

## Remoteness, proximity and few other distance invariants in graphs

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**Abstract.** We establish maximal trees and graphs for the difference of average distance and proximity proving thus the corresponding conjecture posed in M. Aouchiche, P. Hansen, Proximity and remoteness in graphs: results and conjectures, Networks 58 (2) (2011) 95102. We also establish maximal trees for the difference of average eccentricity and remoteness and minimal trees for the difference of remoteness and radius proving thus that the corresponding conjectures posed in M. Aouchiche, P. Hansen, Proximity and remoteness in graphs: results and conjectures, Networks 58 (2) (2011) 95102 hold for trees.

### 1. Introduction

All graphs  $G$  in this paper are simple and connected. A vertex set of graph  $G$  will be denoted by  $V$ , an edge set by  $E$ . A number of vertices in  $G$  is denoted by  $n$ , a number of edges by  $m$ . A path on  $n$  vertices will be denoted by  $P_n$ , while  $C_n$  will denote a cycle on  $n$  vertices. A *tree* is the graph with no cycles, and a *leaf* in a tree is any vertex of degree 1. We say that a tree  $G$  is a *caterpillar tree* if it consists of the path  $P$  and the only vertices outside  $P$  are leaves neighboring to vertices on  $P$ .

The *distance*  $d(u, v)$  between two vertices  $u$  and  $v$  in  $G$  is defined as the length of the shortest path connecting vertices  $u$  and  $v$ . The average distance between all pairs of vertices in  $G$  is denoted by  $\bar{l}$ . The *eccentricity*  $e(v)$  of a vertex  $v$  in  $G$  is the largest distance from  $v$  to another vertex of  $G$ . The *radius*  $r$  of a graph  $G$  is defined as the minimum eccentricity in  $G$ , while the *diameter*  $D$  of  $G$  is defined as the maximum eccentricity in  $G$ . The average eccentricity of  $G$  is denoted by  $ecc$ . That is

$$r = \min_{v \in V} e(v), D = \max_{v \in V} e(v), ecc = \frac{1}{n} \sum_{v \in V} e(v).$$

The *center* of a graph is the vertex  $v$  of minimum eccentricity. It is well-known that every tree has either only one center or two centers which are adjacent. The *diametric path* in  $G$  is the shortest path from  $u$  to  $v$ , where  $d(u, v)$  is equal to the diameter of  $G$ .

The *transmission* of a vertex  $v$  in a graph  $G$  is the sum of the distances between  $v$  and all other vertices of  $G$ . The transmission is said to be *normalized* if it is divided by  $n - 1$ . Normalized transmission of a vertex

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$v$  will be denoted by  $\pi(v)$ . The *remoteness*  $\rho$  is defined as the maximum normalized transmission, while the *proximity*  $\pi$  is defined as the minimum normalized transmission. That is

$$\pi = \min_{v \in V} \pi(v), \quad \rho = \max_{v \in V} \pi(v).$$

In other words, the proximity  $\pi$  is the minimum average distance from a vertex of  $G$  to all others, while the remoteness  $\rho$  of a graph  $G$  is the maximum average distance from a vertex of  $G$  to all others. These two invariants were introduced in [1], [2]. A vertex  $v \in V$  is *centroidal* if  $\pi(v) = \pi(G)$ , and the set of all centroidal vertices is the *centroid* of  $G$ .

Recently, these concepts and relations between them have been extensively studied (see [1], [2], [3], [4], [11], [12]). For example, in [3] the authors established the Nordhaus–Gaddum theorem for  $\pi$  and  $\rho$ . In [4] upper and lower bounds for  $\pi$  and  $\rho$  were obtained expressed in number  $n$  of vertices in  $G$ . Also, relations of both invariants with some other distance invariants (like diameter, radius, average eccentricity, average distance, etc.) were studied. The authors posed several conjectures (one of which was solved in [11]), among which the following.

