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REMARKS ON THE SCHROEDINGER OPERATOR WITH SINGULAR COMPLEX POTE--ETC(U)
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REMARKS ON THE SCHRÖDINGER OPERATOR WITH SINGULAR COMPLEX POTENTIALS

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REMARKS ON THE SCHRODINGER OPERATOR WITH SINGULAR COMPLEX POTENTIALS

Haim Brezis[1,3] and Tosio Kato[2,4]

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ABSTRACT

Let $A = -\Delta + V(x)$ be a Schrödinger operator on an (arbitrary) open set $\Omega \subset \mathbb{R}^m$, where $V \in L^1_{loc}(\Omega)$ is a complex valued function. We consider the "maximal" realization of A in $L^2(\Omega)$ under Dirichlet boundary condition, that is

$$D(A) = \{u \in H^1_0(\Omega);\ Vu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + Vu \in L^2(\Omega)\} .$$

When $\Omega = \mathbb{R}^m$ we also consider the operator

$$A_1 = -\Delta + V$$

with domain

$$D(A_1) = \{u \in L^2(\Omega);\ Vu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + Vu \in L^2(\Omega)\} .$$

A special case of our main results is:

Theorem: Let $m \geq 3$; assume that the function $\max\{-\text{Re}\, V, 0\}$ belongs to $L^\infty(\Omega) + L^{m/2}(\Omega)$ and also to $L^{(m/2)+\varepsilon}_{loc}(\Omega)$ for some $\varepsilon > 0$. Then A (resp. A_1) is closable and $\bar{A} + \lambda$ (resp. $\bar{A}_1 + \lambda$) is m-accretive for some real constant λ.

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SIGNIFICANCE AND EXPLANATION

Schrödinger operators of the form $A = -\Delta + V(x)$, where Δ is the Laplacian and V is a scalar potential, arise in quantum mechanics and other areas. Delicate questions concerning what domain should be assigned to A must be settled in order to have a good theory. These questions are answered here for a very general class of potentials V which may even have complex values.

The responsibilities for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

REMARKS ON THE SCHRÖDINGER OPERATOR WITH SINGULAR COMPLEX POTENTIALS

Haïm Brezis[1,3] and Tosio Kato[2,4]

1. Introduction

Let $A = -\Delta + V(x)$ be a Schrödinger operator on an (arbitrary) open set $\Omega \subset \mathbb{R}^m$, where $V \in L^1_{loc}(\Omega)$ is a complex valued function. We consider the "maximal" realization of A in $L^2(\Omega)$ under Dirichlet boundary condition, that is

$$D(A) = \{u \in H^1_0(\Omega);\ Vu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + Vu \in L^2(\Omega)\}.$$

When $\Omega = \mathbb{R}^m$ we also consider the operator

$$A_1 = -\Delta + V$$

with domain

$$D(A_1) = \{u \in L^2(\Omega);\ Vu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + Vu \in L^2(\Omega)\}.$$

We state now our main results (see Theorems 3.1 and 3.2) in a special case.

Theorem: Let $m \geq 3$; assume that the function $\max\{-\mathrm{Re}\,V,0\}$ belongs to $L^\infty(\Omega) + L^{m/2}(\Omega)$ and also to $L^{(m/2)+\varepsilon}_{loc}(\Omega)$ for some $\varepsilon > 0$. Then A (resp. A_1) is closable and $\bar{A} + \lambda$ (resp. $\bar{A}_1 + \lambda$) is m-accretive for some real constant λ.

We emphasize the fact that $\max\{\mathrm{Re}\,V,0\}$ and $\mathrm{Im}V$ could be arbitrary functions in $L^1_{loc}(\Omega)$.

Our methods rely on some measure theoretic arguments and standard techniques of DeGiorgi-Moser-Stampacchia type, related to the weak form of the maximum principle.

The distributional inequality

$$\Delta|u| \geq \mathrm{Re}[\Delta u \operatorname{sign} \bar{u}]$$

proved in [3] plays a crucial role. We also use a result from [1] concerning a property of Sobolev spaces.

In order to describe our method in a simple case we begin in Section 2 with real valued potentials. The main results in Section 2 are essentially known (see [3], [4], [8]) - except perhaps for Theorem 2.2 when $m \leq 4$.

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In Section 3 we turn to the case of complex potentials. Schrödinger operators with complex potentials have been studied by Nelson [6]. His results were extended in [5]. Here we allow more general singularities.

We thank Professors R. Jensen and B. Simon for useful suggestions and discussions (with the first author).

2. Real valued potentials

Let Ω be an (arbitrary) open subset of $\mathbb{R}^m$ and let $H = L^2 = L^2(\Omega;\mathbb{C})$. Let $q \in L^1_{loc}(\Omega)$ be a real valued function. Set

$$q^+ = \max(q,0), \quad q^- = \max(-q,0) .$$

Assume

$$q^- \in L^\infty(\Omega) + L^p(\Omega) \tag{1}$$

with

$$\begin{cases} p = \frac{m}{2} & \text{when} \quad m \geq 3 \\ p > 1 & \text{when} \quad m = 2 \\ p = 1 & \text{when} \quad m = 1 . \end{cases}$$

Consider the operator A defined in H by

$$A = -\Delta + q(x)$$

with

$$D(A) = \{u \in H^1_0(\Omega);\ qu \in L^1_{loc}(\Omega) \ \text{and} \ -\Delta u + qu \in L^2(\Omega)\} .$$

The main results are the following:

