# REMARKS ON WEAK COMPACTNESS IN $L_1(\mu, X)$

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Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $X$ a Banach space. Denote by $L_1(\mu, X)$ the Banach space of (equivalence classes of) $\mu$-strongly measurable $X$-valued Bochner integrable functions $f: \Omega \to X$ normed by

$$\|f\|_1 = \int_\Omega \|f(\omega)\| \, d\mu(\omega).$$

The problem of characterizing the relatively weakly compact subsets of $L_1(\mu, X)$ remains open. It is known that for a bounded subset of $L_1(\mu, X)$ to be relatively weakly compact it is *necessary* that the set be uniformly integrable; recall that $K \subseteq L_1(\mu, X)$ is uniformly integrable whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mu(E) \leqq \delta$ then $\int_E \|f\| \, d\mu \leqq \varepsilon$, for all $f \in K$. S. Chatterji has noted that in case $X$ is reflexive this condition is also sufficient **[4]**. At present unless one assumes that both $X$ and $X^*$ have the Radon–Nikodym Property (see **[1]**), a rather severe restriction which, for purposes of potential applicability, is tantamount to assuming reflexivity, no good sufficient conditions for weak compactness in $L_1(\mu, X)$ exist. This note puts forth such sufficient conditions; the basic tool is the recent factorization method of W. J. Davis, T. Figiel, W. B. Johnson and A. Pelczynski **[3]**.

First, we present a method of recognizing many weakly compact sets in $L_1(\mu, X)$ *provided* one has a starting point. Later, we present several possible starting points.

Before presenting the first result, recall some basic facts about $L_1(\mu, X)$ and its dual space.

An additive set function $F: \Sigma \to X^*$ (the continuous dual of $X$) is said to be *$\mu$-Lipschitz* whenever there exists $k > 0$ such that $\|F(E)\| \leqq k\mu(E)$, for all $E \in \Sigma$. The space of $\mu$-Lipschitz $X^*$-valued maps is denoted by $V_\infty(\mu, X^*)$. If $F \in V_\infty(\mu, X^*)$ then the Lipschitz norm of $F$ is given by

$$\|F\| = \inf\{k > 0 : \|F(E)\| \leqq k\mu(E), \quad \text{for all} \quad E \in \Sigma\}.$$

If $F \in V_\infty(\mu, X^*)$ and $f = \sum_{i=1}^n x_i \chi_{A_i}$ is an $X$-valued simple function modeled on $\Sigma$ then $\int f \, dF = \sum_{i=1}^n x_i F(A_i)$ is a well-defined scalar satisfying $|\int f \, dF| \leqq \|F\| \, \|f\|_1$. Therefore, $\int dF$ extends in a unique manner to a linear functional $\int dF$ defined on all of $L_1(\mu, X)$. It is well known and easily established that each member of $L_1(\mu, X)^*$ is thus obtained; that is *$V_\infty(\mu, X^*)$ is isometrically isomorphic to $L_1(\mu, X)^*$ with the correspondence between $F \in V_\infty(\mu, X^*)$ and $\varphi \in L_1(\mu, X)^*$ given by*

$$\varphi(f) = \int f \, dF.$$

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For more details regarding $L_1(\mu, X)^*$ we refer the reader to the book of N. Dinculeanu [5].

THEOREM 1. *Let $K$ be a bounded uniformly integrable subset of $L_1(\mu, X)$. Suppose $K$ satisfies the following condition:* (*) *given $\varepsilon > 0$ there exists a measurable set $\Omega_\varepsilon$ such that $\Omega\backslash\Omega_\varepsilon$ has measure no more than $\varepsilon$ and such that $\{f\chi_{\Omega_\varepsilon} : f \in K\}$ is relatively weakly compact in $L_1(\mu|_{\Omega_\varepsilon}, X)$. Then $K$ is relatively weakly compact in $L_1(\mu, X)$.*

