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Remarks on Two Theorems of E. Lieb

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Abstract. The concavity of two functions of a positive matrix A, $\mathrm{Tr}\exp(B+\log A)$ and $\mathrm{Tr}\, A^r K A^p K^*$ (where $B=B^*$ and K are fixed matrices), recently proved by Lieb, can also be obtained by using the theory of Herglotz functions.

In a recent article [1], Lieb has shown, among other things, that, if A_1, A_2, B, K are complex matrices, with $A_1 = A_1^*$, $A_2 = A_2^* > 0$, $B = B^*$, the two functions $t \to \mathrm{Tr}\exp(B + \log(tA_1 + A_2))$ $t \to \mathrm{Tr}(tA_1 + A_2)^r K \cdot (tA_1 + A_2)^p K^*$ (where $0 < r, 0 < p, r + p = s \leqq 1$), are concave functions of the real variable t for sufficient by small t. The object of this note is to indicate how this can also be seen by using the theory of Herglotz functions: in fact, for $A_1 > 0$, the two above mentioned functions can be extended to Herglotz functions holomorphic in the complex plane cut along the real axis from $-\infty$ to $\tau \geqq 0$. Some supplementary work is necessary to study the case of arbitrary self-adjoint A_1. The applicability of the method obviously extends beyond the examples treated here.

Note. in this paper, if A is an element of a C^*-algebra $\mathscr{A}$ with unit, we write $A \geqq 0$ to mean $A = B^*B$ for some $B \in \mathscr{A}$, and $A > 0$ to mean that, for some real number $a > 0$, the inequality $A - a \quad \geqq 0$ holds. Of course $A > 0$ is equivalent to: $A \geqq 0$ and A^{-1} exists as an element of $\mathscr{A}$.

I. Remarks

Let $\mathscr{A}$ be a C^* algebra with unit.

1. Let $A \in \mathscr{A}$ and let $\mathrm{Sp}\, A$ denote its spectrum. Suppose f is a complex function holomorphic in an open set of the complex plane containing $\mathrm{Sp}\, A$. Then $f(A)$ can be defined (as a holomorphic function of A with values in $\mathscr{A}$) by

$$f(A) = \frac{1}{2\pi i} \int_{\mathscr{C}} f(z)\,(z - A)^{-1}\, dz$$

where $\mathscr{C}$ is a contour surrounding $\mathrm{Sp}\, A$. All reasonable definitions of $f(A)$ coincide with this and:

$$\mathrm{Sp}\, f(A) \subset f(\mathrm{Sp}\, A)$$

(see [2], Chapter I, § 4, Proposition 8, p. 47).

2. For any $C \in \mathscr{A}$, denote

$$\operatorname{Re} C = \frac{1}{2}(C + C^*), \qquad \operatorname{Im} C = \frac{1}{2i}(C - C^*).$$

Let $\mathscr{I}^+ = \{C \in \mathscr{A} : \operatorname{Im} C > 0\}$.

Any element C of $\mathscr{I}^+$ is invertible: if $\operatorname{Re} C = A$, $\operatorname{Im} C = B > 0$, $C^{-1} = B^{-\frac{1}{2}}(B^{-\frac{1}{2}} A B^{-\frac{1}{2}} + i)^{-1} B^{-\frac{1}{2}}$.

Moreover $C - z$ is invertible if $\operatorname{Im} z \leqq 0$, so that

$$\operatorname{Sp} C \subset \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

For any $C \in \mathscr{I}^+$, $-C^{-1} \in \mathscr{I}^+$ since:

$$-\operatorname{Im} C^{-1} = C^{-1} \operatorname{Im} C (C^{-1})^*.$$

3. Let $0 < \alpha < 1$. The function $z \to z^\alpha$ will be defined in the cut plane $\mathbb{C} \backslash \mathbb{R}^- = \{z : \operatorname{Im} z \neq 0 \text{ or } \operatorname{Re} z > 0\}$ by the formula:

$$z^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty dt\, t^\alpha \left(\frac{1}{t} - \frac{1}{t+z} \right).$$

If $C \in \mathscr{I}^+$, C^α is defined (since $\operatorname{Sp} C \subset \mathbb{C} \backslash \mathbb{R}^-$) and given by

$$C^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty dt\, t^\alpha (t^{-1} - (t + C)^{-1}).$$

