REMARKS ON THE REALIZABILITY OF WHITEHEAD PRODUCT

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1. A. H. Copeland [2][1] investigated the problem of finding an H-structure of a CW-complex with two non-trivial homotopy groups. In the course of his study, an interesting result is obtained which combine the Eilenberg-MacLane invariant and the Whitehead product of a CW-complex whose non-trivial homotopy groups are of dimensions n and $2n-1$ $(n>1)$ (cf. Proposition 7 of [2]).

The arguments through his paper are true for a connected CW-complex Y with the following properties:

1) the product $Y \times Y$ is a CW-complex whose cells are of the form $E^p \times E^q$ for p-cell E^p and q-cell E^q of Y,

2) for any integer m there exists a CW-complex $X \supset Y$ such that X satisfies the property 1) and the inclusion map induces isomorphisms $\pi_i(X) \approx \pi_i(Y)$ for $1 \leqq i < m$ and $\pi_i(X) = 0$ for $i \geqq m$.

In his paper, it is assumed that Y is a connected locally finite CW-complex. But this may be replaced by a weaker assumption that Y is a connected countable CW-complex[2]. For, if Y is a connected countable CW-complex, then, by Theorem (1.9) of [5], Y has the property 1). On the other hand, by Theorem 13 in § 9 of [7], Y is of the same homotopy type as a locally finite simplex Y'. Hence Y' is connected and so countable[2]. Therefore, using the simplicial approximation theorem we may easily prove that the elements of $\pi_i(Y) \approx \pi_i(Y')$ for each i are countable. Thus we can construct a countable CW-complex $X \supset Y$ such that $\pi_i(X) \approx \pi_i(Y)$ $(1 \leqq i < m)$ and $\pi_i(X) = 0$ $(i \geqq m)$. Since X is countable, its has the property 1). Thus properties 1) and 2) are satisfied for any connected countable CW-complex.

In § 2 we shall prove that Proposition 7 of [2] is also true for any CW-complex and so for any space whose first two non-trivial homotopy groups are of dimensions n and $2n-1 (n>1)$.

In § 3, combining this proposition with results on $H(\Pi, n)$ due to Eilenberg-MacLane [3], we shall give results on the realizability of a given homo-

1) Numbers in brackets refer to the references at the end of the paper.
2) The fact that a connected locally finite CW-complex is countable is noticed in p. 223 of [7].

morphism $T: \Pi \otimes \Pi \to G$ as the Whitehead products in spaces of types $K(\Pi, n; G, 2n-1)$ with $n = 2, 3, 4, 5$.

2. Let Y be an arcwise connected space which has the first two non-trivial homotopy groups Π and G in dimensions n and m with $1 < n < m$. Such a space is said to be of the type $K(\Pi, n; G, m; \ldots\ldots)$ or $K(\Pi, n; G, m; \mathbf{k}; \ldots\ldots)$, where $\mathbf{k} \in H^{m+1}(\Pi, n; G)$ is the Eilenberg-MacLane invariant of Y. As usual, by a space of the type $K(\Pi, n)$ we shall mean a space X such that $\pi_n(X) = \Pi$, $\pi_i(X) = 0$ for $i \neq n$, and it will be denoted by $\mathbf{K}(\Pi, n)$.

Let Π and G be abelian groups. Let $\psi^*, p_1^*, p_2^*: H^{2n}(\Pi, n; G) \to H^{2n}(\Pi + \Pi, n; G)$ be the homomorphisms induced by the maps $\psi, p_1, p_2: \Pi + \Pi \to \Pi$ defined by

$$\psi(a, b) = a + b, \qquad p_1(a, b) = a, \qquad p_2(a, b) = b$$

for $a, b \in \Pi$.

Let $\Theta^*: H^{2n}(\Pi + \Pi, n; G) \to \mathrm{Hom}(\Pi \otimes \Pi, G)$ be the homomorphism determined by the Künneth formula.