<u>Theorem 2.1</u>. A is self-adjoint and $A + \lambda_1$ is m-accretive for some real constant λ_1. Furthermore $u,v \in D(A)$ imply $q|u|^2 \in L^1(\Omega)$, $q|v|^2 \in L^1(\Omega)$ and

$$(Au,v) = \int \text{grad}u\ \text{grad}\bar{v} + \int qu\bar{v} . \tag{2}$$

When $\Omega = \mathbb{R}^m$ we also consider the operator A_1 defined in H by

$$A_1 = -\Delta + q(x)$$

with

$$D(A_1) = \{u \in L^2(\Omega);\ qu \in L^1_{loc}(\Omega) \ \text{and} \ -\Delta u + qu \in L^2(\Omega)\} .$$

<u>Only when $m = 3$ or $m = 4$</u> we will make the additional assumption:

(3) $q^- \in L^{p+\epsilon}_{loc}(\Omega)$ with $p = \frac{3}{2}$ when $m = 3$ and $p = 2$ when $m = 4$, for some arbitrarily small $\epsilon > 0$.

More precisely we assume that for each $x_0 \in \mathbb{R}^m$ there exists a neighborhood U of x_0 and some $\epsilon > 0$ (depending on x_0) such that $q^- \in L^{p+\epsilon}(U)$.

<u>Theorem 2.2</u>: Under the assumptions (1) and (3), $A_1 = A$.

Our first lemma is well known:

__Lemma 2.1__: Assume (1). Then for every $\varepsilon > 0$, there exists a constant λ_ε such that

$$\int q^-|u|^2 \leq \varepsilon\|\text{grad}u\|^2_{L^2} + \lambda_\varepsilon\|u\|^2_{L^2} \qquad \forall u \in H^1_0(\Omega) .$$

In particular

$$\int q^-|u|^2 \leq \|\text{grad}u\|^2_{L^2} + \lambda_1\|u\|^2_{L^2} \qquad \forall u \in H^1_0(\Omega) .$$

__Proof__: Write $q^- = q_1 + q_2$ with $q_1 \in L^\infty(\Omega)$ and $q_2 \in L^p(\Omega)$. Then for each $k > 0$ we have

$$\int q^-|u|^2 \leq \|q_1\|_{L^\infty}\|u\|^2_{L^2} + \int_{[|q_2|>k]} |q_2||u|^2 + k \int_{[|q_2|\leq k]} |u|^2$$

$$\leq (\|q_1\|_{L^\infty} + k)\|u\|^2_{L^2} + \|q_2\|_{L^p([|q_2|>k])}\|u\|^2_{L^t}$$

with

$$\frac{1}{p} + \frac{2}{t} = 1 .$$

In case $m \geq 3$ we find $t = 2^*$ where 2^* is the Sobolev exponent, that is $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{m}$.
By the Sobolev imbedding theorem we have

$$\|u\|_{L^t} \leq C\|\text{grad}u\|_{L^2} \qquad \forall u \in H^1_0(\Omega) .$$

When $m = 2$ we find $2 < t < \infty$ and it is known that

$$\|u\|_{L^t} \leq C(\|\text{grad}u\|_{L^2} + \|u\|_{L^2}) \qquad \forall u \in H^1_0(\Omega) .$$

When $m = 1$ we find $t = \infty$ and it is known that

$$\|u\|_{L^\infty} \leq C(\|\text{grad}u\|_{L^2} + \|u\|_{L^2}) \qquad \forall u \in H^1_0(\Omega) .$$

We reach the conclusion of Lemma 2.1 in all the cases by choosing k large enough so that

$$C^2\|q_2\|_{L^p([|q_2|>k])} < \varepsilon .$$

__Remark 2.1__: Assumption (1) is used in all the results of this paper only through Lemma 2.1 and it may in fact be weakened to a "locally uniform L^p-condition":

(1') $$\|q^-\|_{L^p(\Omega\cap B_r(y))} \to 0 \text{ as } r \to 0 \text{ uniformly in } y \in \Omega ,$$

where

$$B_r(y) = \{x \in \mathbb{R}^m;\ |x - y| \le r\} .$$

Indeed let $\varphi \in \mathcal{D}_+(\mathbb{R}^m)$ with $\operatorname{supp}\varphi \subset B_r(0)$ and $\|\varphi\|_{L^2} = 1$. Then, writing $\varphi_y(x) = \varphi(x - y)$,

$$\int q^-|u|^2 = \int dy \int q^-|u\varphi_y|^2 \le \int \|q^-\|_{L^p(B_r(y))} \|u\varphi_y\|^2_{L^t} dy .$$

Here $\|q^-\|_{L^p(B_r(y))} \le \delta$ for any small δ by (1') if r is chosen small. So

$$\int q^-|u|^2 \le \delta \int \|u\varphi_y\|^2_{L^t} dy \le C\delta \int \|\operatorname{grad}(u\varphi_y)\|^2_{L^2} dy$$

$$\le 2C\delta \int (\|\varphi_y \operatorname{grad} u\|^2_{L^2} + \|u \operatorname{grad}\varphi_y\|^2_{L^2}) dy$$

$$= 2C\delta(\|\operatorname{grad} u\|^2_{L^2} + C_r\|u\|^2_{L^2}) .$$

Choosing δ so that $2C\delta = \varepsilon$, one gets the conclusion of Lemma 2.1. Such a locally uniform L^p-condition was used by Simader [7].

We recall a result of [1] which will be used in the proof of Theorem 2.1[(1)].

