Remarks on the Poisson Stochastic Process (I)

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We adopt in this paper*) the known measure theoretic treatment of probability[1]). In particular we treat a stochastic process as a functional space Ω with a probability-measure. Thus, in the classical Poisson process, each $\omega \epsilon \Omega$ is a function, the value $\omega(t)$ of which is *e. g.* the number of calls in a telephone exchange during the half-open time-intervall $(0,t\rangle$.

Strictly speaking we denote by Ω any set of real valued functions $\omega(t)$ defined for $t \geqslant 0$, by $\boldsymbol{B}_\Omega$ the smallest *σ-field* (*i. e.* a countably additive and complementative class of sets) which contains as elements all the sets of the form

$$A(t,y) = \underset{\omega}{E}[\omega \epsilon \Omega; \omega(t) < y],$$

and by μ a *probability measure* (*i. e.* a non-negative and countably additive set function with $\mu(\Omega)=1$) in $\boldsymbol{B}_\Omega$. The triple $(\Omega, \boldsymbol{B}_\Omega, \mu)$ is called a *stochastic process*[2]).

In particular we denote by Ω_0 the set of all integral valued functions $\omega(t)$ defined for $t \geqslant 0$, which are continuous on the right, non decreasing, and such that $\omega(0)=0$. Finally, we denote by Ω_1 the set of all functions $\omega \epsilon \Omega_0$ possessing only jumps equal to 1.

An important class of stochastic processes is that of *homogeneous differential* ones, *i. e.* fulfilling the following conditions:

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[1]) Cf. Kolmogoroff [8] and Halmos [5], Chapter IX, p. 184-215.

[2]) Cf. Doob [2].

(h) $\mu \underset{\omega}{E}[\omega(t+\tau)-\omega(t)<y]$ does not depend on t (the homogeneity),

(i) $\mu \underset{\omega}{E}[\omega(u_1)-\omega(t_1)<y_1; \ldots; \omega(u_n)-\omega(t_n)<y_n]$

$$= \mu \underset{\omega}{E}[\omega(u_1)-\omega(t_1)<y_1] \cdot \ldots \cdot \mu \underset{\omega}{E}[\omega(u_n)-\omega(t_n)<y_n]$$

for $0 \leqslant t_1 < u_1 \leqslant t_2 < \ldots \leqslant t_n < u_n$ (the independence of increments in non-overlapping intervals).

In the sequel we shall use instead of the condition (i) a formally weaker one:

(j) $\mu \underset{\omega}{E}[\omega(u)-\omega(t)<y; \omega(v)-\omega(u)<z]$

$$= \mu \underset{\omega}{E}[\omega(u)-\omega(t)<y] \cdot \mu \underset{\omega}{E}[\omega(v)-\omega(u)<z]$$

for $0 \leqslant t < u < v$ (the independence of increments in two contiguous intervals).

A process is called *degenerate* if there is an $\omega_0 \epsilon \Omega$ such that $\mu(B)=1$ for each $B \epsilon \boldsymbol{B}_\Omega$ containing ω_0.

Let us suppose $\Omega \subset \Omega_0$ and put

$$P_k(t) = \mu \underset{\omega}{E}[\omega \epsilon \Omega; \omega(t)=k] \qquad (k=0,1,2,\ldots).$$

It is well known that, if (h), (j),

(n) $$\lim_{t \to 0} \frac{1-P_0(t)}{t} = a$$

and

(o) $$\lim_{t \to 0} \frac{1-P_0(t)-P_1(t)}{t} = 0,$$

then the random variable $\omega(t)$ has the Poisson distribution with the mean value at[3]), *i. e.*

(p) $$P_k(t) = e^{-at}\frac{(at)^k}{k!} \qquad (k=0,1,2,\ldots).$$

[3]) Cf. Khintchine [7], p. 19 and 20, Feller [3], p. 405, and [4], p. 364-367. The application of the Poisson distribution in this process goes back to Bortkiewicz [1], § 8, p. 16-19. A generalization for the non-homogeneous case is contained in a recent paper by Rényi [12]. Cf. also Lévy [11], chapter VII.

(For $a=0$ the condition (p) takes the form $P_0(t)=1$, $P_k(t)=0$ for $t\geqslant 0$, $k=1,2,\ldots$, and the process is degenerate).