We shall refer Proposition 7 of [2], i. e.,

PROPOSITION 1. *Let Y be a countable CW-complex of the type $K(\Pi, n; G, 2n-1; \mathbf{k}; \ldots\ldots)$. Then the Whitehead product $W: \Pi \otimes \Pi \to G$ in Y is given by*

$$W = \Theta^*(\psi^* - p_1^* - p_2^*)\mathbf{k}.$$

We shall prove the following

PROPOSITION 2. *Let Y be any space of the type $K(\Pi, n; G, 2n-1, \mathbf{k}; \ldots\ldots)$. Then the Whitehead product $W: \Pi \otimes \Pi \to G$ in Y is given by*

$$W = \Theta\ (\psi^* - p_1^* - p_2^*)\mathbf{k}.$$

Since W, Θ^*, ψ^*, p_1^*, p_2^* are natural, Proposition 2 may be easily proved by Proposition 1 and the following lemmas.

LEMMA 1. *Let Y and Y_0 be spaces of the types $K(\Pi, n; G, m; \mathbf{k}; \ldots\ldots)$ and $K(\Pi_0, n; G_0, m; \mathbf{k}_0; \ldots\ldots)$ respectively. For a map $h: Y_0 \to Y$ we have relation*

$$f^*\mathbf{k} = g^\#\mathbf{k}_0,$$

where $f: \Pi_0 \to \Pi$ and $g: G_0 \to G$ are homomorphism induced by h, and

$$f^*: H^{m+1}(\Pi, n; G) \to H^{m+1}(\Pi_0, n; G),$$
$$g^\#: H^{m+1}(\Pi_0, n; G_0) \to H^{m+1}(\Pi_0, n; G)$$

are homomorphisms induced by f and g, respectively.

PROOF. Let X be a space obtained from Y by attaching i-cells ($i \geqq m + 1$) such that $\pi_i(X) \approx \pi_i(Y)$, $1 \leqq i < m$ and $\pi_i(X) = 0$, $i \geqq m$. Let $\mathbf{k}' \in H^{m+1}(X, Y; G)$ be the first obstruction to retracting X onto Y. Then $\mathbf{k} = j^*\mathbf{k}'$, where $j^*: H^{m+1}(X, Y; G) \to H^{m+1}(X; G)$ is the homomorphism induced by the inclusion map, and $H^{m+1}(X; G)$ is identified with $H^{m+1}(\Pi, n; G)$ under the natural isomorphism. Let X_0, $\mathbf{k}_0'$ and j_0^* be similar to X, $\mathbf{k}'$ and j .

The map $h: Y_0 \to Y$ has an extension $\bar{h}: X_0 \to X$, and we have $\bar{h}_1^*\mathbf{k}' = \bar{g}_1^{\#}\mathbf{k}_0'$, where

$$\bar{h}_1^*: H^{m+1}(X, Y; G) \to H^{m+1}(X_0, Y_0; G),$$
$$\bar{g}_1^{\#}: H^{m+1}(X_0, Y_0; G_0) \to H^{m+1}(X_0, Y_0; G)$$

are homomorphisms induced by $\bar{h}$ and g.

In the following diagram, commutativities hold:

$$\begin{array}{ccccc}
H^{m+1}(X, Y; G) & \xrightarrow{\bar{h}_1^*} & H^{m+1}(X_0, Y_0; G) & \xleftarrow{\bar{g}^{\#}} & H^{m+1}(X_0, Y_0; G) \\
\downarrow j^* & & \downarrow j_1^* & & \downarrow j_0^* \\
H^{m+1}(X; G) & \xrightarrow{\bar{h}^*} & H^{m+1}(X_0; G) & \xleftarrow{\bar{g}^{\#}} & H^{m+1}(X_0; G).
\end{array}$$

Therefore, since $\bar{h}^* = f^*$, $g^{\#} = \bar{g}^{\#}$, we have

$$f^*\mathbf{k} = \bar{h}^* j^*\mathbf{k}' = j_1^*\bar{h}_1^*\mathbf{k}' = j_1^*\bar{g}_1^{\#}\mathbf{k}_0' = \bar{g}^{\#}j_0^*\mathbf{k}_0' = \bar{g}^{\#}\mathbf{k}_0,$$

i. e.,
$$f^*\mathbf{k} = g^{\#}\mathbf{k}_0.$$
q. e. d.