<u>Lemma 2.2</u>: Let $T \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ and let $u \in H^1_0(\Omega)$ be such that a.e. on Ω

$$\operatorname{Re}T \cdot \bar{u} \ge f$$

for some real valued function $f \in L^1(\Omega)$. Then $\operatorname{Re} T \cdot \bar{u} \in L^1(\Omega)$ and

$$\operatorname{Re}\langle T,u\rangle = \int \operatorname{Re}T \cdot \bar{u}$$

where $\langle T,u\rangle$ denotes the Hermitian scalar product in the duality between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

The proof of Theorem 2.1 is divided into 4 steps.

<u>Step 1</u>: $A + \lambda$ is onto for $\lambda > \lambda_1$. Set $q^+_n = \min(q^+,n)$; by a Theorem of Lax-Milgram there exists a unique function $u_n \in H^1_0(\Omega)$ which satisfies

$$(4) \qquad -\Delta u_n + (q^+_n - q^-)u_n + \lambda u_n = f .$$

---

(1) The use of this sort of lemma in this context was suggested by M. Crandall.

(Note that by Lemma 2.1 the sesquilinear form $\int q^- u\bar{v}$ is continuous on $H_0^1(\Omega)$).

Multiplying (4) by $\bar{u}_n$ we find a constant C independent of n such that

(5) $$\|u_n\|_{H^1} \le C ,$$

(6) $$\int q_n^+ |u_n|^2 \le C .$$

Choose a subsequence denoted again by u_n such that $u_n \to u$ weakly in $H_0^1(\Omega)$ and $u_n \to u$ a.e. on Ω. It follows from Fatou's Lemma and (6) that $q^+|u|^2 \in L^1(\Omega)$. We deduce that $qu \in L^1_{loc}(\Omega)$; indeed

$$q^+|u| \le \frac{1}{2} q^+(|u|^2 + 1) \in L^1_{loc}(\Omega) ,$$

$$q^-|u| \le \frac{1}{2} q^-(|u|^2 + 1) \in L^1_{loc}(\Omega) .$$

We pass now to the limit in (4) and prove that $-\Delta u + qu + \lambda u = f$ in $\mathcal{D}'(\Omega)$. It suffices to show that

$$(q_n^+ - q^-)u_n \to qu \text{ in } L^1_{loc}(\Omega) .$$

For this purpose we adapt a device due to W. Strauss [9] and extensively used in the study of strongly nonlinear equations. In view of Vitali's convergence theorem, it suffices to verify that given $\omega \subset\subset \Omega$, then $\forall \varepsilon > 0, \exists \delta > 0$ such that $E \subset \omega$ and $|E| < \delta$ imply $\int_E |q_n^+ - q^-||u_n| < \varepsilon$ for all n. But for every $R > 0$ we have

$$q_n^+|u_n| \le \frac{1}{2} q_n^+(R + \frac{1}{R}|u_n|^2)$$

and thus, by (6),

$$\int_E q_n^+|u_n| \le \frac{1}{2} R \int_E q^+ + \frac{1}{2R} C .$$

We fix R large enough so that $\frac{C}{R} < \varepsilon$ and then $\delta > 0$ so small that $R \int_E q^+ < \varepsilon$. We proceed similarly with $q^-|u_n|$.

Step 2: $A + \lambda_1$ is accretive. Let $u \in D(A)$ and set $T = qu$. Since $T \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ and

$$\text{Re } T\bar{u} = q|u|^2 \ge -q^-|u|^2 \in L^1(\Omega)$$

it follows from Lemma 2.2 that $q|u|^2 \in L^1$ and

$$\text{Re}(T,u) = \int q|u|^2 .$$

But $qu = Au + \Delta u$ and so

$$\mathrm{Re}\langle Au,u\rangle - \int |\mathrm{grad} u|^2 = \int q|u|^2 .$$

Since $Au \in L^2(\Omega)$ we have in fact

$$\mathrm{Re}(Au,u) = \int |\mathrm{grad} u|^2 + \int q|u|^2 \geq -\lambda_1 \int |u|^2$$

by Lemma 2.1.

Step 3: $u \in D(A)$ implies $q|u|^2 \in L^1(\Omega)$ and (2) holds. We have just seen in Step 2 that $u \in D(A)$ implies $q|u|^2 \in L^1(\Omega)$. Now let $u,v \in D(A)$ and set $T = qu$. We have $T \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ and

$$\mathrm{Re}\, T \cdot \bar{v} = \mathrm{Re}\, qu\bar{v} \geq -\frac{1}{2}|q||u|^2 - \frac{1}{2}|q||v|^2 \in L^1(\Omega)$$

and therefore

$$\mathrm{Re}\langle T,v\rangle = \int \mathrm{Re}\, qu\bar{v} .$$

Thus

$$\mathrm{Re}(Au,v) - \mathrm{Re}\int \mathrm{grad} u\, \mathrm{grad}\bar{v} = \mathrm{Re}\int qu\bar{v} .$$

Changing u into iu we find

$$(Au,v) = \int \mathrm{grad} u\, \mathrm{grad}\bar{v} + \int qu\bar{v} .$$

Step 4: A is self-adjoint. Indeed $A + \lambda_1$ is m-accretive and symmetric. Therefore $A + \lambda_1$ is self-adjoint and so is A.

