delta^{(1)}\left(d \theta^{\alpha}\right)$. Additional properties are

$$
\begin{equation*}
\delta\left(d \theta^{\alpha}\right) \wedge \delta\left(d \theta^{\beta}\right)=-\delta\left(d \theta^{\beta}\right) \wedge \delta\left(d \theta^{\alpha}\right), \quad d x \wedge \delta(d \theta)=-\delta(d \theta) \wedge d x \tag{6.16}
\end{equation*}
$$

indicating that actually these are not conventional distributions, but rather de Rham currents.

Given these properties, we retrieve a top form among integral forms as

$$
\begin{equation*}
\omega_{\text {top }}^{(n \mid m)}=\omega(x, \theta) d x^{1} \wedge \ldots \wedge d x^{n} \wedge \delta\left(d \theta^{1}\right) \wedge \ldots \wedge \delta\left(d \theta^{m}\right), \tag{6.17}
\end{equation*}
$$

where $\omega(x, \theta)$ is a superfield. The space of $(n \mid m)$ forms corresponds to the Berezinian bundle since the generator $d x^{1} \wedge \ldots \wedge d x^{n} \wedge \delta\left(d \theta^{1}\right) \wedge \ldots \wedge \delta\left(d \theta^{m}\right)$ transforms as the superdeterminant of the Jacobian.

One can also consider a third class of forms, with non-maximal and non-zero number of delta's: the pseudoforms. A general pseudoform with $q$ Dirac delta's is locally given by

$$
\begin{align*}
\omega^{(p \mid q)}= & \omega_{\left[a_{1} \ldots a_{r}\right]\left(\alpha_{1} \ldots \alpha_{s}\right)\left[\beta_{1} \ldots \beta_{q}\right]}(x, \theta)  \tag{6.18}\\
& d x^{a_{1}} \wedge \ldots \wedge d x^{a_{r}} \wedge d \theta^{\alpha_{1}} \wedge \ldots \wedge d \theta^{\alpha_{s}} \wedge \delta^{\left(t_{1}\right)}\left(d \theta^{\beta_{1}}\right) \wedge \ldots \wedge \delta^{\left(t_{q}\right)}\left(d \theta^{\beta_{q}}\right)
\end{align*}
$$

where $\delta^{(i)}(d \theta) \equiv(\iota)^{i} \delta(d \theta)$. The form number is obtained as

$$
\begin{equation*}
p=r+s-\sum_{i=1}^{q} t_{i}, \tag{6.19}
\end{equation*}
$$

since the contractions carry negative form number. The two quantum numbers $p$ and $q$ in eq. (6.19) correspond to the form number and the picture number, respectively, and they
range as $-\infty<p<+\infty$ and $0 \leq q \leq m$, so the picture number counts the number of delta's. If $q=0$ we have superforms, if $q=m$ we have integral forms, if $0<q<m$ we have pseudoforms.

As in conventional geometry, we can define the integral of a top form on a supermanifold (more rigorously, the integration is on the parity-shifted tangent space $\Pi T \mathcal{S M}$ ) as

$$
\begin{equation*}
I[\omega]=\int_{\mathcal{S} \mathcal{M}} \omega_{\text {top }}^{(n \mid m)}=\int \omega(x, \theta)\left[d^{n} x d^{m} \theta\right], \tag{6.20}
\end{equation*}
$$

where we integrated over the odd variables $d x$ and over the even variables $d \theta$ to obtain an ordinary superspace integral over the variables $(x, \theta)$.

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[^1]:    ${ }^{1}$ Solving the constraints yields that 23 linear independent complex pure spinors $\lambda$ and carrying a nonlinear representation of the Lorentz group.