Hence

$$\operatorname{Im} C^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty dt\, t^\alpha \operatorname{Im}\{-(t + C)^{-1}\}.$$

It is easy to check that the integral is absolutely convergent, and since

$$\operatorname{Im}\{-(t + C)^{-1}\} > 0, \qquad C^\alpha \in \mathscr{I}^+.$$

Let $K = -C^{-1} \in \mathscr{I}^+$; by the preceding argument $K^\alpha \in \mathscr{I}^+$, $(-K^\alpha)^{-1} \in \mathscr{I}^+$, so that

$$\operatorname{Im}(-K^\alpha)^{-1} = -\operatorname{Im} e^{-i\alpha\pi} C^\alpha > 0.$$

Thus:

Lemma 1. *If* $C \in \mathscr{I}^+$ *and if* $0 < \alpha < 1$, *then* $C^\alpha \in \mathscr{I}^+$ *and* $-e^{-i\alpha\pi} C^\alpha \in \mathscr{I}^+$; *in other words*

$$\operatorname{Im} e^{-i\alpha\pi} C^\alpha < 0 < \operatorname{Im} C^\alpha.$$

4. Let $C \in \mathscr{I}^+$. Define $z \to \log z$ in the cut plane $\mathbb{C} \backslash \mathbb{R}^-$ by

$$\log z = \int_0^\infty dt \left(\frac{1}{t+1} - \frac{1}{t+z} \right).$$

Then $\log C$ is well defined and given by the formula

$$\log C = \int_0^\infty dt \left(\frac{1}{t+1} - (t+C)^{-1} \right).$$

This implies that $\operatorname{Im} \log C > 0$. Defining again $K = -C^{-1} = e^{i\pi} C^{-1}$ we find that

$$\operatorname{Im} \log e^{i\pi} C^{-1} = \pi - \operatorname{Im} \log C > 0 .$$

Thus

$$0 < \operatorname{Im} \log C < \pi .$$

By Remark 2, this implies that $\operatorname{Sp} \log C \subset \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$.

5. Let $R = R^* > 0$ be in $\mathscr{A}$ and let C be in $\mathscr{I}^+$. Then for any integer $n > 0$,

$$\operatorname{Sp}\left(R^{\frac{1}{n}} C^{\frac{1}{n}} R^{\frac{1}{n}}\right)^n \quad \text{is contained in} \quad \{z \in \mathbb{C} : \operatorname{Im} z > 0\} . \tag{1}$$

For, by Remark 3,

$$\operatorname{Im} R^{\frac{1}{n}} C^{\frac{1}{n}} R^{\frac{1}{n}} > 0 , \qquad \operatorname{Im} e^{i\left(1 - \frac{1}{n}\right)\pi} R^{\frac{1}{n}} C^{\frac{1}{n}} R^{\frac{1}{n}} > 0 ,$$

which (by Remark 2) implies that $\operatorname{Sp}\left(R^{\frac{1}{n}} C^{\frac{1}{n}} R^{\frac{1}{n}}\right)$ is contained in the angle:

$$\left\{ z = \varrho e^{i\theta} : \varrho > 0, 0 < \theta < \frac{\pi}{n} \right\},$$

from which (1) follows by Remark 1.

6. Let $B = B^* \in \mathscr{A}$ and $C \in \mathscr{I}^+$. Then

$$\operatorname{Sp} \exp(B + \log C) \subset \{z \in \mathbb{C} : \operatorname{Im} z > 0\} . \tag{2}$$

For, by Remark 4,

$$0 < \operatorname{Im}(B + \log C) < \pi .$$

Hence

$$0 < \operatorname{Im} \operatorname{Sp}(B + \log C) < \pi \qquad \text{(by Remark 2)}$$

hence (2) by Remark 1. This can also be seen by using the Trotter product formula

$$\exp(B + \log C) = \lim_{n \to \infty} \left(R^{\frac{1}{n}} C^{\frac{1}{n}} R^{\frac{1}{n}}\right)^n , \qquad \text{with} \qquad R = \exp \frac{B}{2} .$$

This converges in norm, which implies resolvent convergence so that (2) follows from (1).