Proof of Theorem 2.2: Clearly $A \subset A_1$. Let $u \in D(A_1)$ and set $f = A_1 u + \lambda u$ with some $\lambda > \lambda_1$. Let $u^* \in D(A)$ be the unique solution of

$$Au^* + \lambda u^* = f .$$

We have

$$A_1(u - u^*) + \lambda(u - u^*) = 0 .$$

Since $(u - u^*) \in L^1_{loc}(\mathbb{R}^m)$ and $\Delta(u - u^*) \in L^1_{loc}(\mathbb{R}^m)$ we may apply Lemma A in [3] to conclude that

$$\Delta|u - u^*| \geq \mathrm{Re}[\Delta(u - u^*)\, \mathrm{sign}(\bar{u} - \bar{u}^*)] \quad \text{in } \mathcal{D}'(\mathbb{R}^m) ,$$

and thus in $\mathcal{D}'(\mathbb{R}^m)$ we find,

$$\Delta|u - u^*| \geq \mathrm{Re}[(q + \lambda)|u - u^*|] \geq (-q^- + \lambda)|u - u^*| .$$

Using the next lemma we conclude that $u = u^*$ (and hence $D(A_1) = D(A)$).

Lemma 2.3: Assume (1) and (3). Let $v \in L^2(\mathbb{R}^m)$ be a real valued function with $q^- v \in L^1_{loc}(\mathbb{R}^m)$ satisfying

$$-\Delta v - q^- v + \lambda v \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^m)$$

with some $\lambda > \lambda_1$. Then $v \leq 0$ a.e. on $\mathbb{R}^m$.

The proof of Lemma 2.3 relies on the following crucial result. Since we shall need it in Section 3 for a general domain $\Omega \subset \mathbb{R}^m$ we work now again in Ω.

Theorem 2.3: Assume (1). Let $g \in L^2(\Omega) \cap L^\infty(\Omega)$ and let $\psi \in H^1_0(\Omega)$ be the unique solution of

(7) $$-\Delta\psi - q^-\psi + \lambda\psi = g \quad \text{in} \quad \Omega \quad (\lambda > \lambda_1) .$$

Then

a) $g \geq 0$ a.e. on Ω implies $\psi \geq 0$ a.e. on Ω;

b) $\psi \in \bigcap_{2 \leq p < \infty} L^p(\Omega)$.

Proof of Theorem 2.3: a) Multiplying (7) by $-\psi^-$ we find

$$\int |\text{grad } \psi^-|^2 - \int q^- |\psi^-|^2 + \lambda \int |\psi^-|^2 \leq 0$$

and thus $\psi^- = 0$.

b) We have to consider only the case $m \geq 3$ (when $m \leq 2$, $\psi \in H^1_0(\Omega)$ implies $\psi \in \bigcap_{2 \leq p < \infty} L^p(\Omega)$).

We can always assume that $g \geq 0$ a.e. on Ω so that $\psi \geq 0$ a.e. on Ω. We truncate q^- by $q^-_k = \min(q^-, k)$ and define ψ_k to be the unique solution of

$$\begin{cases} \psi_k \in H^1_0(\Omega) \\ -\Delta\psi_k - q^-_k\psi_k + \lambda\psi_k = g \quad \text{in} \quad \Omega . \end{cases}$$

It is clear that $\psi_k \to \psi$ weakly in $H^1_0(\Omega)$ as $k \to \infty$. We shall prove that for every $p \in [2,\infty)$, $\psi_k \in L^p(\Omega)$ and

(8) $$\|\psi_k\|_{L^p} \leq C_p(\|g\|_{L^2} + \|g\|_{L^\infty}) ,$$

where C_p is independent of k, but it depends on q^- through the use of Lemma 2.1. For simplicity we drop now the subscript k on ψ_k and write

(9) $$-\Delta\psi - q^-_k\psi + \lambda\psi = g .$$

Set $\psi_n = \min(\psi, n)$ and let $2 \leq p < \infty$; since $(\psi_n)^{p-1} \in H_0^1(\Omega)$ we can multiply (9) by $(\psi_n)^{p-1}$ and we get

$$(p-1)\int (\psi_n)^{p-2}|\text{grad } \psi_n|^2 \leq \int g(\psi_n)^{p-1} + \int q_k^-(\psi_n)^p + \int_{[\psi>n]} kn^{p-1}\psi ,$$

that is

$$\frac{4(p-1)}{p^2}\int |\text{grad } \psi_n^{p/2}|^2 \leq \|g\|_{L^p}\|\psi_n\|_{L^p}^{p-1} + \int q^-(\psi_n)^p + kn^{p-1}\int_{[\psi>n]} \psi$$

$$\leq \|g\|_{L^p}\|\psi_n\|_{L^p}^{p-1} + \varepsilon\|\text{grad}\psi_n^{p/2}\|_{L^2}^2 + \lambda_\varepsilon\|\psi_n\|_{L^p}^p + k\int_{[\psi>n]} \psi^p$$

by Lemma 2.1 (here $\int_{[\psi>n]} \psi^p$ is possibly infinite). Choosing $\varepsilon > 0$ small enough (for example $\varepsilon = \frac{2(p-1)}{p^2}$) we see that

$$\int |\text{grad } \psi_n^{p/2}|^2 \leq C_p[\|g\|_{L^p}^p + \|\psi\|_{L^p}^p + k\int_{[\psi>n]} \psi^p]$$

where C_p is independent of k and n. Using Sobolev's inequality we find

(10) $$\|\psi\|_{L^{p2^*/2}}^p \leq C_p\left[\|g\|_{L^p}^p + \|\psi\|_{L^p}^p + k\int_{[\psi>n]} \psi^p\right] .$$

<u>Assuming</u> now that $\psi \in L^p(\Omega)$ and passing to the limit in (10) as $n \to \infty$ we obtain that $\psi \in L^{p2^*/2}(\Omega)$ and

$$\|\psi\|_{L^{p2^*/2}} \leq C_p[\|g\|_{L^p} + \|\psi\|_{L^p}] .$$

Iterating this process from $p = 2$ we obtain finally for every $p \in [2,\infty)$

$$\|\psi\|_{L^p} \leq C_p[\|g\|_{L^2} + \|g\|_{L^\infty}] .$$

More precisely we have proved (8). The conclusion of Theorem 2.3 follows since $\psi_k \to \psi$ weakly in $H_0^1(\Omega)$ as $k \to \infty$.

