REMARKS ON THE USE AND MISUSE OF THE SEMI-INVERSE METHOD IN THE NONLINEAR THEORY OF ELASTICITY

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Summary

We show, by considering some specific examples, that in the nonlinear theory of elasticity the application of the semi-inverse method has to be carried out very carefully. Usually, the strategy followed in applying the semi-inverse method while dealing with complex models is to generalize forms of solutions already known within the framework of a simpler theory. This is a smart strategy from a mathematical point of view, but from the physical point of view, it may be imprudent. For example, in the theory of nonlinear elasticity, we are at times motivated by the results obtained in the incompressible case to arrive at an understanding of what happens in the compressible case, and by doing so, many important exact solutions for special classes of compressible elastic materials have been obtained successfully. Sometimes, the admissibility of a given deformation field is considered to delineate special classes of constitutive laws. We wish to point out that the classes of constitutive equations thus identified from the standpoint that it may admit a type of deformation may lead to models that exhibit physically unacceptable mechanical behavior. To illustrate the dangers inherent to merely turning the mathematical crank to determine classes of constitutive equations where a certain class of deformations are possible, we consider the torsion of a cylindrical shaft and the propagation of transverse waves in a compressible nonlinear elastic material and show that care has to be exercised in appealing to the semi-inverse method.

1. Introduction, basic equations and background of the problem

The aim of this paper is to emphasize that great care has to be exercised in using the semi-inverse method in continuum mechanics to delineate classes of constitutive equations that admit a particular

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class of deformations and motions to be possible. This method has been misused often in nonlinear elasticity, and hence, we will appeal to examples within the context of nonlinear elasticity to illustrate our thesis. To simplify the exposition, we consider the theory of *unconstrained* (compressible) *nonlinear isotropic elasticity*, but our remarks are completely general and apply (with some modifications) in general to the use of semi-inverse method in continuum mechanics.

It is well known that nonlinear theories lead to counterintuitive solutions: for example, shearing a nonlinear elastic solid leads to forces normal to the plane of shear. In general, the application of certain given loads produces deformation fields that may be completely different from the ones that we may expect based on our intuition or our knowledge of simplified material models such as linearized elasticity. One needs to exercise a great deal of prudence in ensuring that the results obtained by using the semi-inverse method make sense.

To this end, let us consider a material body $\mathcal{B}$. A particle $P \in \mathcal{B}$ is identified by its position $\boldsymbol{X}(P)$ in a given reference frame at some reference time. The deformation or motion of the body is described by the mapping $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$, which identifies the position, $\boldsymbol{x}$, of P at time t. The deformation gradient is defined as $\boldsymbol{F} = \partial \boldsymbol{x}/\partial \boldsymbol{X}$ and the left Cauchy–Green strain tensor is given by $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$. The principal invariants of $\boldsymbol{B}$ are $I_1 = \operatorname{tr} \boldsymbol{B}$, $I_2 = [I_1^2 - \operatorname{tr}(\boldsymbol{B}^2)]/2$ and $I_3 = \det \boldsymbol{B}$.

The constitutive equation for the Cauchy stress tensor $\boldsymbol{T}$ in an unconstrained isotropic hyperelastic (Green) solid is

$$\boldsymbol{T} = \beta_0 \boldsymbol{I} + \beta_1 \boldsymbol{B} + \beta_{-1} \boldsymbol{B}^{-1}, \tag{1.1}$$

where the response coefficients, β_Γ ($\Gamma = 0, 1, -1$), are related to the strain-energy density $\Sigma(I_1, I_2, I_3)$ by

$$\beta_0(I_1, I_2, I_3) = \frac{2}{\sqrt{I_3}}\left[I_2 \frac{\partial \Sigma}{\partial I_2} + I_3 \frac{\partial \Sigma}{\partial I_3}\right], \quad \beta_1(I_1, I_2, I_3) = \frac{2}{\sqrt{I_3}} \frac{\partial \Sigma}{\partial I_1},$$

$$\beta_{-1}(I_1, I_2, I_3) = -2\sqrt{I_3} \frac{\partial \Sigma}{\partial I_2}.$$

The Piola–Kirchhoff stress tensor, $\boldsymbol{S}$, is related to the Cauchy stress tensor by the formula

$$\boldsymbol{S} = \sqrt{I_3} \boldsymbol{T} \boldsymbol{F}^{-T}.$$

The balance equations in the absence of body forces are given by

$$\operatorname{div} \boldsymbol{T} = \rho \boldsymbol{a} \tag{1.2}$$

or, equivalently, by

$$\operatorname{Div} \boldsymbol{S} = \rho_{\mathrm{R}} \ddot{\boldsymbol{x}}, \tag{1.3}$$

where div denotes the divergence operator with respect to $\boldsymbol{x}$, Div denotes the divergence operator with respect to $\boldsymbol{X}$, ρ is the density in the current configuration, ρ_{R} is the density in the reference configuration, $\boldsymbol{a}$ is the Eulerian acceleration vector and $\ddot{\boldsymbol{x}}$ is the Lagrangian acceleration vector.