**Conjecture 1.1.** *Among all connected graphs  $G$  on  $n \geq 3$  vertices with average distance  $\bar{l}$  and proximity  $\pi$ , the difference  $\bar{l} - \pi$  is maximum for a graph  $G$  composed of three paths of almost equal lengths with a common end point.*

**Conjecture 1.2.** *Let  $G$  be a connected graph on  $n \geq 3$  vertices with remoteness  $\rho$  and average eccentricity  $ecc$ . Then*

$$ecc - \rho \leq \begin{cases} \frac{3n+1}{4} \frac{n-1}{n} - \frac{n}{2} & \text{if } n \text{ is odd,} \\ \frac{n-1}{4} - \frac{1}{4n-4} & \text{if } n \text{ is even,} \end{cases}$$

with equality if and only if  $G$  is a cycle  $C_n$ .

**Conjecture 1.3.** *Let  $G$  be a connected graph on  $n \geq 3$  vertices with remoteness  $\rho$  and radius  $r$ . Then*

$$\rho - r \geq \begin{cases} \frac{3-n}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4n-4} - \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The inequality is best possible as shown by the cycle  $C_n$  if  $n$  is even and by the graph composed by the cycle  $C_n$  together with two crossed edges on four successive vertices of the cycle.

In this paper we prove Conjecture 1.1, and find the extremal trees for  $ecc - \rho$  and  $\rho - r$  (maximal and minimal trees respectively) showing thus that Conjectures 1.2 and 1.3 hold for trees.

All these conjectures were obtained with the use of AutoGraphiX, a conjecture-making system in graph theory (see for example [6] and [7]). Some results on center and centroidal vertices will be used which are already known in literature since those concepts were also quite extensively studied (see for example [5], [8], [9], [10]).

## 2. Preliminaries

Let us introduce some additional notation for trees and state some auxiliary results known in literature. First, we will often use the notion of the diametric path. So, if a tree  $G$  of diameter  $D$  has diametric path  $P$ , we will suppose that vertices on  $P$  are denoted by  $v_i$  so that  $P = v_0 v_1 \dots v_D$ . When deleting edges of  $P$  from  $G$ , we obtain several connected components which are subtrees rooted in vertices of  $P$ . Now,  $G_i$  will denote the connected component of tree  $G \setminus P$  rooted in vertex  $v_i$  of  $P$  and  $V_i$  will denote the set of vertices of  $G_i$ .

Furthermore, for a tree  $G$  let  $e \in E$  be an edge in  $G$  and  $u \in V$  a vertex in  $G$ . With  $G_u(e)$  we will denote the connected component of  $G - e$  containing  $u$ . We denote  $V_u(e) = V(G_u(e))$  and  $n_u(e) = |V_u(e)|$ . Now the following lemma holds.

**Lemma 2.1.** *The following statements hold for a tree  $G$ :*

1. a vertex  $v \in V(G)$  is a centroidal vertex if and only if for any edge  $e$  incident with  $v$  it holds that  $n_v(e) \geq \frac{n}{2}$ ,
2.  $G$  has at most two centroidal vertices,
3. if there are two centroidal vertices in  $G$ , then they are adjacent,
4.  $G$  has two centroidal vertices if and only if there is an edge  $e$  in  $G$ , such that the two components of  $G - e$  have the same order. Furthermore, the end vertices of  $e$  are the two centroidal vertices of  $G$ .

*Proof.* See [11].  $\square$

Since we will often use transformation of tree  $G$  to  $G'$ , for the sake of notation simplicity we will write  $D'$  for  $D(G')$ ,  $\rho'$  for  $\rho(G')$ ,  $\pi'(v)$  for  $\pi(v)$  in  $G'$ , etc.

### 3. Average distance and proximity

To prove Conjecture 1.1 for trees, we will use graph transformations which transform tree to either:

- 1) path  $P_n$ ,
- 2) a tree consisting of four paths of equal length with a common end point,
- 3) a tree consisting of three paths of almost equal length with a common end point.

So let us first prove that among those graphs the difference  $\bar{l} - \pi$  is maximum for the last.