In recent works the condition (n) is omitted; it tourns out namely that, *under the assumptions* (h) *and* (j), *if* (o), *then there is a number* $a\geqslant 0$ *such that* (p) (*and conversely*)[4].

The purpose of the §1 of this paper is to prove directly that the condition (o) may be replaced by

(q) $\mu(\Omega-\Omega_1)=0$

or, in other words, that (q) *if and only if* (p) *for a number* $a\geqslant 0$ (Theorem 2).

Obviously the condition (q) is fulfilled in the important case of $\Omega\subset\Omega_1$, *i. e.* if all $\omega\in\Omega$ have only jumps equal to 1.

It seems that condition (q) has a more expressive probabilistic sense than the analytic condition (o). In the case of telephone calls the condition (q) says that two simultaneous calls are almost impossible.

In §2 we prove in a very simple way that *every probability-measure in* Ω_0 *or* Ω_1 *vanishing for all one point sets is point-isomorphic to the Lebesgue measure* (Theorem 3).

§ 1. The Poisson distribution. A known reasoning[4]) gives directly the following

Theorem 1. *If* $(\Omega,\boldsymbol{B}_\Omega,\mu)$ *is a stochastic process with* $\Omega\subset\Omega_0$, *and fulfills* (h) *and* (j), *then there are two non-negative numbers* a *and* b *such that*

(1) $$P_0(t)=e^{-at},\qquad P_1(t)=bte^{-at}.$$

We shall prove the

Theorem 2. *If* $(\Omega,\boldsymbol{B}_\Omega,\mu)$ *is a stochastic process with* $\Omega\subset\Omega_0$, *and fulfills* (h) *and* (j), *then* (p) *for a number* $a\geqslant 0$ *if and only if* (q).

1. (q)→(p). In view of Theorem 1 we have (1), and we shall prove that

(2) $$a=b.$$

Let us establish a number $T>0$ and put

$$\Xi=\underset{\omega}{E}[\omega\in\Omega;\omega(T)>1],$$

[4]) Feller [3], p. 365, footnote 5, and Jánossy, Rényi and Aczél [6], § 1, p. 211-213.

$$\Xi_n=\underset{\omega}{E}\left[\omega\in\Xi;\omega\left(\frac{j}{2^n}T\right)-\omega\left(\frac{j-1}{2^n}T\right)\leqslant 1\quad\text{for } j=1,2,\ldots,2^n\right].$$

Obviously $\lim\limits_n \Xi_n=\Xi\Omega_1$, whence, in view of (q),

(3) $$\lim_n \mu(\Xi_n)=\mu(\Xi)=1-P_0(T)-P_1(T).$$

Next, putting

$$\Xi_n^j=\underset{\omega}{E}\left[\omega\in\Omega;\omega\left(\frac{j-1}{2^n}T\right)=1;\omega\left(\frac{j}{2^n}T\right)=2\right],$$

we obtain

(4) $$\Xi_n\subset\sum_{j=1}^{2^n}\Xi_n^j\subset\Xi\quad\text{and}\quad\Xi_n^i\cdot\Xi_n^j=0\quad\text{for}\quad i\neq j.$$

By virtue of (j) and (h) we have

$$\mu(\Xi_n^j)=\mu\underset{\omega}{E}\left[\omega\left(\frac{j-1}{2^n}T\right)-\omega(0)=1\right]\cdot\mu\underset{\omega}{E}\left[\omega\left(\frac{j}{2^n}T\right)-\omega\left(\frac{j-1}{2^n}T\right)=1\right]$$
$$=P_1\left(\frac{j-1}{2^n}T\right)P_1\left(\frac{1}{2^n}T\right),$$

whence, following (3) and (4),

(5) $$\lim_n P_1\left(\frac{1}{2^n}T\right)\sum_{j=1}^{2^n}P_1\left(\frac{j-1}{2^n}T\right)=1-P_0(T)-P_1(T).$$

Since, by (1)

$$\lim_n\frac{2^n}{T}P_1\left(\frac{1}{2^n}T\right)=b$$

and since obviously

$$\lim_n\frac{T}{2^n}\sum_{j=1}^{2^n}P_1\left(\frac{j-1}{2^n}T\right)=\int_0^T P_1(t)\,dt=\int_0^T bte^{-at}\,dt,$$

we deduce from (5)

$$b^2\int_0^T te^{-at}\,dt=1-e^{-aT}-bte^{-aT},$$

and by differentiation

$$b^2te^{-at}=ae^{-at}-b^{-at}+abte^{-at}.$$

Hence, putting $t=0$ we obtain (2).