*Proof.* Let $(f_n)$ be a sequence of members of $K$. By (*) there exists a set $\Omega_1 \subseteq \Omega$ such that $\mu(\Omega\backslash\Omega_1) < 1$ and such that $\{f\chi_{\Omega_1} : f \in K\}$ is relatively weakly compact in $L_1(\mu|_{\Omega_1}, X)$. Choose a subsequence $(f_n^{(1)})$ of $(f_n)$ and an $f^1 \in L_1(\mu|_{\Omega_1}, X)$ such that

$$f_n^{(1)}\chi_{\Omega_1} \to f^1 \quad \text{weakly in} \quad L_1(\mu|_{\Omega_1}, X).$$

We may clearly assume that $f^1$ is defined on all of $\Omega$ and vanishes off $\Omega_1$. Now use (*) to manufacture a measurable set $\Omega_2$ such that $\mu(\Omega\backslash\Omega_2) < 1/2$ and $\{f\chi_{\Omega_2} : f \in K\}$ is relatively weakly compact in $L_1(\mu|_{\Omega_2}, X)$. We may assume that $\Omega_1 \subseteq \Omega_2$. There then exists a subsequence $(f_n^{(2)})$ of $(f_n^{(1)})$ and a function $f^2 \in L_1(\mu|_{\Omega_2}, X)$ such that

$$f_n^{(2)}\chi_{\Omega_2} \to f^2 \quad \text{weakly in} \quad L_1(\mu|_{\Omega_2}, X).$$

Clearly, we may assume $f^2$ is defined on all of $\Omega$, vanishes off $\Omega_2$ and $f^2(x) = f^1(x)$ $(x \in \Omega_1)$.

The procedure is now clear. We obtain by repeated use of (*) a sequence $(\Omega_n)$ of measurable subsets of $\Omega$ with $\Omega_n \subseteq \Omega_{n+1}$ for all $n$, $\mu(\Omega\backslash\Omega_n) < 1/n$ and a sequence of subsequences $(f_n^{(k)})$ where each $(f_n^{(k+1)})$ is a subsequence of $(f_n^{(k)})$ with $f_n^0 = f_n$ and a sequence $f^k$ of functions defined on $\Omega$ such that for each $k$, $f^{k+1}(x) = f^k(x)$ $(x \in \Omega_k)$, $f^k$ vanishes off $\Omega_k$ and

$$f_n^{(k)}\chi_{\Omega_k} \to f^k\chi_{\Omega_k} \quad \text{weakly in} \quad L_1(\mu|_{\Omega_k}, X).$$

Define $f : \Omega \to X$ in the obvious fashion; that is $f$ is the almost everywhere limit of $f^k$'s. We claim that $f \in L_1(\mu, X)$ and $f_n^{(n)} \to f$ weakly in $L_1(\mu, X)$.

By $K$'s uniform integrability, $\overline{\text{co}}\,\{f\chi_E : f \in K, E \in \Sigma\} = H$ is also uniformly integrable. By Mazur's theorem, each $f^k$ belongs to $H$. Therefore $(f^k)$ is a bounded uniformly integrable sequence of functions in $L_1(\mu, X)$ which converges almost everywhere to $f$. It follows from Vitali's convergence theorem (see [2]) that $f \in L_1(\mu, X)$ and $\|f - f^k\|_1 \to 0$.

Now let us show that $(f_n^{(n)})$ converges weakly to $f$. To this end, let $F \in V_\infty(\mu, X^*)$. From the previous paragraph $\{f - f_n^{(n)} : n \in N\}$ is uniformly integrable and so, given $\varepsilon > 0$ there is $\delta > 0$ such that $\mu(E) \leqq \delta$ implies $\int_E \|f - f_n^{(n)}\|\, d\mu \leqq \varepsilon/2(1 + \|F\|)$. Choose $m$ so that $\mu(\Omega\backslash\Omega_m) \leqq \delta$. Then for $n \geqq m$,

$$\left|\int_{\Omega\backslash\Omega_m} [f - f_n^{(n)}]\, dF\right| \leqq \|F\| \int_{\Omega\backslash\Omega_m} \|f - f_n^{(n)}\|\, d\mu < \varepsilon/2.$$