<u>Proof of Lemma 2.3</u>: By assumption $q^- v \in L_{loc}^1(\mathbb{R}^m)$ and

$$\int v(-\Delta\varphi - q^-\varphi + \lambda\varphi) \leq 0 \quad \forall\varphi \in D_+(\mathbb{R}^m) .$$

An easy density argument (smoothing by convolution) shows that

(11) $$\int v(-\Delta\varphi - q^-\varphi + \lambda\varphi) \leq 0 \quad \forall\varphi \in H^2(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m),\ \text{supp}\,\varphi \text{ compact},\ \varphi \geq 0 \text{ a.e.} .$$

Fix $g \in \mathcal{D}_+(\mathbb{R}^m)$ and let $\psi_k \in H^1(\mathbb{R}^m)$ be the unique solution of

(12) $$-\Delta\psi_k - q_k^-\psi_k + \lambda\psi_k = g \quad \text{in} \quad \mathbb{R}^m .$$

We know by Theorem 2.3 that $\psi_k \geq 0$ a.e.

$$\psi_k \in \bigcap_{2 \leq p < \infty} L^p(\mathbb{R}^m) \quad \text{with} \quad \|\psi_k\|_{L^p} \leq C_p ,$$

and also $\|\text{grad}\,\psi_k\|_{L^2} \leq C$. In addition we derive from (12) that

$$\psi_k \in H^2(\mathbb{R}^m) \cap L^\infty_{loc}(\mathbb{R}^m) .$$

Fix $\zeta \in \mathcal{D}_+(\mathbb{R}^m)$ satisfying $\zeta(x) = 1$ for $|x| \leq 1$ and set $\zeta_n(x) = \zeta(\frac{x}{n})$. In (11) we choose $\varphi = \psi_k\zeta_n$. Note that by (12)

$$-\Delta\varphi - q^-\varphi + \lambda\varphi = \zeta_n g - (\Delta\zeta_n)\psi_k - 2\,\text{grad}\zeta_n\,\text{grad}\psi_k - \zeta_n\psi_k(q^- - q_k^-) ,$$

and therefore

$$\int v\zeta_n g \leq \frac{C}{n^2} + \frac{C}{n} + \int v\zeta_n\psi_k(q^- - q_k^-) .$$

First we <u>fix n</u> and let <u>$k \to \infty$</u>. We distinguish two cases:

a) $m \geq 5$,

b) $m < 5$.

a) When $m \geq 5$ we have $q^- - q_k^- \to 0$ in $L^{m/2}_{loc}(\mathbb{R}^m)$. Let $p \in [2,\infty)$ be such that $\frac{1}{2} + \frac{2}{m} + \frac{1}{p} = 1$; we have

$$\left|\int v\zeta_n\psi_k(q^- - q_k)\right| \leq \|v\|_{L^2}\|\psi_k\|_{L^p}\|\zeta_n(q^- - q_k)\|_{L^{m/2}} \to 0 .$$

Consequently

$$\int v\zeta_n g \leq \frac{C}{n^2} + \frac{C}{n} .$$

b) When $m < 5$ we use the assumption (3) (or (1)): $q^- \in L^{m/2+\varepsilon}_{loc}(\mathbb{R}^m)$ with some $\varepsilon > 0$. It follows from (12) that ψ_k remains bounded in $W^{2,q}_{loc}(\mathbb{R}^m)$ for some $q > \frac{m}{2}$ (when $m \geq 2$) as $k \to \infty$. We conclude that ψ_k remains bounded in $L^\infty_{loc}(\mathbb{R}^m)$ as $k \to \infty$ (in case $m = 1$, ψ_k is bounded in $L^\infty(\mathbb{R})$ since it is bounded in $H^1(\mathbb{R})$). Therefore

$$\int v\zeta_n\psi_k(q^- - q_k^-) \to 0 \quad \text{as} \quad k \to \infty$$

since $\|\zeta_n v(q^- - q_k^-)\|_{L^1} \to 0$ by the dominated convergence theorem (recall that

$q^- v \in L^1_{loc}(\mathbb{R}^m))$. In both cases we find

$$\int v\zeta_n g \le \frac{C}{n^2} + \frac{C}{n} \quad \forall n .$$

As $n \to \infty$ we see that

$$\int vg \le 0 \quad \forall g \in \mathcal{D}_+(\mathbb{R}^m)$$

and therefore $v \le 0$ a.e. on $\mathbb{R}^m$.