LEMMA 2. *Let Y be a CW-complex of the type $K(\Pi, n; G, m, \mathbf{k}; \ldots\ldots)$ and abelian groups Π_0, G_0 and homomorphisms $f: \Pi_0 \to \Pi$, $g: G_0 \to G$ be given. Let a cocycle $k_0 \in Z^{m+1}(\Pi_0, n; G_0)$, such that $f^*k = g^{\#}k_0$ for some cocycle k belonging to $\mathbf{k}$, be given, where $f^*: Z^{m+1}(\Pi, n; G) \to Z^{m+1}(\Pi_0, n; G)$, $g^{\#}: Z^{m+1}(\Pi_0, n; G_0) \to Z^{m+1}(\Pi_0, n; G)$ be homomorphisms induced by f and g. Then there exist a CW-complex Y_0 of the type $K(\Pi_0, n; G_0, m; \mathbf{k}_0; \ldots\ldots)$ and a map $h: Y_0 \to Y$ which induces f and g, where $\mathbf{k}_0$ is the cohomology class of k_0. Moreover, if Π_0, G_0 are countable groups, then Y_0 may be chosen to be a countable CW-complex.*

PROOF. We shall consider a CW-complex $|K(G, m + 1)|$ which is the geometric realization of the Eilenberg-MacLane complex $K(G, m + 1)$. Let E be the space of paths in $|K(G, m + 1)|$ terminating in the unique 0-cell of $|K(G, m + 1)|$ with the fibre map $p: E \to |K(G, m + 1)|$ and the fibre $\mathbf{K}(G, m)$. Let $b \in Z^{m+1}(G, m + 1; G)$ be the basic cocycle and $\mathbf{b} \in H^{m+1}$

$(G, m+1 ; G)$ be its cohomology class. By Theorem 5. 1 of [4], there exists a c. s. s. map $\lambda: K(\Pi, n) \to K(G, m+1)$ such that $\lambda^*(b) = k$, where λ^* denotes the cochain map induced by λ. Then λ defines a map $|\lambda| : |K(\Pi, n)| \to |K(G, m+1)|$ and $|\lambda|$ induces a space Y' and maps q, F such that the diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\quad F \quad} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ |K(\Pi, n)| & \xrightarrow{\quad |\lambda| \quad} & |K(G, m+1)| \end{array}$$

is commutative and Y' is a fibre space over $|K(\Pi, n)|$. Since $|K(\Pi, n)|$ is a space of the type $K(\Pi, n)$ and $|\lambda|^*(\mathbf{b}) = \mathbf{k}$, Y' is a space of the type $K(\Pi, n ; G, m ; \mathbf{k} ; \ldots\ldots)$. (cf. Proof of Proposition 9 of [2]). Therefore the geometric realization $|S(Y')|$ of the singular complex of Y' is also a space of the type $K(\Pi, n ; G, m ; \mathbf{k} ; \ldots\ldots)$. Hence $|S(Y')|$ and Y are of the same homotopy type and so there exists a map

$$h_1 : |S(Y')| \to Y$$

which induces the identities on homotopy groups.

Similarly we shall consider the diagram

$$\begin{array}{ccc} Y_0' & \xrightarrow{\quad F_0 \quad} & E_0 \\ {\scriptstyle q_0}\downarrow & & \downarrow{\scriptstyle p_0} \\ |K(\Pi_0, n)| & \xrightarrow{\quad |\mu| \quad} & |K(G_0, m+1)| \end{array}$$

where $\mu : K(\Pi_0, n) \to K(G_0, m+1)$ is a c.s.s. map such that $\mu^*(b_0) = k_0$ for the basic cocycle $b_0 \in Z^{m+1}(G_0, m+1 ; G_0)$. The space Y_0' is also of the type $K(\Pi_0, n ; G_0, m ; \mathbf{k}_0 ; \ldots\ldots)$.

The homomorphisms f and g induce c.s.s. maps $K(\Pi_0, n) \to K(\Pi, n)$ and $K(G_0, m+1) \to K(G, m+1)$, and these maps are denoted again by f and g respectively. Then $|g| : |K(G_0, m+1)| \to |K(G, m+1)|$ induces a map $\bar{g} : E_0 \to E$ such that $p \circ \bar{g} = |g| \circ p_0$.