Now choose $n_0 \geqq m$ so that if $n \geqq n_0$ then $\left|\int_{\Omega_m} f^n - f_n^{(n)}\, dF\right| < \varepsilon/2$. Then for $n \geqq n_0$ we have

$$|F(f)-F(f_n^{(n)})| \leqq \left| \int_{\Omega_m} [f-f_n^{(n)}]\,dF \right| + \left| \int_{\Omega\backslash\Omega_m} [f-f_n^{(n)}]\,dF \right|$$

$$= \left| \int_{\Omega_m} [f^n-f_n^{(n)}]\,dF \right| + \left| \int_{\Omega\backslash\Omega_m} [f-f_n^{(n)}]\,dF \right| < \varepsilon/2+\varepsilon/2 = \varepsilon.$$

THEOREM 2. *Let $K$ be a weakly compact convex subset of $X$ and consider the set $\tilde{K} = \{f \in L_1(\mu, X) : f(\omega) \in K$ for almost all $\omega \in \Omega\}$. $\tilde{K}$ is weakly compact in $L_1(\mu, X)$.*

*Proof.* It is obvious that $\tilde{K}$ is convex and closed (hence by Mazur's theorem weakly closed) in $L_1(\mu, X)$.

By [3], there exists a reflexive Banach space $Y$ and a one-to-one continuous linear operator $T: Y \to X$ such that $K$ is the image under $T$ of some weakly compact convex set $J \subseteq Y$. Note that $T$ is weakly continuous; hence $T|_J$ is a weak homeomorphism. Next, note that $T$ " lifts " in a natural way to a continuous linear operator $\tilde{T}$ from $L_1(\mu, Y)$ to $L_1(\mu, X)$. Moreover, the lifting of $T$ to $\tilde{T}$ takes $\{g \in L_1(\mu, Y) : g(\omega) \in J$ for a.a. $\omega \in \Omega\} = \tilde{J}$ onto $\tilde{K}$. (This needs proof: it is clear that $\tilde{T}$ takes $\tilde{J}$ into $\tilde{K}$. Let $f \in \tilde{K}$. Then define $g : \Omega \to J$ by $g(\omega) = T^{-1}f(\omega)$ if $f(\omega) \in K$ and $g(\omega) = 0$ otherwise; $g(\omega) \in J$ for a.a. $\omega \in \Omega$. Also, $g$ is strongly measurable. In fact, $f$ is strongly measurable and, therefore, is weakly measurable and essentially (weakly) separably valued. $T^{-1}$ is a weak homeomorphism on $K$, and so $T^{-1}f$, which coincides with $g$ ($\mu$ almost everywhere), is weakly measurable and has weakly (hence norm) separable essential range.) By Chatterji's result, $\tilde{J}$ is relatively weakly compact in $L_1(\mu, Y)$, and so $\tilde{K} = \tilde{T}\tilde{J}$ is weakly compact.

REMARK. The above result also holds for $1 < p < \infty$; in this case, $L_p(\mu, Y)$ is reflexive.

It follows from the Krein–Šmulian theorem that *if $K$ is a relatively weakly compact set in $X$ then* the closed convex hull of $K$ is weakly compact and so by Theorem 2 or the previous remark we have

$$\tilde{K}_p = \{f \in L_p(\mu, X) : f(\omega) \in K \quad \text{for a.a. } \omega \in \Omega\}$$

*is relatively weakly compact.*

The next result follows immediately from Theorems 1 and 2.

COROLLARY 3. *Let $\tilde{K}$ be a bounded uniformly integrable subset of $L_1(\mu, X)$. Suppose that given $\varepsilon > 0$ there exists a measurable set $\Omega_\varepsilon$ and a weakly compact set $K_\varepsilon \subseteq X$ such that $\mu(\Omega\backslash\Omega_\varepsilon) < \varepsilon$ and for each $f \in \tilde{K}$, $f(\omega) \in K_\varepsilon$ for almost all $\omega \in \Omega_\varepsilon$. Then $\tilde{K}$ is a relatively weakly compact subset of $L_1(\mu, X)$.*

Proceeding similarly as in Theorem 2 one can prove the next theorem. However, the proof can be given without use of factorization and so we do it in that way.