The formula (p) is thus proved for $k=0$ and $k=1$. By a known reasoning we prove successively the same formula for $k=2,3,\ldots$

2. (p)$\rightarrow$(q)[5]. The formula (p) for $k=0$ and $k=1$ implies directly the equality (o).

Let us establish a number $T>0$ and put

$$\Delta=\underset{\omega}{E}[\omega\,\epsilon\,\Omega;\ \text{there is } t<T \text{ with } \omega(t)-\omega(t-0)\geqslant 2],$$

$$\Delta_n^j=\underset{\omega}{E}\left[\omega\,\epsilon\,\Omega;\omega\left(\frac{j}{n}\,T\right)-\omega\left(\frac{j-1}{n}\,T\right)\geqslant 2\right].$$

Obviously $\Delta\subset\Delta_n^1+\Delta_n^2+\ldots+\Delta_n^n$ for $n=1,2,\ldots$ When $n\rightarrow\infty$

$$\mu(\Delta_n^1)+\mu(\Delta_n^2)+\ldots+\mu(\Delta_n^n)=n\left[1-P_0\left(\frac{1}{n}\right)-P_1\left(\frac{1}{n}\right)\right]\rightarrow 0,$$

following (o). Consequently $\mu(\Delta)=0$, whence, since T was chosen arbitrarily and μ is countably additive, we obtain the equality (q).

Theorem 2 is thus proved.

Let us remark that it is possible to deduce the same result from the general form of the distribution function for all homogeneous differential processes with $\Omega\subset\Omega_0$[6].

§ 2. The point isomorphism with the Lebesgue measure. Let R denote the set of all rational non-negative numbers. For each set Ω of real functions defined for $t\geqslant 0$ we denote by $\Omega|R$ the set of all partial functions $\omega|R$, where $\omega\,\epsilon\,\Omega$.

Let us treat the space $\mathcal{C}^R$ of all real functions of a rational variable $r\geqslant 0$ as the denumerable Cartesian power of the set of real numbers, and consequently as a complete and separable metric space[7].

It is easy to see that

(a) If, for $\omega_1,\omega_2\,\epsilon\,\Omega_0$, $\omega_1|R=\omega_2|R$, then $\omega_1=\omega_2$.

(b) For every $\Omega\subset\Omega_0$ and every decreasing sequence $r_n\,\epsilon\,R$ tending to t

$$A(t,y)=\sum_{m=1}^{\infty}\prod_{n=m}^{\infty}A(r_n,y).$$

[5]) Cf. Lévy [11], p. 173.

[6]) See *e. g.* Jánossy, Rényi and Aczél [6], § 2, p. 213-217, in particular the remark of Kolmogoroff, formulated in the same paper, p. 216.

[7]) See *e. g.* Kuratowski [9], p. 231 and 313.

Next we shall prove

(c) $\Omega_0|R$ and $\Omega_1|R$ are Borel subsets of $\mathcal{C}^R$.

The set I of all $\omega\,\epsilon\,\mathcal{C}^R$ with non-negative integral values is obviously closed in $\mathcal{C}^R$. Obviously $\omega\,\epsilon\,\Omega_0|R$ if and only if $\omega\,\epsilon\,I$, and ω is non-decreasing and continuous on the right in the set R. Consequently

$$\Omega_0|R=I\cdot\underset{\omega}{E}\Big\{\prod_{r_1<r_2}[\omega(r_1)\leqslant\omega(r_2)]\cdot\prod_{r_1}\sum_{r_2}(r_1<r_2)[\prod_{r_3}(r_1<r_3<r_2)\big(\omega(r_3)=\omega(r_1)\big)]\Big\},$$

where Σ and Π are quantifiers. It follows that $\Omega_0|R$ is an $F_{\sigma\delta}$-set in $\mathcal{C}^R$.