Since $g^* b = g^\# b_0$, by $f^* k = g^\# k_0$, we have

$$\begin{aligned} (\lambda f)^* b &= (f^* \lambda^*) b = f^* k = g^\# k_0 \\ &= g^\# \mu^*(b_0) = \mu^* g^\# b_0 = \mu^* g^* b = (g\mu)^* b, \end{aligned}$$

i. e.,

$$(\lambda f)^* b = (g\mu)^* b.$$

Therefore, by Theorem 5.1 of [4], we know that $\lambda f = g\mu$, hence $|\lambda|\circ|f| = |g|\circ|\mu|$. Therefore we can define a map

$$h_0 : Y_0' \to Y$$

by $h_0(r, s) = (|f|(r), |g|(s))$ for $r \in |K(\Pi_0, n)|$, $s \in E_0$ ($p_0(s) = |\mu|(r)$). Then we have a commutative diagram:

$$\begin{array}{ccccccc}
 & & & \bar{g} & & & \\
E_0 & \xleftarrow{F_0} & Y_0' & \xrightarrow{h_0} & Y' & \xrightarrow{F} & E \\
\downarrow p_0 & & \downarrow q_0 & & \downarrow q & & \downarrow p \\
K(G_0, m+1) & \xleftarrow{|\mu|} & K(\Pi_0, n) & \xrightarrow{|f|} & K(\Pi, n) & \xrightarrow{|\lambda|} & K(G, m+1). \\
 & & & |g| & & &
\end{array}$$

Therefore it is easily seen that h_0 induces the homomorphisms f and g on homotopy groups. Thus if we put $Y_0 = |S(Y_0')|$ and $h = h_1\circ|\bar{h}_0|$, then Y_0 and h have the required properties, where $\bar{h}_0 : S(Y_0') \to S(Y')$ is the c.s.s. map induced by h.

If Π_0 and G_0 are countable groups, then by Theorem (5.1) of [1], we know that the minimal subcomplex M of $S(Y_0')$ is countable. Therefore $|M|$ and $h|\,|M|$ have the required properties. q. e. d.

LEMMA 3. *Let Π, G be abelian groups and we assume that Π is countable. For any element $\mathbf{k} \in H^{m+1}(\Pi, n ; G)$ there exist a countable subgroup $G_0 \subset G$ and an element $\mathbf{k}_0 \in H^{m+1}(\Pi\ n ; G_0)$ such that $\mathbf{k} = g^{\#}\mathbf{k}_0$, where $g^{\#} : H^{m+1}(\Pi, n ; G_0) \to H^{m+1}(\Pi, n; G)$ is the homomorphism induced by the inclusion map $G_0 \subset G$.*

PROOF. By the universal coefficient theorem $H^{m+1}(\Pi, n ; G) = \mathrm{Hom}\ (H_{m+1}(\Pi, n), G) + \mathrm{Ext}\ (H_m(\Pi, n)\ G)$, hence we have $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ for some $\mathbf{k}_1 \in \mathrm{Hom}\ (H_{m+1}(\Pi, n), G)$ and $\mathbf{k}_2 \in \mathrm{Ext}\ (H_m(\Pi, n), G)$. Since Π is countable, the complex $K(\Pi, n)$ is countable, hence $H_i(\Pi, n)$ for each i is a countable group. Hence $G_1 = \mathbf{k}_1(H_{m+1}(\Pi, n))$ is countable.

Next, we shall consider an exact sequence

$$0 \to R \xrightarrow{i} F \xrightarrow{j} H_m(\Pi, n) \to 0,$$

where F is a free group. Since $H_m(\Pi, n)$ is countable, we may assume that F and also R are countable. By the definition of Ext,

$$\mathrm{Ext}\ (H_m(\Pi, n), G) = \mathrm{Hom}\ (R, G)/i^*\mathrm{Hom}\ (F, G),$$

hence we can choose an element $a \in \mathrm{Hom}\,(R, G)$ which represents $\mathbf{k}_2$. Then $a(R) = G_2$ is countable. Hence $G_0 = G_1 \cup G_2$ is countable and it is obvious that there exists an element $\mathbf{k}_0 \in H^{m+1}(\Pi, n\,;\, G_0)$ such that $\mathbf{k} = g^{\#}\mathbf{k}_0$.

q. e. d.