THEOREM 4. *Let $K$ be a weakly compact subset of $X$ and $J$ a bounded uniformly integrable subset of $L_1(\mu)$. Then $\tilde{K} = \{f(\cdot)x : f \in J, x \in K\}$ is a relatively weakly compact subset of $L_1(\mu, X)$.*

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*Proof.* We note the following characterization of Banach spaces $X$ possessing the Dunford–Pettis property [7]: *a Banach space $X$ has the Dunford–Pettis property if and only if given any Banach space $Y$ and any weakly compact sets $K, J$ in $Y$ and $X$ respectively $K \otimes J = \{k \otimes j : k \in K, j \in J\}$ is weakly compact in $Y \hat{\otimes} X$*—the projective tensor product of $Y$ with $X$. Pertinent remarks here are that all $L_1(\mu)$ spaces have the Dunford–Pettis property [7] and Grothendieck [8] has shown that $L_1(\mu, X)$ is identifiable with $L_1(\mu) \hat{\otimes} X$ in a natural manner.

Remark. A word or two on the aforementioned characterization of the Dunford–Pettis property is appropriate. It is well known that a Banach space $X$ has the Dunford–Pettis property if and only if given weakly convergent sequences $(x_n)$ in $X$ and $(x_n^*)$ in $X^*$ one of which has limit zero, then $\lim_n x_n^* x_n = 0$ (cf. [9, pp. 263–6]). With this in mind, suppose $X$ has the Dunford–Pettis property. To prove that the tensor of weakly compact sets, one factor from each of $X$ and $Y$, is weakly compact, it suffices to show that if $(x_n)$ is weakly null in $X$ and $y_n$ is weakly null in $Y$, then $x_n \otimes y_n$ is weakly null in $X \hat{\otimes} Y$. However, the dual of $X \hat{\otimes} Y$ is identifiable with the space of continuous linear operators from $Y$ to $X^*$, where the action of such an operator $T$ on $x \otimes y$ is given by $T(y)(x)$. If $(y_n)$ is weakly null, then $(Ty_n)$ is weakly null in $X^*$ and so $(Ty_n)(x_n)$ is null by the assumption of the Dunford–Pettis property for $X$. The converse is even easier since one need only test $Y = X^*$ and evaluate the trace functional.

Corollary 5. *Let $\tilde{K}$ be a bounded uniformly integrable subset of $L_1(\mu, X)$. Suppose that given $\varepsilon > 0$ there exist a measurable set $\Omega_\varepsilon$ with $\mu(\Omega \backslash \Omega_\varepsilon) < \varepsilon$, a bounded uniformly integrable subset $J_\varepsilon$ of $L_1(\Omega_\varepsilon, \mu|_{\Omega_\varepsilon})$ and a weakly compact set $K_\varepsilon \subseteq X$ such that if $f \in \tilde{K}$ then $f$ admits a representation $f(\omega) = \Sigma_n \lambda_n f_n(\omega) x_n$, for almost all $\omega \in \Omega_\varepsilon$, for some sequence $(\lambda_n)$ of scalars with $\Sigma_n |\lambda_n| \leqq 1$, $f_n \in J_\varepsilon$, $x_n \in K_\varepsilon$. Then $\tilde{K}$ is relatively weakly compact in $L_1(\mu, X)$.*

Remark. One might hope that Corollary 3 contains the sought after necessary condition for weak compactness in $L_1(\mu, X)$. This hope is destined to doom. Professor J. J. Uhl has noted that if $X$ is not reflexive but $X^*$ has the Radon–Nikodym property then proceeding as in [1], the sequence $(r_n x_n)$ tends to zero weakly (where $(r_n)$ is the sequence of Rademacher functions and $x_n$ is any bounded sequence without a weakly convergent subsequence) in $L_1(\mu, X)$, where $\mu$ is Lebesgue measure on $(0, 1)$ yet $(r_n x_n)$ does not satisfy the criteria set forth in Corollary 3.

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