Remark 2.2: The conclusion of Lemma 2.3 fails in $\mathbb{R}^3$ and in $\mathbb{R}^4$ if we do not assume (3). Ancona (personal communication) has constructed in $\mathbb{R}^3$ and in $\mathbb{R}^4$ functions $q^- \in L^{m/2}(\mathbb{R}^m)$ and $u \in L^{m/m-2}(\mathbb{R}^m) \cap L^2(\mathbb{R}^m)$ such that

$$-\Delta u - q^- u + u = 0 \quad \text{in} \quad \mathcal{D}'$$

with $\|q^-\|_{L^{m/2}}$ as small as we please and $u \neq 0$.

3. Complex potentials

Let Ω be an (arbitrary) open subset of $\mathbb{R}^m$. Assume $q(x)$ and $q'(x)$ are real valued functions such that $q, q' \in L^1_{loc}(\Omega)$ and set

$$V(x) = q(x) + iq'(x).$$

We assume

$$\text{(13)} \qquad \underline{\text{either}}\ q' \in L^{1+\varepsilon}_{loc}(\Omega)\ \underline{\text{or}}\ q^- \in L^{(m/2)+\varepsilon}_{loc}(\Omega) \quad \text{when } m \geq 2,$$

for some arbitrarily small $\varepsilon > 0$. Define

$$A = -\Delta + V(x)$$

with

$$D(A) = \{u \in H^1_0(\Omega);\ Vu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + Vu \in L^2(\Omega)\}.$$

The main results are the following

Theorem 3.1: Assume (1) and (13). Then A is closable in $L^2(\Omega)$ and $\bar{A} + \lambda_1$ is m-accretive. In addition $u \in D(\bar{A})$ implies that $u \in H^1_0(\Omega)$, $q|u|^2 \in L^1(\Omega)$ and

$$\text{(14)} \qquad \operatorname{Re}(\bar{A}u, u) = \int |\text{grad}\, u|^2 + \int q|u|^2.$$

Remark 3.1: In case we assume

$$\text{(15)} \qquad |q'(x)| \leq Mq^+(x) + h(x) \quad \text{for a.e. } x \in \Omega$$

with $h \in L^{2m/(m+2)}_{loc}(\Omega)$ and $m \geq 3$ then A is closed in $L^2(\Omega)$. (Note that (15) corresponds essentially with the assumption made in [5]). Indeed let $u_n \in D(A)$ be such that $u_n \to u$ in $L^2(\Omega)$ and $Au_n \to f$ in $L^2(\Omega)$. It follows from Lemma 2.1 and (14) that $u_n \to u$ in $H^1_0(\Omega)$ and $\sqrt{q^+}u_n \to \sqrt{q^+}u$ in $L^2(\Omega)$. From (15) we deduce easily that $Vu \in L^1_{loc}(\Omega)$ and that $-\Delta u + Vu = f$ in $\mathcal{D}'(\Omega)$. Therefore $u \in D(A)$ and $Au = f$.

When $\Omega = \mathbb{R}^m$ we consider also the operator A_1 defined in $L^2(\mathbb{R}^m)$ by

$$A_1 = -\Delta + V(x)$$

with

$$D(A_1) = \{u \in L^2(\mathbb{R}^m);\ Vu \in L^1_{loc}(\mathbb{R}^m) \text{ and } -\Delta u + Vu \in L^2(\mathbb{R}^m)\}.$$

Theorem 3.2: Assume (1), (3) and (13). Then A_1 is closable and $\overline{A_1} = \bar{A}$.

In the proof of Theorem 3.1 we shall use the following

Lemma 3.1: Let $v \in H_0^1(\Omega)$ be a real valued function. Assume (1) and

$$-\Delta v - q^- v + \lambda v \le 0 \quad \text{in} \quad D'(\Omega)$$

with $\lambda > \lambda_1$. Then $v \le 0$ a.e. on Ω.

Proof of Lemma 3.1: We have, for every $\varphi \in D_+(\Omega)$

$$\int \text{grad}v \, \text{grad}\varphi - \int q^- v\varphi + \lambda \int v\varphi \le 0 .$$

Now we use the fact (pointed out by G. Stampacchia) that $D_+(\Omega)$ is dense in $\{u \in H_0^1(\Omega);\ u \ge 0$ a.e. on $\Omega\}$ for the H^1 norm[(1)] to derive that

$$\int \text{grad}v \, \text{grad}\varphi - \int q^- v\varphi + \lambda \int v\varphi \le 0 \quad \forall \varphi \in H_0^1(\Omega), \quad \varphi \ge 0 .$$

Choosing $\varphi = v^+$ we obtain

$$\int |\text{grad}v^+|^2 - \int q^- |v^+|^2 + \lambda \int |v^+|^2 \le 0$$

and therefore $v^+ = 0$.

The proof of Theorem 3.1 is divided into five steps.

Step 1: $R(A + \lambda) \supset L^2(\Omega) \cap L^\infty(\Omega)$ for $\lambda > \lambda_1$.

Indeed let $f \in L^2(\Omega) \cap L^\infty(\Omega)$ and let $u_n \in H_0^1(\Omega)$ be the unique solution of

$$(16) \qquad -\Delta u_n + V_n u_n + \lambda u_n = f$$

where $V_n = q_n^+ - q^- + iq_n'$ and

$$q_n' = \begin{cases} n & \text{if} \quad q' > n \\ q' & \text{if} \quad |q'| \le n \\ -n & \text{if} \quad q' \le -n . \end{cases}$$

The existence of u_n follows from a theorem of Lax-Milgram. Multiplying (16) by $\overline{u_n}$ we find

$$(17) \qquad \|u_n\|_{H^1} \le C$$

$$(18) \qquad \int q_n^+ |u_n|^2 \le C .$$

---

(1) Indeed let $u \in H_0^1(\Omega)$ with $u \ge 0$ a.e. on Ω; let $u_n \in D(\Omega)$ be such that $u_n \to u$ in $H^1(\Omega)$. We claim that $|u_n| \to |u| = u$ in $H^1(\Omega)$ because $\| |u_n| \|_{H^1} = \|u_n\|_{H^1}$ and $|u_n| \to |u|$ weakly in $H^1(\Omega)$. On the other hand $|u_n|$ can be smoothed by convolution and for fixed n, $\rho_\varepsilon * |u_n| \to |u_n|$ in $H^1(\Omega)$ as $\varepsilon \to 0$.