3. Let Π, G be abelian groups. Then for integers n, m with $1 < n < m$ and for each element $\mathbf{k} \in H^{m+}(\Pi, n\,;\, G)$ there exists a space of the type $K(\Pi, n\,;\, G, m\,;\, \mathbf{k}\,;\ldots\ldots)$. Therefore, by Proposition 2, in order that a given homomorphism $W: \Pi \otimes \Pi \rightarrow G$ is realizable as the Whitehead product in a space of the type $K(\Pi, n\,;\, G, 2n-1\,;\ldots\ldots)$ it is necessary and sufficient that $W \in \Theta^*(\psi^* - p_1^* - p_2^*)H^{2n}(\Pi, n\,;\, G)$. For $n = 2, 3, 4, 5$, $\Theta^*(\psi^* - p_1^* - p_2^*)$ $H^{2n}(\Pi, n\,;\, G)$ are computable and we have the following

THEOREM 1. *In order that a given homomorphism* $W: \Pi \otimes \Pi \rightarrow G$ *is realizable as the Whitehead product in a space of the type* $K(\Pi, n\,;\, G, 2n - 1\,;\ldots\ldots)$ *for* $n = 2$ *or* 4, *it is necessary and sufficient that there exists a map* $\eta: \Pi \rightarrow G$ *such that* $\eta(x) = \eta(-x)$, *and* $W(x \otimes y) = \eta(x + y) - \eta(x) - \eta(y)$ *for any* $x, y \in \Pi$.

THEOREM 2.[3)] *In order that a given homomorphism* $W: \Pi \otimes \Pi \rightarrow G$ *is realizable as the Whitehead product in a space of the type* $K(\Pi, n\,;\, G, 2n - 1\,;\ldots\ldots)$ *for* $n = 3$ *or* 5, *it is necessary and sufficient that* $W(x \otimes x) = 0$ *for any* $x \in \Pi$.

PROOF OF THEOREM 1. We shall consider the following commutative diagram which is seen in the proof of Theorem 21. 1 of [3]:

$$\begin{array}{ccc} \Gamma(\Pi_1) \otimes \Gamma(\Pi_2) & \xrightarrow{\theta_{1,\Gamma} \otimes \theta_{2,\Gamma}} & H(\Pi_1, n) \otimes H(\Pi_2, n) \\ \Big\downarrow g & & \Big\downarrow \pi_* \\ \Gamma(\Pi_1 + \Pi_2) & \xrightarrow{\theta_\Gamma} & H(\Pi_1 + \Pi_2, n). \end{array} \qquad (n : \text{even})$$

If we restrict to the subgroups of degree 4 and if we put $n = 2$, this diagram gives the following commutative diagram:

$$\begin{array}{ccc} \Gamma_4(\Pi_1) + \Gamma_4(\Pi_2) + \Pi_1 \otimes \Pi_2 & \xrightarrow{\Psi} & H_4(\Pi_1, 2) + H_4(\Pi_2, 2) + \Pi_1 \otimes \Pi_2 \\ \Big\downarrow g & & \Big\downarrow \pi^* \\ \Gamma_4(\Pi_1 + \Pi_2) & \xrightarrow{\theta_4} & H_4(\Pi_1 + \Pi_2, 2), \end{array}$$

3) Theorem 2 for $n=3$ covers Theorem 8 of [6].

where $\Gamma_4(\Pi_1)\otimes 1$, $H_4(\Pi_1, 2)\otimes H_0(\Pi_2, 2)$ and $\Gamma_0(\Pi_1)$ etc. are naturally identified with $\Gamma_4(\Pi_1)$, $H_4(\Pi_1, 2)$ and Π_1 etc. respectively. And under these identifications, g and Ψ are defined by

$$\begin{cases} g(\gamma_4(x)) = \gamma_4(x, 0), \\ g(\gamma_4(y)) = \gamma_4(0, y), \qquad (x \in \Pi_1, y \in \Pi_2) \\ g(x\otimes y) = \gamma_4(x, y) - \gamma_4(x, 0) - \gamma_4(0, y), \end{cases}$$

$$\Psi = \begin{cases} \theta_{1,\Gamma} & \text{on } \Gamma_4(\Pi_1), \\ \theta_{2,\Gamma} & \text{on } \Gamma_4(\Pi_2), \\ \text{identity} & \text{on } \Pi_1\otimes\Pi_2. \end{cases}$$