Our starting point is a result due to Ericksen (**1**) on universal solutions in unconstrained isotropic elastic materials. Ericksen proved[1] that homogeneous deformations, $\boldsymbol{x} = \boldsymbol{F}_0 \boldsymbol{X}$, where $\boldsymbol{F}_0$ is a constant tensor, are the only *controllable* static deformations possible in every hyperelastic material

[1] For a modern and simple proof of Ericksen's theorem, we refer to (**2**, Chapter 3).

(1.1). A controllable deformation is a deformation that is produced in a material by the application of surface tractions alone. A controllable deformation that can be effected in every homogeneous isotropic material is referred to as a *universal* solution. We point out that universal solutions are the *only* solutions wherein the geometry associated with the deformation does not depend on the response functions β_Γ. This fundamental property of universal solutions deserves special mention.

Ericksen's result concerning universal deformations has had a profound influence on the development of nonlinear elasticity and for many years there has been 'the false impression that the only deformations possible in an elastic body are the universal deformations' (**3**). However, around the same time as the publication of Ericksen's result, there was considerable activity in trying to find solutions for nonlinear elastic materials using the semi-inverse method. A summary of these earlier results may be found in the monograph by Green and Adkins (**4**). The purpose of the semi-inverse method is to reduce a formidable set of nonlinear partial differential equations to a system of ordinary differential equations or to a simpler system of compatible partial differential equations[2] that are more amenable to mathematical analysis but can yet capture the salient features of the physics of the problem under consideration. If this reduced system can be solved in closed form, then it is possible to obtain some exact solutions to boundary-value problems that hopefully are meaningful within the framework of the theory that is being employed.[3] Of course, even if it cannot be solved exactly, the semi-inverse method leads to a simpler set of equations that can be resolved numerically.

In any case, Currie and Hayes (**3**) were right to point out that after some initial interest in the search for exact solutions in nonlinear elasticity, there followed a long period of inactivity concerning such an endeavor, with some notable exceptions (see, for example, Ogden (**5**)). At the end of the 1970s after two important papers by Knowles (**6**, **7**) about the anti-plane shear problem and the paper by Currie and Hayes (**3**), the search for possible solutions outside universal ones was revitalized.

Knowles used Cartesian coordinates (X_1, X_2, X_3) in the reference configuration and Cartesian coordinates (x_1, x_2, x_3) in the current configuration to write down the anti-plane shear deformation: $x_1 = X_1$, $x_2 = X_2$ and $x_3 = X_3 + u(X_1, X_2)$. For this class of deformations, the general balance equations reduce to an overdetermined system of partial differential equations for the out-of-plane displacement $u = u(X_1, X_2)$. For the anti-plane shear problem considered by Knowles (**6**, **7**), it is possible to find the restrictions on $\Sigma(I_1, I_2, I_3)$ such that this overdetermined system of scalar partial differential equations admits nontrivial solutions, so that the anti-plane shear deformation is controllable. Clearly in this case, the function $u(X_1, X_2)$ depends on the response coefficients because the determining partial differential equation contains β_Γ.

In Currie and Hayes (**3**), the strategy proposed to search for exact solutions starts from a different point of view. They search for special solutions by choosing a deformation whose geometry is completely known *a priori*; in doing so they are solving Ericksen's problems *in miniature*: they are searching for universal solutions for a subclass of isotropic elastic materials.[4]

The influential papers by Knowles, Currie and Hayes have stimulated the development of a large amount of research on closed-form solutions in nonlinear elasticity. Beatty, Boulanger, Carroll, Chadwick, Hill, Horgan, Murphy, Ogden, Polignone, Rajagopal, Saccomandi, Wineman and many

[2] Because the semi-inverse method is essentially heuristic, it is nearly impossible to give a rigorous definition of what is meant by the method.

[3] This is not always the case, as it is well known in the framework of the Navier–Stokes theory where the exact solutions found by the semi-inverse method are often not compatible with the canonical no-slip boundary conditions.