7. Let A and B be elements of $\mathscr{A}$, with

$$A = A_1 + iA_2 , \qquad B = B_1 + iB_2 ,$$

$$A_1 = A_1^* , \qquad A_2 = A_2^* , \qquad B_1 = B_1^* , \qquad B_2 = B_2^* ,$$

satisfying

$$\operatorname{Im} A > 0\,, \quad \operatorname{Im} e^{-i\alpha} A < 0\,, \quad \operatorname{Im} B > 0\,, \quad \operatorname{Im} e^{-i\beta} B < 0\,, \tag{3}$$

where $0 < \alpha$, $0 < \beta$, $\alpha + \beta < \pi$. Then

$$\operatorname{Im} \operatorname{Sp} AB \geqq 0\,. \tag{4}$$

To see this, note that (3) means

$$A_2 > 0\,, \quad A_1 > A_2 \cot\alpha\,, \quad B_2 > 0\,, \quad B_1 > B_2 \cot\beta\,.$$

Consider the following two analytic functions:

$$\xi \to Z(\xi) = A_1 \sin\alpha - A_2 \cos\alpha + e^{\xi} A_2\,, \quad (\xi \in \mathbb{C})$$

$$\xi \to W(\xi) = B_1 \sin\beta - B_2 \cos\beta + e^{\xi} B_2\,, \quad (\xi \in \mathbb{C})\,.$$

For real ξ, $Z(\xi)$ and $W(\xi)$ are positive in $\mathscr{A}$; for $0 < \operatorname{Im} \xi < \pi$, $\operatorname{Im} Z(\xi) > 0$ and $\operatorname{Im} W(\xi) > 0$. Finally $Z(i\alpha) = A \sin\alpha$, $W(i\beta) = B \sin\beta$. Denote

$$R(w, z_1, z_2) = (w - Z(z_1)\, W(z_2))^{-1}\,.$$

This a holomorphic function of three complex variables. Fix w with $\operatorname{Im} w < 0$ and z_1 real; then

$$w - Z(z_1)\, W(z_2) = Z(z_1)^{\frac{1}{2}} \left[w - Z(z_1)^{\frac{1}{2}}\, W(z_2)\, Z(z_1)^{\frac{1}{2}}\right] Z(z_1)^{-\frac{1}{2}}\,.$$

Hence this is invertible if $0 \leqq \operatorname{Im} z_2 \leqq \pi$. Similarly, it is invertible if z_2 is real and if $0 \leqq \operatorname{Im} z_1 \leqq \pi$; in other words, for $\operatorname{Im} w < 0$, the domain of holomorphy of $R(w, z_1, z_2)$ contains an open neighborhood of the "flattened tube":

$$\{z_1, z_2 : \operatorname{Im} z_1 = 0, 0 \leqq \operatorname{Im} z_2 \leqq \pi\} \cup \{z_1, z_2 : \operatorname{Im} z_2 = 0, 0 \leqq \operatorname{Im} z_1 \leqq \pi\}\,.$$

So that, by the "local tube theorem" (see, e.g. [3]), $R(w, z_1, z_2)$ is holomorphic in a neighborhood of:

$$\{w, z_1, z_2 : \operatorname{Im} w < 0, 0 \leqq \operatorname{Im} z_1, 0 \leqq \operatorname{Im} z_2, \operatorname{Im}(z_1 + z_2) \leqq \pi\}\,.$$

In particular, taking $z_1 = i\alpha$, $z_2 = i\beta$, we get (4).

Suppose now A' and B' are elements of $\mathscr{A}$ such that

$$\operatorname{Im} A' < 0\,, \quad \operatorname{Im} B' < 0\,, \quad \operatorname{Im} e^{i\alpha} A' > 0\,, \quad \operatorname{Im} e^{i\beta} B' > 0\,. \tag{5}$$

Applying the preceding result to A'^* and B'^* yields:

$$\operatorname{Im} \operatorname{Sp} A' B' \leqq 0\,.$$

If we take now $A' = e^{-i\alpha} A$, $B' = e^{-i\beta} B$, we find

$$-\operatorname{Im} e^{-i(\alpha+\beta)} \operatorname{Sp} AB \geqq 0\,. \tag{6}$$

Actually (4) and (6) can be sharpened to strict inequalities: our hypothesis (3) implies that $0 \notin \operatorname{Sp} AB$ since A and B are both invertible; moreover there is a $\delta > 0$ so small that $e^{\pm i\delta}A$ and $e^{\pm i\delta}B$ still satisfy (3), so that the spectrum of AB is actually contained in

$$\{z = \varrho e^{i\theta} \in \mathbb{C} : 0 < \varrho, 2\delta < \theta < \alpha + \beta - 2\delta\}$$