By Theorem 18.4 and Theorem 21.1 of [3], g and Ψ are onto isomorphisms. Let $i: \Pi_1\otimes\Pi_2 \to H_4(\Pi_2, 2) + H_4(\Pi_2, 2) + \Pi_1\otimes\Pi_2$ and $i: \Pi_1\otimes\Pi_2\to\Gamma_4(\Pi_1) + \Gamma_4(\Pi_2) + \Pi_1\otimes\Pi_2$ be the inclusion maps. Then the composition homomorphism $i\circ\pi_*$ induces the homomorphism

$$\mathrm{Hom}\,(i\circ\pi_*): \mathrm{Hom}\,(H_4(\Pi_1 + \Pi_2, 2), G) \to \mathrm{Hom}\,(\Pi_1\otimes\Pi_2, G).$$

Since $H_3(\Pi, 2) = 0$, by the universal coefficient theorem we have

$$H^4(\Pi_1 + \Pi_2, 2;\ G) = \mathrm{Hom}\,(H_4(\Pi_1 + \Pi_2, 2), G),$$

and if we put $\Pi = \Pi_1 = \Pi_2$, then $\mathrm{Hom}\,(i\circ\pi_*)$ is Θ^* in Proposition 2. Thus, from the above diagram and the naturality of θ_4 we have the following commutative diagram:

$$\begin{array}{ccc}
H^4(\Pi, 2;\ G) & \xrightarrow{\mathrm{Hom}\,(\theta_4)} & \mathrm{Hom}\,(\Gamma_4(\Pi), G) \\
\downarrow{\scriptstyle \psi^* - p_1^* - p_2^*} & & \downarrow{\scriptstyle \bar{\psi}^* - \bar{p}_1^* - \bar{p}_2^*} \\
H^4(\Pi+\Pi, 2; G) = \mathrm{Hom}(H_4(\Pi+\Pi, 2), G) & \xrightarrow{\mathrm{Hom}\,(\theta_4)} & \mathrm{Hom}\,(\Gamma_4(\Pi + \Pi), G) \\
\downarrow{\scriptstyle \mathrm{Hom}\,(\pi_*)} & & \downarrow{\scriptstyle \mathrm{Hom}\,(g)} \\
\mathrm{Hom}\,(H_4(\Pi, 2) + H_4(\Pi, 2) + \Pi\otimes\Pi, G) & \xrightarrow{\mathrm{Hom}\,(\Psi)} & \mathrm{Hom}\,(\Gamma_4(\Pi) + \Gamma_4(\Pi) + \Pi\otimes\Pi, G) \\
& {\scriptstyle \mathrm{Hom}\,(i)} \searrow \quad \swarrow {\scriptstyle \mathrm{Hom}\,(i)} & \\
& \mathrm{Hom}\,(\Pi\otimes\Pi, G) &
\end{array}$$

where $\bar{\psi}$, $\bar{p}_i: \Gamma_4(\Pi + \Pi) \to \Gamma_4(\Pi)$ are homomorphisms induced by ψ, p_i and $\bar{\psi}^* = \mathrm{Hom}\,(\bar{\psi})$, $\bar{p}_i^* = \mathrm{Hom}\,(\bar{p}_i)$.

Therefore we have

$$\Theta^*(\psi^* - p_1^* - p_2^*)$$
$$= \mathrm{Hom}\,(g\circ i)\circ(\bar{\psi}^* - p_1^* - p_2^*)\circ\mathrm{Hom}\,(\theta_4).$$

Since θ_4 is an onto isomorphism, we can identify $H^4(\Pi, 2; G)$ with Hom $(\Gamma_4(\Pi), G)$ under the isomorphism Hom (θ_4). Then we have

$$\Theta^*(\psi^* - p_1^* - p_2^*) = \mathrm{Hom}\,(g\circ i)\circ(\bar{\psi}^* - \bar{p}_1^* - \bar{p}_2^*).$$