[4] The expression 'in miniature' is taken from a paper by Knowles (**8**) where the author tries to find nonhomogeneous universal solutions in the family of anti-plane shear deformations, a similar but not identical problem.

others have determined a long list of exact solutions for special classes of constitutive equations. We refer to the recent books edited by Fu and Ogden (**9**) and Hayes and Saccomandi (**2**) for an overview of such activity. Here, we wish to point out that although these researches have helped in obtaining a better understanding of the behavior of nonlinear elastic bodies, some of them nonetheless have been the source of possible confusion in the field; some are even incorrect in their use of the semi-inverse method. Indeed, in some of the papers that followed the papers of Knowles concerning anti-plane shear, the authors investigate the restrictions that the strain-energy density function must fulfil for a given class of deformation to be admissible. For example, they determine the most general constitutive equation for a compressible isotropic elastic material such that the deformation, of pure torsion[5] or pure axial shear, is controllable. This requirement is interesting because it allows one the possibility to determine exact solutions for special classes of materials. Our point is that the constitutive equations determined by such special mathematical characterizations may lead to constitutive models that exhibit response that is too special from the point of view of physics and therefore their use in modeling of real materials is highly questionable.

To examine what kind of problems one may encounter, let us recall that it has been possible to determine the most general class of compressible materials for which pure torsion is a controllable deformation in the case of a circular solid cylinder. This means that for the constitutive equations that allow the deformation in question, the balance equations are satisfied for the pure torsion deformation. The next step is to ensure that the lateral surface of the circular cylinder is traction free. Now, because simple torsion is an isochoric deformation, we have to ensure that the lateral boundary is traction free while the volume remains constant. There is no reason to expect that this situation is automatically complied with in a compressible material. It is more natural to expect that when the lateral boundary of the cylinder is traction free, the volume change has to be nonzero. In some sense, the behavior of a class of compressible materials such that pure torsion is controllable is extraordinary. In this paper, we wish to investigate quantitatively the meaning of this sort of unusual possibility.

It is well known that the general theory of unconstrained nonlinear elasticity predicts that new types of surface loads need to be applied in order to effect a given deformation; these loads may be absent in the linearized theory or in the case of nonlinear incompressible materials.[6] Conversely, given loads will produce new types of deformations that are not possible within the context of linearized elasticity or incompressible nonlinear elasticity theory. Let us consider a solid cylinder and let us subject it to a pure twisting couple. In the real world, a material will not only twist but also extend or contract and undergo a change of volume. (This is the well-known Poynting effect.) Therefore, a pure torsion deformation (an isochoric deformation) as the consequence of a pure twisting couple happens only within the context of very special constitutive theories for compressible materials.

We recall that for incompressible nonlinear elastic materials, there exist some families of *inhomogeneous* universal solutions (see, for example, (**2**, Chapter 3)). Therefore, a strategy to find some exact solutions for compressible elastic materials may be to take inspiration from these isochoric deformations and to seek similar solutions in compressible elastic materials. We point out that any inhomogeneous solution admissible in a compressible material must be nonuniversal because of

[5] We point out that pure or simple torsion is by definition an *isochoric* deformation (see, for example, page 21 of Atkin and Fox (**10**)).

[6] In constrained nonlinear materials such as incompressible bodies, the arbitrariness of the pressure allows one a great deal of flexibility in effecting certain deformations; this luxury is not available when one considers unconstrained materials.

Ericksen's result. It must be clear that the isochoric deformations of an incompressible elastic body have to be modified because the same loads will in general produce changes in volume in compressible materials. It is hard to justify, from the point of view of mechanics, a compressible material for which the given load will, in the case of an incompressible material, once again produce isochoric deformations. Clearly in some cases, the volume variations may be *small* but in nonlinear mechanics the idea that small is negligible has to be handled delicately. For example, the papers by Fosdick and Kao (**11**), Rajagopal and Zhang (**12**) and Mollica and Rajagopal (**13**) on secondary deformations in nonconcentric hollow circular tubes or in tubes of elliptic cross-section or the recent paper concerning the deformation of a cork by De Pascalis *et al.* (**14**) are good examples of the ambiguity of the word 'small' in this framework.

The aim of the present work is to show by examples the importance of a clear description of the complex deformation field that may be engendered by a simple state of surface load. The complex deformation may be split into a basic field that may be denoted as the *primary* field, whereas the remaining deformation may be denoted as *secondary*. The exact reasons for calling them primary and secondary fields will become clear as we discuss the examples.

The plan of the paper is the following. In section 2, we consider some static deformations with the help of which we can lay bare the confusion that has been created in seeking semi-inverse solutions. By considering torsional deformation of a cylindrical shaft, we discuss step by step our criticism concerning the mistakes that have been made as well as the possible errors that can be committed. Here, the primary deformation is pure torsion and the secondary deformation is radial expansion. Then in section 3, we consider the propagation of transverse bulk waves (primary motion) that according to general nonlinear elasticity theory must always be coupled to a longitudinal wave (secondary motion). Instead of considering what happens within the context of the linearized theory, a second-order theory and then the general nonlinear setting, we consider a top to bottom approach. We derive the general equations and, assuming that the amplitude of the displacements is of order ϵ, we show that at the first-order we recover the results of the linearized theory and that at a higher-order of approximation we may have some insight into the coupling between the various modes of deformation. Here, the interesting point is the occurrence of the phenomena of resonance between the primary and secondary fields. Section 4 is devoted to concluding remarks.

