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# REMARKS ON THE KURAMOTO-SIVASHINSKY EQUATION

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We report here a joint work in progress on the Kuramoto-Sivashinsky equation. The question we address is the analytical study of the following fourth order nonlinear evolution equation:

$$\frac{\partial u}{\partial t} + \Delta^2 u + \Delta u + \tfrac{1}{2}|\nabla u|^2 = 0 \quad . \tag{0.1}$$

This equation has been obtained by Sivashinsky [8] in the context of combustion and independently by Kuramoto [3] in the context of reaction diffusion-systems. Both were motivated by (nonlinear) stability of travelling waves. Numerical calculations have been done on this equation: we should mention the work of Michelson-Sivashinsky [6], Aimar [2], Manneville [5], and Hyman [7]. All the results seem to indicate a "chaotic" behavior of the solution. Therefore, the analytical study is of interest in analogy with the Burger's and Navier-Stokes equations. Here we give some existence and uniqueness results for equation (0.1), in space dimension one (§1), and we also study a fractional step method of numerical resolution (§2). In a forthcoming joint paper with R. Temam, we will study the asymptotic behavior, as $t \to +\infty$, of the solution of (0.1) and give an estimate on the number of "determining modes" (see [9]).

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## §1. Existence and Uniqueness in Space Dimension One.

We are considering the following initial-value problem (for $0 < T < \infty$)

$$(1.1)\qquad \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 = 0 \qquad \text{in } [0,1] \times [0,T]$$

$$(1.2)\qquad u(0) = \theta$$

with periodic boundary conditions (the unit interval can be therefore identified with the one dimensional torus). The case of the boundary conditions:

$$(1.3)\qquad \frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial x^3} = 0 \qquad \text{on } x = 0 \text{ and } x = 1$$

could be handled in the same way. Rewriting the problem (1.1) as an integral equation:

$$u(t) = S(t)\theta + \frac{1}{2}\int_0^t S(t-s)\left|\frac{\partial u}{\partial x}(s)\right|^2 ds \quad ,$$

where S(t) denote the semigroup associated to $\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial x^2}$, we easily obtain local in time existence and uniqueness results. It is sufficient to apply the standard Picard iteration scheme. We therefore need a priori estimates on u in order to get existence on $[0,T]$ (instead of $[0,T_*]$, $T_* < T$). Indeed we will prove ($\|\cdot\|$ is the usual $L^2$ norm):

Theorem 1. Let $\theta$ satisfy $\frac{\partial^k}{\partial x^k}\theta \in L^2$, $0 < k \leq 2$. Then, for any solution of

(1.1), (1.2), one has:

$$(1.4) \qquad \sup_{0\le t\le T} \left\|\frac{\partial^k U}{\partial x^k}(t)\right\| \quad , \int_0^T \left\|\frac{\partial^{k+2} U}{\partial x^{k+2}}(t)\right\|^2 dt < \text{Cste} \quad ,$$

where $0 \le k \le 2$, the constant depends only on $\vartheta$ and T and $U \equiv u(x,t) - \int_0^1 u(x,t)\, dx$.

Corollary 1. Under the hypothesis of Theorem 1, the problem (1.1), (1.2) admits a unique solution u such that: $u \in L^2(0,T;H^4) \subset L^\infty(0,T;H^2)$, $\frac{du}{dt} \in L^2(0,T;L^2)$. (Here $H^2$ is the Sobolev's space of functions u such that $\frac{\partial^j u}{\partial x^j} \in L^2$, $0 \le j \le k$.)

Proof of Theorem 1.

If the space dimension is one, we make the following remark. Set $v \equiv \frac{\partial u}{\partial x}$, then v satisfies a "Burger's like" equation:

$$(1.5) \qquad \frac{\partial v}{\partial t} + \frac{\partial^4 v}{\partial x^4} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} = 0 \quad .$$

In other words, we multiply (1.1) by $-\frac{\partial^2 u}{\partial x^2}$; after integrating by parts we get:

$$(1.6) \qquad \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial x}(t)\right\|^2 + \left\|\frac{\partial^3 u}{\partial x^3}(t)\right\|^2 - \left\|\frac{\partial^2 u}{\partial x^2}(t)\right\|^2 = 0$$

(here and in what follows, we denote by $\|\cdot\|$ the usual $L^2$ norm). This identity reflects the fact that, in (1.1) or (1.3), the term $\frac{\partial^4 u}{\partial x^4}$ (resp.