Similarly

$$\Omega_1|R=(\Omega_0|R)\cdot\underset{\omega}{E}[\prod_{r_1}\sum_{r_2}(r_2<r_1)(\omega(r_1)-\omega(r_2)\leqslant 1)],$$

which implies that $\Omega_1|R$ is also an $F_{\sigma\delta}$-set in $\mathcal{C}^R$.

Now we shall prove the following

Lemma 1. *If Ω is a non-denumerable set of functions of a non-negative variable t, such that*

(α) *if $\omega_1|R=\omega_2|R$, where $\omega_1,\omega_2\,\epsilon\,\Omega$, then $\omega_1=\omega_2$,*

(β) *for each $t\geqslant 0$ the sets $A(t,y)$ belong to the smallest σ-field containing all the sets $A(r,z)$, where $r\,\epsilon\,R$,*

(γ) *$\Omega|R$ is a Borel subset of $\mathcal{C}^R$,*

then the field $\boldsymbol{B}_\Omega$ is point-isomorphic with the field of all Borel subsets of the unit interval.

Proof. Let us associate with every function $\omega\,\epsilon\,\Omega$ the function $\omega|R$. In view of (α) we obtain in this way a ~~one-one~~ mapping h_1 of Ω onto $\Omega'=\Omega|R$.

In view of (β) this mapping transforms the class $\boldsymbol{B}_\Omega$ onto the class $\boldsymbol{B}'$ of all Borel subsets of Ω'.

Finally, in view of (γ), there is a measurable (B) one-one mapping h_2 of Ω' onto the unit interval[8]). It transforms the class $\boldsymbol{B}'$ onto the field $\boldsymbol{B}$ of all Borel subsets of the unit interval.

[8]) Every non-denumerable Borel subset of a separable and complete metric space is the image of the unit interval by a measurable (B) one-one mapping (Kuratowski [9], p. 358, Theorem 2).

The mapping $x = h_2(h_1(\omega))$ is a point-isomorphism of $\boldsymbol{B}_\Omega$ and $\boldsymbol{B}$.

Lemma 2. *If $(\Omega, \boldsymbol{B}_\Omega, \mu)$ is a stochastic process satisfying the conditions (α), (β) and (γ), and such that μ vanishes for every one-point set, then μ is point-isomorphic with the Lebesgue measure in the field of all Borel subsets in the unit interval.*

Proof. The condition (α) implies that $\omega \in \boldsymbol{B}_\Omega$ for every $\omega \in \Omega$. The Lemma 2 follows from Lemma 1 and from the fact that every probability measure in the field $\boldsymbol{B}$, which vanishes for all one point sets is point-isomorphic with the Lebesgue measure in $\boldsymbol{B}$[9]).

The propositions (a), (b), (c) and Lemma 2 imply directly the

Theorem 3. *Every stochastic process $(\Omega, \boldsymbol{B}_\Omega, \mu)$ with $\Omega = \Omega_0$ or $\Omega = \Omega_1$, and such that μ vanishes for each one point set, is point-isomorphic with the Lebesgue measure in the field of Borel subsets of the unit interval*[10]).

In particular the hypotheses of Theorem 3 are fulfilled by every non-degenerate process of the form $(\Omega_0, \boldsymbol{B}_{\Omega_0}, \mu)$ or $(\Omega_1 \boldsymbol{B}_{\Omega_1}, \mu)$, satisfying the conditions (h) and (j). In fact, in view of Theorem 1 we have the equalities (1). If $a=0$, then $P_0(t)=1$ and the process is degenerate, and if $a>0$ it is easy to prove that $\mu((\omega))=0$ for every $\omega \in \Omega$.

Lemma 2 implies an analogous theorem for the space Ω_c of all continuous real functions of a non-negative variable (and so particularly in the case of the Brownian motion[11])) and for many other spaces considered in the theory of stochastic processes.

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[9]) See *e. g.* Marczewski [10], p. 57.

[10]) For the case of processes having the Poisson distribution cf. Wiener [14], p. 51.

[11]) Wiener's proof of the existence of the measure for Brownian motions furnishes at the same time the construction of such an isomorphism (Wiener [13], p. 216). Obviously our theorems do not imply the existence of a measure for the considered stochastic processes. For the existence proofs, see Doob [2], in particular p. 120-123.

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