2. Statement of the problem

In this section, we illustrate our point of view by considering the works of Polignone and Horgan (**15** to **17**), Beatty and Jiang (**18**, **19**) and Jiang and Beatty (**20**). In these papers, the authors investigate when certain deformations are admissible (that is, controllable) for an unconstrained nonlinear elastic material. We wish to point out that our criticism is not about the mathematical results contained in these papers. These results can and do lead to useful exact solutions if the correct subclass of materials is picked. However, with regard to the whole class of materials that are identified in the papers, one has to exercise a great deal of caution because models that are obtained on the basis of purely mathematical arguments may exhibit highly questionable physical behavior.

We use cylindrical polar coordinates (R, Θ, Z) in the reference configuration and (r, θ, z) in the deformed configuration. There are several interesting deformation fields that may be examined within this setting, one of them being the torsion problem

$$r = f(R), \quad \theta = \Theta + \tau Z, \quad z = Z, \tag{2.1}$$

the axial shear problem

$$r = f(R), \quad \theta = \Theta, \quad z = Z + w(R) \tag{2.2}$$

or the azimuthal shear problem

$$r = f(R), \quad \theta = \Theta + g(R), \quad z = Z. \tag{2.3}$$

In the above equations, $\tau > 0$ is the twist per unit undeformed length and $f(R)$, $w(R)$ and $g(R)$ are the radial, axial and azimuthal deformations, respectively, that must be determined by solving the balance equations. It is clear that helical shear deformation may be easily obtained by combining axial and azimuthal shear.

In the case of a cylinder composed of an *incompressible* isotropic elastic material, it is well known that pure torsion, pure axial shear and pure azimuthal shear are admissible deformations. This means that the deformed configuration is again a solid circular cylinder which undergoes no volume change. There is no physical reason to presume that in a compressible material such *pure* deformations can occur, but from a mathematical point of view, we may ask for which subclass of unconstrained isotropic elastic materials are the deformations (2.1), (2.2) and (2.3) admissible with $r = R$.

To minimize the algebra involved in our discussion, we focus on the case of deformation (2.1). This case has been examined in a detailed and exhaustive manner by Kirkinis and Ogden (**21**). In this paper, the authors formulate the equilibrium equations for a general compressible material and then specialize their computations to the case of pure torsion. One of the important results obtained by Kirkinis and Ogden (**21**) is the necessary and sufficient condition on the strain energy for the material to sustain pure torsion with zero traction on the lateral surface. The fundamental difficulty of ensuring the absence of traction on the lateral surface of the cylinder was completely ignored by the previous authors. Here, we consider the general results in the paper by Kirkinis and Ogden (**21**) with the aim to illustrate our thesis within the context of a simple example.

In the case of deformation (2.1), we have

$$\boldsymbol{F} = f'\boldsymbol{e}_r \otimes \boldsymbol{E}_R + (f/R)\boldsymbol{e}_\theta \otimes \boldsymbol{E}_\Theta + \tau f \boldsymbol{e}_\theta \otimes \boldsymbol{E}_Z + \boldsymbol{e}_z \otimes \boldsymbol{E}_Z, \tag{2.4}$$

and the invariants of $\boldsymbol{B}$ are computed to be

$$I_1 = 1 + f'^2 + f^2/R^2 + \tau^2 f^2, \quad I_2 = f'^2 + f^2(1 + f'^2)/R^2 + \tau^2 f'^2 f^2, \quad I_3 = (ff'/R)^2.$$

Since $r = f(R)$, it is possible to consider the stress as a function of the reference coordinate R, $\boldsymbol{T} = \boldsymbol{T}(R)$ instead of $\boldsymbol{T} = \boldsymbol{T}(r)$. In this case, the balance equations reduce to a single equation

$$\frac{dT_{rr}}{dr} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0,$$

and using the chain rule, we obtain

$$\frac{d}{dR}\left[\frac{Rf'}{f}(\Sigma_1 + \Sigma_2) + \frac{ff'}{R}(\Sigma_3 + \Sigma_2) + \tau^2 Rf' f \Sigma_2\right] + \left(\frac{Rf'^2}{f^2} - \frac{1}{R}\right)(\Sigma_1 + \Sigma_2) - \tau^2 R \Sigma_1 = 0. \tag{2.5}$$

This is the equation that is to be used for determining the radial expansion $f(R)$. The usual boundary conditions for a solid circular cylinder of undeformed radius A subjected to twisting moments at its end are that the outer surface is traction free, $T_{rr}(A) = 0$, and that $\boldsymbol{F}$ is bounded, so that $r(R) = O(R)$ as $R \to 0$. We point out that in this case

$$T_{rr} = \beta_0 + \beta_1 f'^2 + \beta_{-1} f'^{-2}.$$

Clearly, the solution of (2.5) is not universal and therefore the solution will depend on the material properties. Here, we refer to the deformation associated with torsion as primary; the radial expansion is treated as the secondary field.