On the other hand we have

$$\Delta|u_n| \geq \mathrm{Re}[\Delta u_n \text{ sign } \overline{u_n}] \quad \text{in} \quad \mathcal{D}'(\Omega)$$

which leads to

$$-\Delta|u_n| - q^-|u_n| + \lambda|u_n| \leq |f| \quad \text{in} \quad \mathcal{D}'(\Omega) .$$

Let $\psi \in H_0^1(\Omega)$ be the solution of

(19) $$-\Delta\psi - q^-\psi + \lambda\psi = |f| .$$

It follows from Lemma 3.1 that

(20) $$|u_n| \leq \psi \quad \text{a.e. on } \Omega .$$

By Theorem 2.3 we know that $\psi \in L^p(\Omega)$ for every $p \in [2,\infty)$. We extract a subsequence, denoted again by u_n such that $u_n \to u$ weakly in $H_0^1(\Omega)$, $u_n \to u$ a.e. on Ω. We see as in the proof of Theorem 2.1 (Step 1) that $(q_n^+ - q^-)u_n \to qu$ in $L^1_{loc}(\Omega)$. Therefore we have only to verify that $q'_n u_n \to q'u$ in $L^1_{loc}(\Omega)$. We distinguish two cases:

a) $q' \in L^{1+\varepsilon}_{loc}(\Omega)$,

b) $q^- \in L^{(m/2)+\varepsilon}_{loc}(\Omega)$.

<u>Case a)</u> From (20) we deduce that $u_n \to u$ in every L^p space, $2 \leq p < \infty$ and so $q'_n u_n \to q'u$ in $L^1_{loc}(\Omega)$.

<u>Case b)</u> Since $q^-\psi \in L^q_{loc}(\Omega)$ for some $q > \frac{m}{2}$, it follows from (19) that $\psi \in L^\infty_{loc}(\Omega)$. We deduce from the dominated convergence theorem that $q'_n u_n \to q'u$ in $L^1_{loc}(\Omega)$.

<u>Step 2</u>: $A + \lambda_1$ is accretive. Let $u \in D(A)$ and set $T = Vu$. We have $T \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ and

$$\mathrm{Re}\, T \cdot \bar{u} = q|u|^2 \geq -q^-|u|^2 \in L^1(\Omega) .$$

It follows from Lemma 2.2 that $q|u|^2 \in L^1(\Omega)$ and

$$\int q|u|^2 = \mathrm{Re}\langle T,u\rangle = \mathrm{Re}\langle Au + \Delta u, u\rangle .$$

Therefore

(21) $$\mathrm{Re}(Au,u) = \int |\mathrm{grad}\, u|^2 + \int q|u|^2 \geq -\lambda_1 \int |u|^2 .$$

<u>Step 3</u>: $D(A)$ is dense in $L^2(\Omega)$. Given $f \in L^2(\Omega) \cap L^\infty(\Omega)$ we solve for large n the equation

(22) $$u_n + \frac{1}{n} Au_n = f.$$

We shall prove that $u_n \to f$ in $L^2(\Omega)$ as $n \to \infty$ — and as a consequence $D(A)$ is dense in $L^2(\Omega)$. By (21) we have

$$\int |u_n|^2 + \frac{1}{n}\int |\mathrm{grad}\, u_n|^2 + \frac{1}{n}\int q|u_n|^2 = \mathrm{Re}(f,u_n) .$$

In particular we deduce that

$$\limsup_{n\to\infty} \|u_n\|_{L^2} \le \|f\|_{L^2} \tag{23}$$

$$\frac{1}{n}\int q^+|u_n|^2 \le C \tag{24}$$

$$\frac{1}{n}\int |\mathrm{grad}\, u_n|^2 \le C . \tag{25}$$

Next we have (as in the proof of Step 1)

$$|u_n| - \frac{1}{n}\Delta|u_n| - \frac{1}{n}q^-|u_n| \le |f| \quad \text{in} \quad D'(\Omega) .$$

On the other hand let $\psi \in H_0^1(\Omega)$ be the solution of

$$-\Delta\psi - q^-\psi + \lambda\psi = |f|$$

for some fixed $\lambda > \lambda_1$. Since $|u_n| \ge \lambda\left|\frac{u_n}{n}\right|$ for $n \ge \lambda$, we deduce from Lemma 3.1 that $\left|\frac{u_n}{n}\right| \le \psi$ a.e. Choose a subsequence, denoted again by u_n such that $u_n \to u$ weakly in $L^2(\Omega)$, $\frac{1}{n}u_n \to 0$ a.e. (this is possible since $\frac{1}{n}u_n \to 0$ in $L^2(\Omega)$). For every $\varphi \in D(\Omega)$ we have

$$\int u_n\bar\varphi - \frac{1}{n}\int u_n\Delta\bar\varphi + \frac{1}{n}\int Vu_n\bar\varphi = \int f\bar\varphi \quad . \tag{26}$$