Lemma 2. *Let A and B be elements of $\mathscr{A}$ verifying* (3). *Then:*

$$\operatorname{Sp} AB \subset \{z = \varrho e^{i\theta} \in \mathbb{C} : 0 < \varrho, 0 < \theta < \alpha + \beta\}.$$

As a corollary, if A and B are complex $N \times N$ matrices satisfying (3),

$$\operatorname{Tr} AB \subset \{z = \varrho e^{i\theta} : \varrho > 0, 0 < \theta < \alpha + \beta\}.$$

This can also be seen more simply by noticing that

$$\operatorname{Im} \operatorname{Tr} AB = \operatorname{Tr} A_1 B_2 + \operatorname{Tr} A_2 B_1,$$
$$\operatorname{Tr} A_1 B_2 = \operatorname{Tr} B_2^{\frac{1}{2}} A_1 B_2^{\frac{1}{2}} > \operatorname{Tr} B_2^{\frac{1}{2}} A_2 B_2^{\frac{1}{2}} \cot\alpha = \operatorname{Tr} A_2 B_2 \cot\alpha$$
$$\operatorname{Tr} A_2 B_1 > \operatorname{Tr} A_2 B_2 \cot\beta,$$
$$\operatorname{Tr}(A_1 B_2 + A_2 B_1) > (\cot\alpha + \cot\beta) \operatorname{Tr} A_2 B_2 > 0,$$

(since $\cot\alpha + \cot\beta = \sin(\alpha+\beta)/\sin\alpha \sin\beta$). From this one concludes that $\operatorname{Im} \operatorname{Tr} e^{-i(\alpha+\beta)} AB < 0$ by the same substitutions as in the proof of the lemma.

8. *Estimate of* $\|A^\alpha\|$ for $0 < \alpha < 1$.

Let $A \in \mathscr{A}$ with $A = V + iW$, $V = V^* > 0$, $W = W^*$, then

$$A^{-1} = V^{-\frac{1}{2}}(1 + iV^{-\frac{1}{2}} W V^{-\frac{1}{2}})^{-1} V^{-\frac{1}{2}}$$
$$= V^{-\frac{1}{2}}(1 + iT)^{-1} V^{-\frac{1}{2}},$$
$$\|(1 + iT)^{-1}\|^2 = \|(1 + T^2)^{-1}\| \leqq 1.$$

Hence $\|A^{-1}\| \leqq \|V^{-\frac{1}{2}}\|^2 = \|V^{-1}\|$. Let $a = \|A\|$.

$$\frac{\pi}{\sin\alpha\pi} A^\alpha = \int_0^{2a} t^\alpha (t^{-1} - (t + A)^{-1})\, dt$$
$$+ \int_{2a}^{\infty} t^\alpha \left(\sum_{n=1}^{\infty} (-1)^n \frac{A^n}{t^{n+1}} \right) dt.$$

The first integral is bounded in norm by

$$\int_0^{2a} 2t^{\alpha-1} dt = \frac{2(2a)^\alpha}{\alpha} \text{(using } \|(t + A)^{-1}\| \leqq \|(t + V)^{-1}\| \leqq t^{-1}).$$

The second integral is bounded in norm by

$$\int_{2a}^{\infty} t^{\alpha}\left(\frac{1}{t-a}-\frac{1}{t}\right)dt = a^{\alpha}\int_{2}^{\infty} t^{\alpha}\left(\frac{1}{t-1}-\frac{1}{t}\right)dt$$

$$\leqq a^{\alpha}\int_{2}^{\infty} dt\, 2t^{\alpha-2} = \frac{2^{\alpha}a^{\alpha}}{1-\alpha}.$$

Thus $\|A^{\alpha}\| \leqq \left(\frac{2}{\alpha}+\frac{1}{1-\alpha}\right)\|2A\|^{\alpha}\frac{\sin\alpha\pi}{\pi}$.

9. **Lemma 3.** *Let D denote the domain in $\mathscr{A}$ given by*

$$D = \bigcup_{-\frac{\pi}{2}\leqq\theta\leqq\frac{\pi}{2}} \ \bigcup_{0<\varepsilon\in\mathbb{R}} \{A\in\mathscr{A} : \operatorname{Re} e^{-i\theta}A \geqq \varepsilon\}.$$

Let f be a complex valued function on D such that

(i) *f is holomorphic on D.*

(ii) *If* $\operatorname{Im} A > 0$ *then* $\operatorname{Im} f(A) \geqq 0$, *and if* $\operatorname{Im} A < 0$, *then* $\operatorname{Im} f(A) \leqq 0$.