If in (2.5), we set *a priori* $r = R$, then we obtain

$$\frac{d}{dR}[\Sigma_1 + 2\Sigma_2 + \Sigma_3 + \tau^2 R^2 \Sigma_2] - \tau^2 R \Sigma_1 = 0, \tag{2.6}$$

which is an equation that expresses a restriction on the strain-energy density. The determination of the most general strain-energy density satisfying (2.6) is the main question studied by Polignone and Horgan (**15**). In the paper, a necessary condition that the strain energy Σ has to satisfy for pure torsion to be possible is determined, then a subclass of materials for which this condition is satisfied is identified. Kirkinis and Ogden (**21**) have shown that the condition obtained by Polignone and Horgan is not sufficient to ensure that pure torsion is possible in a compressible material with zero traction on the mantle of the cylinder.

To make this claim quantitative, let us observe that any idealized material characterized by special mathematical properties cannot be clearly identified in the real world. That is, all mathematical models have to be viewed as approximations and one has to evaluate how well such models represent reality. We have to make some determination of what we will find acceptable in terms of an approximate answer. Such a determination cannot be totally subjective and one has to have some sort of agreement among those developing and using such models. Whether our criticism concerning the inapplicability of certain models is appropriate or otherwise needs to be judged by the reader.

For example, let us suppose that we wish to consider the mathematical assumption $\Sigma = \Sigma(I_2, I_3)$ with regard to a specific body. This is exactly the constitutive assumption of the celebrated Blatz and Ko model for foamed polyurethane elastomeric foams (**22**). It is imperative, when we make such an assumption, to check whether the experimental data back the validity of the mathematical relationship

$$\partial\Sigma/\partial I_1 = 0. \tag{2.7}$$

Because the first derivatives of the strain energy are the mechanical quantities directly related to the stress, the relation (2.7) is indeed the correct way to check the constitutive assumption $\Sigma = \Sigma(I_2, I_3)$, for example in a biaxial experiment. It is clear that in the real world, our measurement in itself introduces an uncertainty with regard to the measured quantity and that the accuracy of measurement is such that any measurement of the mechanical quantity $\partial\Sigma/\partial I_1$ to check (2.7) will deliver a real number ϵ different from zero. It is not merely the prerogative of the modeler to say when ϵ is sufficiently small enough to be considered zero but, and as always, any theoretical assumption is an approximation and making such an approximation is an art. Roughly speaking, in a nonlinear theory, just because a certain quantity is small it does not follow that everything else connected with this quantity is or remains small. For this reason, we must be very careful in considering constitutive assumptions generated by purely mathematical arguments as the ones arising from the semi-inverse method.[7]

On the other hand, it is clear that approximations must be consistent and for the specific problem under consideration the following problem arises. If a given problem depends on various parameters $\alpha_i, i = 1, \ldots, n$, and depends on a small parameter ϵ such that for $\epsilon = 0$ the secondary deformation may be ignored, the small-ϵ approximation is consistent if for $\epsilon \ll 1$ the secondary field is negligible for any admissible value of the parameters α_i. For pure torsion, this means

$$\max_{R\in[0,A]} \left| \frac{f(R)}{R} - 1 \right| \approx O(\epsilon)$$

or $\sqrt{I_3} \approx 1$ for all $R \in [0, A]$ and for any possible value of α_i.

Now, let us consider the classical Blatz–Ko material (**22**) whose stored energy is given by

$$\Sigma = \frac{\mu}{2}\left[\left(\frac{I_2}{I_3} - 3\right) + 2(\sqrt{I_3} - 1)\right], \tag{2.8}$$

where μ is a constant. This model is of the form $\Sigma = \Sigma(I_2, I_3)$ and it is well known that for the class of materials whose stored energy is given by (2.8), the isochoric simple torsion deformation is controllable.

Let us consider a more general strain energy than (2.8),

$$\Sigma = k(I_1 - 3) + \frac{\mu}{2}\left[\left(\frac{I_2}{I_3} - 3\right) + 2(1 - 2k/\mu)(\sqrt{I_3} - 1)\right], \tag{2.9}$$

where k and μ are constants. The strain energy (2.9) differs from (2.8) by a linear term in I_1 and a *null lagrangian* term $\sqrt{I_3}$ (**23**) such that the usual restrictions imposed by the normalization conditions are satisfied. Clearly as $k \to 0$, from (2.9) we recover (2.8).