We claim that $\frac{1}{n}\int Vu_n\bar\varphi \to 0$ as $n \to \infty$. Indeed by (24) and (25) we have

$$\frac{1}{n}\left|\int q^+u_n\bar\varphi\right| \le \frac{C}{\sqrt{n}} \quad \text{and} \quad \frac{1}{n}\left|\int q^-u_n\bar\varphi\right| \le \frac{C}{\sqrt{n}} .$$

Thus we have only to verify that $\frac{1}{n}\int q'u_n\bar\varphi \to 0$. We distinguish two cases:

a) if $q^- \in L_{loc}^{(m/2)+\varepsilon}(\Omega)$, we have $\psi \in L_{loc}^{\infty}(\Omega)$ and we deduce from the dominated convergence theorem that $\frac{1}{n}\int q'u_n\bar\varphi \to 0$;

b) if $q' \in L_{loc}^{1+\varepsilon}(\Omega)$ we use the fact that $\left|\frac{u_n}{n}\right| \le \psi \in L^p(\Omega)$ for every $2 \le p < \infty$ to deduce that $\frac{u_n}{n} \to 0$ in $L^p(\Omega)$ and so $\frac{1}{n}\int q'u_n\bar\varphi \to 0$.

In all the cases, we derive from (26) that

$$\int u\bar{\varphi} = \int f\bar{\varphi} \quad \forall\varphi \in D$$

and consequently $u = f$. We conclude using (23) that $u_n \to f$ in $L^2(\Omega)$.

Step 4: A is closable and $\bar{A} + \lambda_1$ is m-accretive. This is a standard fact, see e.g. Theorem 3.4 in [2].

Step 5: $u \in D(\bar{A})$ implies that $u \in H_0^1(\Omega)$, $q|u|^2 \in L^1(\Omega)$ and (14).

We already know (Step 2) that $v \in D(A)$ implies $q|v|^2 \in L^1(\Omega)$ and

$$\text{(27)} \qquad \operatorname{Re}(Av,v) = \int |\operatorname{grad} v|^2 + \int q|v|^2 .$$

Now let $u \in D(\bar{A})$ and let $u_n \in D(A)$ be such that $u_n \to u$, $Au_n \to \bar{A}u$. It follows from (27) applied to $v = u_n - u_m$ that $u_n \to u$ in $H_0^1(\Omega)$ and $\int q^+|u_n - u|^2 \to 0$ (since u_n is a Cauchy sequence in $H_0^1(\Omega)$ and in $L^2(\Omega)$ with weight q^+). In particular $q|u|^2 \in L^1(\Omega)$ and (14) holds.

Proof of Theorem 3.2: Clearly $A \subset A_1$. Now let $u \in D(A_1)$ and let $\lambda > \lambda_1$. Set $f = A_1 u + \lambda u$, and let u^* be the unique solution of

$$\bar{A}u^* + \lambda u^* = f .$$

Thus, there exists a sequence $u_n^* \to u^*$ in $L^2(\mathbb{R}^m)$ with $u_n^* \in D(A)$ and

$$Au_n^* + \lambda u_n^* = f_n \to f \quad \text{in} \quad L^2(\mathbb{R}^m) .$$

In particular we have

$$A_1(u_n^* - u) + \lambda(u_n^* - u) = f_n - f$$

and therefore

$$-\Delta|u_n^* - u| - q^-|u_n^* - u| + \lambda|u_n - u| \le |f_n - f| \quad \text{in} \quad D'(\mathbb{R}^m) .$$

We deduce from Lemma 2.3 that $|u_n^* - u| \le \psi_n$ a.e. on $\mathbb{R}^m$ where $\psi_n \in H^1(\mathbb{R}^m)$ is the solution of

$$-\Delta\psi_n - q^-\psi_n + \lambda\psi_n = |f_n - f| .$$

Hence $\|\psi_n\|_{H^1} \to 0$ and in particular $u_n^* - u \to 0$ in $L^2(\mathbb{R}^m)$. It follows that $u^* = u$, that is $A_1 \subset \bar{A}$. We have $A \subset A_1 \subset \bar{A}$ and therefore A_1 is closable with $\overline{A_1} = \bar{A}$.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let $A = -\Delta + V(x)$ be a Schrödinger operator on an (arbitrary) open set $\Omega \subset \mathbb{R}^m$, where $V \in L^1_{loc}(\Omega)$ is a complex valued function. We consider the "maximal" realization of A in $L^2(\Omega)$ under Dirichlet boundary condition, that is

20. ABSTRACT - cont'd.

$$D(A) = \{u \in H^1_0(\Omega);\ Vu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + Vu \in L^2(\Omega)\}.$$

When $\Omega = \mathbb{R}^m$ we also consider the operator

$$A_1 = -\Delta + V$$

with domain

$$D(A_1) = \{u \in L^2(\Omega);\ Vu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + Vu \in L^2(\Omega)\}.$$

A special case of our main results is:

<u>Theorem</u>: Let $m \geq 3$; assume that the function $\max\{-\mathrm{Re}\, V, 0\}$ belongs to $L^\infty(\Omega) + L^{m/2}(\Omega)$ and also to $L^{(m/2)+\varepsilon}_{loc}(\Omega)$ for some $\varepsilon > 0$. Then A (resp. A_1) is closable and $\bar{A} + \lambda$ (resp. $\bar{A}_1 + \lambda$) is m-accretive for some real constant λ.