(iii) *For every real $\varrho > 0$ and every $A \in D$*

$$f(\varrho A) = \varrho^{s} f(A)$$

where $0 < s \leqq 1$ (s being independent of ϱ and A).

Then the restriction of f to $\mathscr{A}^{+} = \{A\in\mathscr{A} : A = A^{*} > 0\}$ is concave. More precisely, let $A_1 = A_1^{*}$ and $A_2 = A_2^{*} > 0$ be elements of $\mathscr{A}$. Then, for all sufficiently small real t, and for all integer $n \geqq 1$,

$$\frac{d^{2n}}{dt^{2n}} f(A_2 + tA_1) \leqq 0.$$

(Remark: a function f satisfying the conditions (i), (ii) and (iii) with $s=0$ is a constant).

Proof. Note that condition (ii) implies in particular that $f(A) = f(A^{*})^{*}$. Let $A_1 = A_1^{*}$ and $A_2 = A_2^{*} > 0$ be fixed elements of $\mathscr{A}$, with $A_1 \neq 0$, and let $\tau = \|A_2^{-1}\|\,\|A_1\|$. Denote, for $z\in\mathbb{C}$,

$$F(z) = f(A_2 + zA_1),$$

$$G(z) = f(A_1 + zA_2).$$

$G(z)$ is well defined and analytic when $\operatorname{Im} z \neq 0$ or $\operatorname{Re} z > \tau$. $F(z)$ is well defined and analytic when $|z| < \tau^{-1}$. In the region where $\operatorname{Re} z > \tau$, we have, by analytic continuation of (iii),

$$G(z) = z^{s} F(z^{-1}). \tag{7}$$

Hence this relation extends every where. In particular it shows that $G(z)$ could be analytically continued across the real axis from $-\infty$ to $-\tau$.

Furthermore, G is a Herglotz function, i.e. $\operatorname{Im} G(z)$ has the sign of $\operatorname{Im} z$. This guarantees the existence of boundary values of G, in the sense of tempered distributions, on either side of the real axis. We symbolically denote $G(z \pm i0)$ these distributions, i.e. for every $\varphi \in \mathcal{S}(\mathbb{R})$, we write

$$\int_{-\infty}^{\infty} G(x \pm i0)\, \varphi(x)\, dx = \lim_{\substack{y \to 0 \\ y > 0}} \int_{-\infty}^{\infty} G(x \pm iy)\, \varphi(x)\, dx\,.$$

The Herglotz condition shows that $\operatorname{Im} G(x+i0)$ is, in fact, a positive measure which we denote symbolically by $h(x)$. It is clear from (7) that, for $|z| > 2\tau$, there is a constant K such that $|G(z)| < K|z|^s$.

Let $A \in D$ be such that $\operatorname{Re} e^{-i\theta} A > 0$ for some θ with $-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}$. By analytic continuation of (iii) we have

$$f(A) = e^{is\theta} f(e^{-i\theta} A)\,.$$

Let $\operatorname{Im} A > 0$. Then

$$f(A) = e^{is\pi} f(-A)\,,$$

$$e^{-is\pi} f(A) = f(-A^*)^*\,,$$

so that

$$\operatorname{Im} e^{-is\pi} f(A) \leqq 0\,,$$

(implying, in particular the triviality of the case $s = 0$).

Applying this to $A = A_1 + zA_2$ shows that

$$\operatorname{Im} z > 0 \Rightarrow \operatorname{Im} e^{-is\pi} G(z) \leqq 0 \tag{8}$$

or:

$$\sin s\pi \operatorname{Re} G(z) - \cos s\pi \operatorname{Im} G(z) \geqq 0\,.$$