The derivatives of the strain energy (2.9) with respect to the invariants are

$$\Sigma_1 = k, \quad \Sigma_2 = \frac{\mu}{2I_3}, \quad \Sigma_3 = \frac{\mu}{2}\left(\frac{1 - 2k/\mu}{\sqrt{I_3}} - \frac{I_2}{I_3^2}\right). \tag{2.10}$$

Now, it is possible to evaluate via a suitable experiment the magnitude of the parameter k with respect to the parameter μ and to decide if the assumption $\Sigma_1 = 0$ is reasonable on the basis of fitting the experimental data. If $k = 0$, the model (2.9) reduces to (2.8); our point is that this model is so special that it is not possible to ensure that the predictions of the mechanical response are not in contradiction with the assumption $k = 0$.

To make this point more quantitative, the next step is to introduce the dimensionless independent variable $\zeta = R/A \in [0, 1]$, the dimensionless dependent variable $F(\zeta) = f/A$ and the quantities $\hat{\tau} = A\tau$ and $\hat{k} = k/\mu$. The introduction of (2.10), evaluated for the specific deformation under consideration, in (2.5) after some simple algebra leads to the equation

$$\hat{k}\left(\frac{\zeta F''}{F} + \frac{F'}{F} - \hat{\tau}^2\zeta - \frac{1}{\zeta}\right) + \frac{3}{2}\frac{\zeta F''}{FF'^4} + \frac{\zeta^3}{2F^4} - \frac{1}{2FF'^3} = 0. \tag{2.11}$$

[7] We point out that this procedure is exactly the reverse of the constitutive assumption that comes out from a rigorous mathematical definition of some physical intuition. Notable examples of this last situation are the concept of frame indifference and material symmetry. In this case, we start by the evidence provided by our observations in the real world and then try to translate this into mathematics; in the former case, we force mathematics to fit into the real world.

(Here $F' = dF/d\zeta$.) Moreover, the dimensionless radial stress component, associated with the deformation, for the model (2.9) is given by

$$\hat{T}_{\zeta\zeta}(\zeta) = 1 - 2\hat{k} + 2\hat{k}\frac{F'^2}{\sqrt{I_3}} - \frac{F'^{-2}}{\sqrt{I_3}}. \tag{2.12}$$

Therefore, for a solid circular cylinder of undeformed radius A subjected to end torques only, the boundary-value problem of interest here is given by (2.11), subject to the conditions $\hat{T}_{\zeta\zeta}(1) = 0$ (that is, $T_{rr}(A) = 0$) and $F(\zeta) \to 0$ as $\zeta \to 0$. We point out that the isochoric solution $F(\zeta) = \zeta$ is controllable for the model (2.9) if and only if $k = 0$ and in this case $\hat{T}_{\zeta\zeta}(1) = 0$.

It seems unlikely that one can obtain an explicit exact solution for (2.11), and even a numerical solution for the boundary-value problem under investigation is not an easy task because the boundary condition on $\zeta = 1$ is nonlinear and of mixed type. For this reason, we consider an approximate $O(\hat{k})$ solution. A straightforward computation gives

$$F(\zeta) \approx \zeta + \hat{k}\frac{\hat{\tau}^2\zeta}{24}(2\zeta^2 - 5) \tag{2.13}$$

and the $O(\hat{k})$ volume approximation is

$$J \approx 1 + \mathcal{V}(\hat{\tau}^2, \zeta)\hat{k}, \tag{2.14}$$

where $\mathcal{V}(\hat{\tau}^2, \zeta) = \frac{1}{12}(4\zeta^2 - 5)\hat{\tau}^2$ is the variation of volume at order $\hat{k}$. The maximum of this variation is

$$|\mathcal{V}(\hat{\tau}^2, 0)| = \frac{5}{12}\hat{\tau}^2. \tag{2.15}$$

Because (2.13) and (2.15) depend not only on $\hat{k}$ but also on τ^2, and because the two parameters are independent, it is clear that the approximation $\hat{k} = 0$ may not be consistent.

Therefore, imagine that you are able to evaluate via an experiment the parameter $\hat{k}$ and that you discover that this parameter is small. It is clear that the experimentally determined number may be never small enough to justify the model corresponding to $\hat{k} = 0$ and only the modeler can choose to set $\hat{k} = 0$, or do otherwise. Our computation shows that such an assumption might be dangerous under certain circumstances. Indeed, while the limiting model for $\hat{k} \to 0$ predicts that during torsion the variation of volume is null, this is not always the case even for very small $\hat{k}$. Therefore, the use of the Blatz–Ko model (2.8) is fraught with danger because it is too special. This situation is peculiar to all the constitutive models that are identified by enforcing via purely mathematical properties special mechanical behaviors such as the controllability of isochoric deformations within the context of a theory to describe the response of compressible bodies.