$\frac{\partial^2 u}{\partial x^2}$) . sink (resp. source) of energy. We now distinguish two cases.

a) The sink term is dominant. In other words, by Poincaré's inequality:

$$(1.7) \qquad ||\frac{\partial^2 u}{\partial x^2}(t)||^2 \le \Lambda_1 \; ||\frac{\partial^3 u}{\partial x^3}(t)||^2$$

where $\Lambda_1$ is the constant of Poincaré (notice that $\frac{\partial^2 u}{\partial x^2}$ has 0 mean value). From (1.6) and (1.7) we conclude, by Gronwall's lemma:

$$(1.8) \qquad \sup_{0\le t\le T} ||\frac{\partial u}{\partial x}(t)|| \quad , \int_0^T ||\frac{\partial^3 u}{\partial x^3}(t)||^2 \, dt < C(||\frac{\partial \vartheta}{\partial x}||)$$

where the bound is uniform in T.

b) The sink term is not dominant. We use the interpolation inequality

$$(1.9) \qquad ||\frac{\partial^2 u}{\partial x^2}(t)||^2 \le \tfrac{1}{2}||\frac{\partial^3 u}{\partial x^3}(t)||^2 + C||\frac{\partial u}{\partial x}(t)||^2$$

and again we obtain, from (1.6) and Gronwall's lemma, the bound (1.8). But now the constant depends on T.

Coming back to (1.1), we note that $\bar{u}(t) \equiv \int_0^1 u(x,t)\,dx$ satisfy:

$$(1.10) \qquad \frac{d}{dt}\bar{u}(t) + \tfrac{1}{2}\int_0^1 \left|\frac{\partial u}{\partial x}\right|^2 dx = 0 \quad ,$$

therefore $U(x,t) \equiv u(x,t) - \bar{u}(t)$ satisfy:

$$(1.11) \qquad \frac{\partial}{\partial t}U + \frac{\partial^4}{\partial x^4}U + \frac{\partial^2 U}{\partial x^2} + \tfrac{1}{2}\left|\frac{\partial U}{\partial x}\right|^2 - \tfrac{1}{2}\int_0^1 \left|\frac{\partial U}{\partial x}\right|^2 dx = 0 \quad ,$$

where now, $\int_0^1 U(x,t)\,dx = 0$. Multiplying (1.11) by U and integrating, we get:

$$\frac{1}{2}\frac{d}{dt}\,||U(t)||^2 + ||\frac{\partial^2 u}{\partial x^2}(t)||^2 = ||\frac{\partial U}{\partial x}(t)||^2 - \frac{1}{2}\int_0^1 |\frac{\partial U}{\partial x}|^2\, U\,dx \quad . \tag{1.12}$$

The first term in the right hand side of (1.12) is bounded using an inequality similar to (1.9). For the second term we use Hölder inequality:

$$\int_0^1 \left|\frac{\partial U}{\partial x}\right|^2 U\,dx \leq ||\frac{\partial U}{\partial x}(t)||\;||\frac{\partial U}{\partial x}(t)||_{L^4}\;||U(t)||_{L^4} \quad . \tag{1.13}$$

If the space dimension is one, by the Sobolev imbedding theorem and interpolation (see for instance [1], [4]), we know that:

$$[H^2,L^2]_{\frac{7}{8}} = H^{\frac{1}{4}} \quad L^4 \tag{1.14}$$

$$[H^2,L^2]_{\frac{3}{8}} = H^{\frac{5}{4}} \quad . \tag{1.15}$$

Using the norm inequalities associated with (1.14), (1.15) we deduce finally from (1.13):

$$\int_0^1 \left|\frac{\partial u}{\partial x}\right|^2 U\,dx \leq ||\frac{\partial U}{\partial x}(t)||\;\{C_\varepsilon||U(t)||^2 + \varepsilon||\frac{\partial^2 U}{\partial x^2}(t)||^2\} \tag{1.16}$$

for $\varepsilon > 0$ small. Thanks to (1.8), we can control $||\frac{\partial U}{\partial x}(t)|| = ||\frac{\partial u}{\partial x}(t)|| \leq \sup_{0<t\leq T} ||\frac{\partial u}{\partial x}(t)||$, and obtain from (1.12) and (1.16):

$$(1.17)\qquad \tfrac{1}{2}\frac{d}{dt}||U(t)||^2 + C_\varepsilon'||\frac{\partial^2 U}{\partial x^2}(t)||^2 \le C_\varepsilon''||U(t)||^2 \ .$$

By Gronwall's lemma, we get:

$$(1.18)\qquad \sup_{0\le t\le T} ||u(t) - u(t)|| \ , \int_0^T ||\frac{\partial^2 u}{\partial x^2}(t)||^2\, dt < C_T(||\vartheta-\bar{\vartheta}|| \ , ||\frac{\partial\vartheta}{\partial x}||).$$

The remaining needed estimates are proved by multiplying (1.1) by $\frac{\partial^4 u}{\partial x^4}$; after integration by parts we get:

$$(1.19)\qquad \tfrac{1}{2}\frac{d}{dt}||\frac{\partial^2 u}{\partial x^2}(t)||^2 + ||\frac{\partial^4 u}{\partial x^4}(t)||^2 = ||\frac{\partial^3 u}{\partial x^3}(t)||^2 - \tfrac{1}{2}\int_0^1 \left|\frac{\partial^2 u}{\partial x^2}\right|^2 \frac{\partial^2 u}{\partial x^2}\, dx.$$