Denote

$$M(z) = z^{1-s} G(z)\,.$$

This function is identical to G if $s = 1$. If $s < 1$, we have $\sin s\pi > 0$ and, for $\varrho > 0, 0 < \theta < \pi$,

$$\begin{aligned} \operatorname{Im} M(\varrho e^{i\theta}) &= \varrho^{1-s}[\sin(1-s)\,\theta \operatorname{Re} G(\varrho e^{i\theta}) + \cos(1-s)\,\theta \operatorname{Im} G(\varrho e^{i\theta})] \\ &\geqq \varrho^{1-s} (\sin s\pi)^{-1} \sin[(1-s)\,\theta + s\pi] \operatorname{Im} G(\varrho e^{i\theta})\,, \end{aligned}$$

and, since $0 < (1-s)\,\theta + s\pi = \theta + s(\pi - \theta) < \pi$, we find that

$$\operatorname{Im} z > 0 \Rightarrow \operatorname{Im} M(z) \geqq 0\,,$$

a conclusion which, of course, also holds for $s = 1$. Thus, for all s with $0 < s \leqq 1$, M is a Herglotz function. Furthermore, since $M(z) = zF(z^{-1})$, it is analytic in the complement of the cut $\{z : \operatorname{Im} z = 0, |z| \leqq \tau\}$ and, at infinity, is bounded by const. $|z|$. We denote $k(x) = \operatorname{Im} M(x + i0)$

(symbolically) the positive measure with support in $[-\tau, \tau]$ which is the boundary value of $\operatorname{Im} M$, on the real axis, from the upper half plane. Then

$$M(z) = \frac{1}{\pi} \int_{-\tau}^{\tau} \frac{k(t)}{t-z} dt + az + b.$$

It follows that

$$F(z) = zM(z^{-1}) = \frac{1}{\pi} \int_{-\tau}^{\tau} \frac{z^2 k(t)\, dt}{zt-1} + a + bz,$$

for all z in the complement of $\{z : \operatorname{Im} z = 0 \text{ and } |z| \geqq \tau^{-1}\}$. Since $z^2(zt-1)^{-1} = -[-zt^{-1} - t^{-2} + t^{-2}(1-zt)^{-1}]$, we have, for $n \geqq 2$

$$\frac{d^n}{dz^n} F(z) = -\frac{n!}{\pi} \int_{-\tau}^{\tau} \frac{t^{n-2} k(t)}{(1-tz)^{n+1}} dt$$

which is $\leqq 0$ for all even n, and real z such that $|z| < \tau^{-1}$.

II. Applications to Matrices

In this section, we restrict our attention to the case when $\mathscr{A}$ is the set of all complex $N \times N$ matrices. However, our discussion would also hold in more general situations: for example a von Neumann algebra with a finite trace; note that, in the latter case, the trace of an element A is contained in the convex hull of $\operatorname{Sp} A$ ([4], p. 108, Corollary).

Let $B = B^*$ and K be fixed elements of $\mathscr{A}$. We consider the following functions $\mathscr{A} \to \mathbb{C}$:

f_1, given by $f_1(A) = \operatorname{Tr} \exp[B + \log A]$

f_2, given by $f_2(A) = \operatorname{Tr} \left[e^{\frac{B}{2n}} A^{\frac{1}{n}} e^{\frac{B}{2n}} \right]^n$

f_3, given by $f_3(A) = [\operatorname{Tr} A^r K A^p K^*]^{\frac{1}{s}}$

f_4, given by $f_4(A) = \operatorname{Tr} A^r K A^p K^*$,

where n is a positive integer, r and p are real, $0 \leqq r$, $0 \leqq p$, $s = r + p \leqq 1$. From Remarks 6, 5, 7, it follows that $f_j (j = 1, 2, 3, 4)$ satisfies all the conditions of lemma 3. (It is worth noting that, in view of the estimate in Remark 8, f_2, f_3, f_4 are bounded in modulus, on D, by const. $\|A\|^\alpha$. Using this fact would slightly simplify the proof of Lemma 3.) Let $A_1 = A_1^*$ and $A_2 = A_2^* > 0$ be elements of $\mathscr{A}$, denote

$$F_j(z) = f_j(A_2 + zA_1), \quad (j = 1, 2, 3, 4),$$

we find that, for real t with $|t| \leqq \tau^{-1}$,

$$\frac{d^{2m}F_j(t)}{dt^{2m}} \leqq 0 \quad \text{for all integer} \quad m \geqq 1 .$$

In particular, all the functions f_j are concave on $\mathscr{A}^+$.

Using the properties described in Section I, it is easy to construct other examples; examples involving functions of several variables can be constructed by considering $(A_1, \ldots, A_n)$ (where $A_j \in \mathscr{A}_j$) as an element of $\mathscr{A}_1 \oplus \cdots \oplus \mathscr{A}_n$, etc....

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