3. Transverse and longitudinal waves

Another important example is given by the propagation of longitudinal and transverse waves. Introducing the Cartesian coordinates (X_1, X_2, X_3) in the undeformed configuration and the Cartesian coordinates (x_1, x_2, x_3) in the current configuration, we consider the motion given by

$$x_1 = u(X_1, t), \quad x_2 = X_2 + v(X_1, t), \quad x_3 = X_3, \tag{3.1}$$

where the longitudinal wave u and the transverse wave v must be determined from the balance equation. The principal invariants, I_1, I_2 and I_3, are given by

$$I_1 = 2 + u_{X_1}^2 + v_{X_1}^2, \quad I_2 = 1 + 2u_{X_1}^2 + v_{X_1}^2, \quad I_3 = u_{X_1}^2. \tag{3.2}$$

The equations of motion (1.2) reduce to the two scalar equations

$$\rho_{\mathrm{R}} \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial X_1}\left[2(\Sigma_1 + 2\Sigma_2 + \Sigma_3) u_{X_1}\right], \quad \rho_{\mathrm{R}} \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial X_1}\left[2(\Sigma_1 + \Sigma_2) v_{X_1}\right]. \tag{3.3}$$

Here, the strain energy Σ is a function of $u_{X_1}^2$ and $v_{X_1}^2$.

We remark that in the linearized limit, (3.3) reduces to the classical *uncoupled* systems of linear wave equations (**10**).

If we consider the case $u(X_1, t) \equiv X_1$, (3.3) reduces, in the general case, to an *overdetermined* system of two differential equations in the single unknown v. Therefore, it seems, at least at first sight, that it is not possible to ensure the existence of a transverse wave in the nonlinear theory for any material within the constitutive class (1.1). It is possible that for *special* classes of materials, this overdetermined system may have a solution. For example, this is the case for Hadamard materials whose stored energy is given by

$$\Sigma = c_1(I_1 - 3) + c_2(I_2 - 3) + H(I_3), \tag{3.4}$$

where c_1 and c_2 are constants and $H(I_3)$ is an arbitrary function to be specified on the basis of constitutive arguments. The connection of these parameters with the usual Lamé constants μ and λ is $c_1 = \mu + H'(1)$, $c_2 = -\mu - H'(1)$ and $2H''(1) = \lambda + 2\mu$. In the case of Hadamard materials, because $u \equiv X_1$ and $I_3 = 1$, we find that (3.3) reduces to

$$\rho_{\mathrm{R}} \frac{\partial^2 v}{\partial t^2} = 2\mu \frac{\partial^2 v}{\partial X_1^2}. \tag{3.5}$$

In this case, the system is compatible and the transverse wave solution may be computed by solving a linear differential equation, as in the linearized theory of elasticity.

Now, let us consider for the Hadamard material, the case that the longitudinal wave $u(X_1, t)$ is of order ϵ, where $\epsilon \ll 1$. Then it is possible to consider $H(I_3) = (\lambda+\mu)(I_3-1) - 2(\lambda+2\mu)(\sqrt{I_3}-1)$ as proposed by Levinson and Burgess (**24**). Now, (3.3) becomes

$$\rho_{\mathrm{R}} \frac{\partial^2 u}{\partial t^2} = 2(\lambda + 2\mu) \frac{\partial^2 u}{\partial X_1^2}, \quad \rho_{\mathrm{R}} \frac{\partial^2 v}{\partial t^2} = 2\mu \frac{\partial^2 v}{\partial X_1^2}. \tag{3.6}$$

In this case, we find that the equations are the same as in the linearized theory and therefore they are uncoupled.

We take a further step and we consider a very small coupling, that is, we modify the constitutive equation (3.4) to be

$$\Sigma = c_1(I_1 - 3) + c_2(I_2 - 3) + (\lambda + \mu)(I_3 - 1) - 2(\lambda + 2\mu)(\sqrt{I_3} - 1) + kI_3(I_1 - I_3 - 2), \tag{3.7}$$

where k is the coupling parameter. In this case, we compute

$$\rho_{\mathrm{R}} \frac{\partial^2 u}{\partial t^2} = 2(\lambda + 2\mu) \frac{\partial^2 u}{\partial X_1^2} + 2k \frac{\partial}{\partial X_1}\left(v_{X_1}^2 u_{X_1}\right) \tag{3.8}$$

and

$$\rho_{\mathrm{R}} \frac{\partial^2 v}{\partial t^2} = 2\mu \frac{\partial^2 v}{\partial X_1^2} + 2k \frac{\partial}{\partial X_1}\left(u_{X_1}^2 v_{X_1}\right). \tag{3.9}$$

Clearly, the term $\partial\left(u_{X_1}^2 v_{X_1}\right)/\partial X_1$ in the right-hand side of (3.9) may be (at least, at first sight) ignored because the amplitude u is very small. This means that we may consider the system of (3.8) and (3.9) as being decoupled. This is, indeed, a way to justify the Hadamard material (3.4) which is a model that predicts an exact decoupling. Indeed, as we have already remarked any experimental determination of the coupling k may lead to k being small but never zero.