The right hand side is bounded using (1.8), the Hölder inequality and the following version of Poincaré's inequality $||\frac{\partial^2 u}{\partial x^2}(t)||_{L^\infty} \le 2\Lambda_1||\frac{\partial^3 u}{\partial x^3}(t)||$.
We conclude then:

$$(1.20)\qquad \sup_{0\le t\le T} ||\frac{\partial^2 u}{\partial x^2}(t)|| \ , \int_0^T ||\frac{\partial^4 u}{\partial x^4}(t)||^2\, dt \le C_T(||\vartheta||, ||\frac{\partial\vartheta}{\partial x}|| \ , ||\frac{\partial^2\vartheta}{\partial x^2}||.$$

This completes the proof of (1.4). □

## §2. A Fractional Step Method.

It is natural to decompose equation (1.1) in two parts corresponding to the nonlinear term and the linear terms. Precisely, we can split (1.1) in the following way:

$$(2.1)\qquad \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \tfrac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2 = 0$$

$$(2.2) \qquad \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + 2\,\frac{\partial^2 u}{\partial x^2} = 0 \quad .$$

According to (2.1), (2.2) we define the following scheme. For N given large set $\tau = \frac{T}{N}$ and $u^n = u(n\tau)$, $0 \leq n \leq N$. For $u^n$ given ($u^0 = \theta$) $u^{n+1}$ is obtained via:

$$(2.3) \qquad u^{n+\frac{1}{2}} - \tau\,\frac{\partial u^{n+\frac{1}{2}}}{\partial x^2} + \frac{\tau}{2}\left|\frac{\partial u^{n+\frac{1}{2}}}{\partial x}\right|^2 = u^n \text{ in } [0,1]$$

$$(2.4) \qquad u^{n+\frac{1}{2}} + \tau\,\frac{\partial^4 u^{n+1}}{\partial x^4} + 2\,\frac{\partial^2 u^{n+1}}{\partial x^2} = u^{n+\frac{1}{2}} \text{ in } [0,1] \quad .$$

A convenient equivalent form of (2.4) is:

$$(2.5) \qquad u^{n+1} + \tau\left(\frac{\partial^2}{\partial x^2} + 1\right)^2 u^{n+1} - \tau\, u^{n+1} = u^{n+\frac{1}{2}} \quad .$$

We still suppose periodic boundary conditions. From (2.3), multiplying by $-\frac{\partial^2 u^{n+\frac{1}{2}}}{\partial x^2}$ and integrating by parts, we get:

$$(2.6) \qquad \left\|\frac{\partial u^{n+\frac{1}{2}}}{\partial x}\right\|^2 - \left\|\frac{\partial u^n}{\partial x}\right\|^2 + \left\|\frac{\partial u^{n+\frac{1}{2}}}{\partial x} - \frac{\partial u^n}{\partial x}\right\|^2 + 2\tau\left\|\frac{\partial^2 u^{n+\frac{1}{2}}}{\partial x^2}\right\|^2 = 0 \quad .$$

Similarly from (2.5) we get:

$$(2.7) \qquad (1 - 2\tau)\left\|\frac{\partial u^{n+1}}{\partial x}\right\|^2 - \left\|\frac{\partial u^{n+\frac{1}{2}}}{\partial x}\right\|^2 + \left\|\frac{\partial u^{n+1}}{\partial x} - \frac{\partial u^{n+\frac{1}{2}}}{\partial x}\right\|^2$$

$$+ 2\tau\left\|\left(\frac{\partial^2}{\partial x^2} + 1\right)\frac{\partial u^{n+1}}{\partial x}\right\| = 0 \quad .$$

Therefore, combining (2.6) and (2.7):

$$(2.8) \qquad \left\|\frac{\partial u^{n+1}}{\partial x}\right\|^2 \leq (1 - 2\tau)^{-1} \left\|\frac{\partial u^n}{\partial x}\right\|^2 \quad .$$

Consequently, for $\tau$ small enough:

$$(2.9) \qquad \left\|\frac{\partial u^{n+\frac{1}{2}}}{\partial x}\right\|^2 \leq \left\|\frac{\partial \theta}{\partial x}\right\|^2$$

$$(2.10) \qquad \left\|\frac{\partial u^{n+1}}{\partial x}\right\|^2 \leq e^{2T}\left\|\frac{\partial \theta}{\partial x}\right\|^2 \quad .$$

Coming back to (2.3), (2.4) we can find bounds, as in §1, for $||u^{n+\frac{1}{2}}||$, $||u^{n+1}||$, $\left\|\frac{\partial^2 u^{n+1}}{\partial x^2}\right\|$. Then, thanks to the maximum principle we can estimate in $L^\infty$, $u^{n+\frac{1}{2}}$ and $\frac{\partial u^{n+\frac{1}{2}}}{\partial x}$; as a consequence we can pass to the limit in the non-linearity. So are the main estimates in order to prove stability and convergence results for the scheme defined by (2.3) - (2.4). Details will be given elsewhere.

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