Thus, for $k \in \mathrm{Hom}\,(\Gamma_4(\Pi), G)$ and $x, y \in \Pi$ we have

$$[\Theta^*(\psi^* - p_1^* - p_2^*)k](x \otimes y)$$
$$= [\mathrm{Hom}\,(g\circ i)\circ(\bar{\psi}^* - \bar{p}_1^* - \bar{p}_2^*)k](x \otimes y)$$
$$= k\gamma_4(x + y) - k\gamma_4(x) - k\gamma_4(y).$$

Therefore, if we put $\eta(x) = k\gamma_4(x)$, then we have

$$[\Theta^*(\psi^* - p_1^* - p_2^*)k](x \otimes y) = \eta(x + y) - \eta(x) - \eta(y),$$

and since $\gamma_4(x) = \gamma_4(-x)$, $\eta(x)$ satisfies the condition $\eta(x) = \eta(-x)$.

Conversely, let $T: \Pi \otimes \Pi \to G$ be a given homomorphism and if $T(x \otimes y) = \eta(x + y) - \eta(x) - \eta(y)$ for some map $\eta: \Pi \to G$ such that $\eta(x) = \eta(-x)$, then $T(x \otimes (y + z)) = T(x \otimes y) + T(x \otimes z)$ implies the relation

$$\eta(x + y + z) - \eta(y + z) - \eta(z + x) - \eta(x + y)$$
$$+ \eta(x) + \eta(y) + \eta(z) = 0.$$

Therefore, there exists a homomorphism $k: \Gamma_4(\Pi) \to G$ such that $k\gamma_4(x) = \eta(x)$. Hence $\Theta^*(\psi^* - p_1^* - p_2^*)k = T$. Thus the proof for $n = 2$ is complete.

By Theorems 24. 1, 24. 2 and 27. 3 of [3]

$$\theta_7: {}_2\Pi \simeq H_7(\Pi, 4),$$
$$\theta_8: \Gamma_4(\Pi) + \Pi/3\Pi \simeq H_8(\Pi, 4),$$
$$\theta^8: H^8(\Pi, 4; G) \simeq \mathrm{Hom}\,({}_2\Pi, G/2G)$$
$$+ \mathrm{Hom}\,(\Gamma_4(\Pi), G) + \mathrm{Hom}\,(\Pi/3\Pi, G).$$

But it is easily seen that $\Theta^*(\psi^* - p_1^* - p_2^*)$ is trivial on the first and third summands of $H^8(\Pi, 4; G)$. Therefore the proof for $n = 4$ is reduced to the above proof for $n = 2$. Thus the proof of Theorem 1 is complete.

PROOF OF THEOREM 2. The proof is similar to that of Theorem 1, and so we shall sketch the proof. We shall consider an isomorphism

$$g: \Lambda_2(\Pi_1) + \Lambda_2(\Pi_2) + \Pi_1 \otimes \Pi_2 \to \Lambda_2(\Pi_1 + \Pi_2) \qquad (\Pi = \Pi_1 = \Pi_2)$$

defined by

$$g(x \wedge x') = (x, 0) \wedge (x', 0),$$
$$g(y \wedge y') = (0, y) \wedge (0, y'),$$
$$g(x \otimes y) = (x, 0) \wedge (0, y)$$

for $x, x' \in \Pi_1$, $y, y' \in \Pi_2$.
This isomorphism is the restriction of g on the subgroup of degree 4 which is defined in Theorem 19. 2 of [3].

Let $i: \Pi_1 \otimes \Pi_2 \to \Lambda_2(\Pi_1) + \Lambda_2(\Pi_2) + \Pi_1 \otimes \Pi_2$ be the inclusion map. Then, by the similar argument with that in the proof of Theorem 1 we know that

$$\Theta^*(\psi^* - p_1^* - p_2^*)H^{2n}(\Pi, n; G)$$
$$= \mathrm{Hom}\,(g \circ i) \circ (\bar{\psi}^* - \bar{p}_1^* - \bar{p}_2^*) \circ \mathrm{Hom}\,(\Lambda_2(\Pi), G)$$

for $n = 3$ or 5.
Since $\Lambda_2(\Pi)$ is $\Pi \otimes \Pi$ modulo the diagonal, this proves Theorem 2.

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