To make the idea rigorous, we must (at least) require that, given a set of boundary conditions (for example, $u = v = 0$ at $X_1 = 0$ and L), the initial condition is such that $u(X_1, 0) \approx O(\epsilon^n)$ with suitable $n \geqslant 1$ and that we have a suitable *a priori* bound on the solution such that for any time, we are ensured $u(X_1, t) \approx O(\epsilon)$. Then if this *priori* bound exists, the initial conditions satisfy the requirements and when k is small it is possible to consider the transverse waves as decoupled from the longitudinal waves.

The point is that from the structure of the equations it is clear that this bound cannot exist for all the admissible range of parameters. Let $k \approx O(\epsilon)$, then when the longitudinal motion is small a better approximation than the linear one is to neglect in (3.9) the term $2\epsilon^2 k\partial_{X_1}\left(u_{X_1}^2 v_{X_1}\right)$ (which is $O(\epsilon^3)$) but to retain the coupling term in (3.8) (which is $O(\epsilon)$). In this case, (3.9) is a classical linear wave equation and introducing $c_T^2 = 2\mu/\rho_R$ this equation admits solutions of the usual form,

$$v(X_1, t) = \sum_{n=1}^{\infty}[A_n \cos(k_n^T t) + B_n \sin(k_n^T t)] \sin(n\pi X_1/L),$$

where $k_n^T (= n\pi c_T/L)$ is the transverse wave number of the nth mode. If we introduce this solution into (3.8), we obtain for $u(X_1, t)$ a linear but nonautonomous equation for which it is possible to search for solutions of the form

$$u(X_1, t) = \sum_{n=1}^{\infty} \eta_n(t) \sin(n\pi X_1/L).$$

Using standard methods of nonlinear oscillations (**25**), we obtain a reduction of the equations to an infinite system of coupled ordinary differential equations in the unknowns η_n. These equations are nonautonomous and they display autoparametric resonance phenomena for some values of the various parameters. Therefore, an *a priori* bound is impossible. This means that it does not matter how small the longitudinal motions are, after a certain time the amplitude of such waves cannot be neglected and a full coupling between transverse and longitudinal motion must be considered. Therefore, the Hadamard model is much too special to be considered as a reasonable idealization of real elastic bodies.

Phenomena of this kind are quite common in classical mechanics. For example in the framework of the elementary and classical theory for holonomic systems, it is well known that unstable normal modes may not contribute to the approximate linear theory. This happens for such modes that are latent at the initial time. Nevertheless, the higher orders neglected in the Lagrangian are able to awaken these latent unstable modes, which brings the system away from equilibrium. A simple and clear example of a mechanical system displaying wake-up of latent modes is reported in page 133 of Biscari *et al.* (**26**).

4. Concluding remarks

The semi-inverse method is the principal tool for obtaining closed-form solutions in continuum mechanics. Usually, we are concerned with the mathematical difficulties that this method may present.

Here, we have discussed certain subtle physical issues that can come into play in using the semi-inverse method.

Our simple examples have to be added to the list of contributions that have recognized some of the issues involved, namely, the paper by Fosdick and Kao (**11**), where the presence of normal stress differences in anti-plane shearing of cylinders of nonlinear materials stimulates secondary deformations, the papers by Rajagopal and Zhang (**12**) and Rajagopal and Mollica (**13**), where the secondary deformation comes into play because of the geometry of the domain, the paper by Horgan and Saccomandi (**27**), where it is shown that the coupling between different deformation fields may also exist when the governing equations are uncoupled because of the boundary conditions, and the paper by De Pascalis *et al.* (**14**), where a detailed study of latent deformation is carried out for a complex deformation field and where a small change in the constitutive assumptions produces a dramatic change of the nature of the solution.

Here, we have considered two simple and well-known deformations to show that if we ignore the full scope of the deformation, we may be misguided and we may miss real and interesting phenomena.

Nonetheless, we must acknowledge the value of simple exact solutions, and we must remember that such solutions have been obtained by inverse or semi-inverse methods in many cases. Therefore, we have to be particularly aware that inverse and semi-inverse methods, while they may lead to solutions of important boundary-value problems, may also lead to the solutions with unsuspected discontinuities or certain other limitations that are more a mathematical by-product than a representation of real-world phenomena.

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