**Lemma 3.1.** *The difference  $\bar{l} - \pi$  is greater for a tree  $G$  on  $n \geq 4$  vertices consisting of three paths of almost equal length with a common end point than for path  $P_n$ .*

*Proof.* For a path  $P_n$  we have  $\bar{l}(P_n) = \frac{1+n}{3}$ , while  $\pi(P_n) = \frac{n^2}{4(n-1)}$  for  $n$  even and  $\pi(P_n) = \frac{n+1}{4}$  for  $n$  odd. Therefore the difference  $\bar{l}(P_n) - \pi(P_n)$  equals  $\frac{n^2-4}{12(n-1)}$  for  $n$  even and  $\frac{n+1}{12}$  for  $n$  odd. Now, let  $G$  be a tree on  $n$  vertices consisting of three paths of almost equal length with a common end point. Here we have

$$\bar{l}(G) - \pi(G) = \begin{cases} \frac{7n^2+13n-2}{27n} - \frac{2+n}{6} & \text{for } n = 3k + 1 \\ \frac{(7n-8)(1+n)^2}{27n(n-1)} - \frac{n(n+1)}{6(n-1)} & \text{for } n = 3k + 2 \\ \frac{7n^2+6n-9}{27(n-1)} - \frac{n(n+1)}{6(n-1)} & \text{for } n = 3k + 3 \end{cases}$$

where  $k \in \mathbb{N}$ . Now, one has to show that the difference  $\bar{l}(G) - \pi(G)$  is greater than  $\bar{l}(P_n) - \pi(P_n)$  in each of the six possible cases. For example, if  $n$  is even and  $n = 3k + 1$ , then

$$\left(\bar{l}(G) - \pi(G)\right) - \left(\bar{l}(P_n) - \pi(P_n)\right) = \frac{(n+2)^3}{108n(n-1)} > 0$$

and the claim holds. In a similar way it can be seen that the claim holds in each of the remaining five cases.  $\square$

**Lemma 3.2.** *The difference  $\bar{l} - \pi$  is greater for the tree  $G$  on  $n \geq 9$  vertices, where  $n \equiv 1 \pmod{4}$ , consisting of three paths of almost equal length with a common end point than for the tree  $G'$  on  $n$  vertices consisting of four paths of equal length.*

*Proof.* First note that we already established the value of  $\bar{l}(G) - \pi(G)$  in the proof of Lemma 3.1. Now, let us establish the value of  $\bar{l}(G') - \pi(G')$ . Note that  $\bar{l}(G') = \frac{5n^2+14n-3}{24n}$ , while  $\pi(G') = \frac{n+3}{8}$ . Therefore,  $\bar{l}(G') - \pi(G') = \frac{2n^2+5n-3}{24n}$ . Now, one has to show that the difference  $\bar{l}(G) - \pi(G)$  is greater than  $\bar{l}(G') - \pi(G')$  in each of the three possible cases. For example, if  $n = 3k + 1$  then

$$\left(\bar{l}(G) - \pi(G)\right) - \left(\bar{l}(G') - \pi(G')\right) = \frac{2n^2-13n+11}{216n} > 0.$$

In a similar way it can be seen that the claim holds in each of the remaining two cases.  $\square$

**Lemma 3.3.** *Let  $G$  be a tree on  $n \geq 6$  vertices with at least four leaves. Then there is a tree  $G'$  on  $n$  vertices with three leaves for which the difference  $\bar{l} - \pi$  is greater or equal than for  $G$ .*

*Proof.* Let  $u$  be a centroidal vertex of  $G$ , let  $v$  be the branching vertex furthest from  $u$ . We distinguish two cases.

CASE I:  $u \neq v$ . Let  $G_v$  be the subtree of  $G$  rooted in  $v$  consisting of all vertices  $w$  such that path from  $u$  to  $w$  leads through  $v$ . Since  $v$  is a branching vertex furthest from  $u$ , tree  $G_v$  consists of paths with common end  $v$ . Let  $P_1$  and  $P_2$  be two such paths. For  $i = 1, 2$  let  $x_i$  be a vertex in  $P_i$  adjacent to  $v$  and let  $y_i$  be a leaf in  $P_i$ . Let  $G'$  be the tree obtained from  $G$  by deleting edge  $vx_2$  and adding edge  $x_2y_1$ . This transformation is illustrated in Figure 1. Note that  $G'$  has one leaf less than  $G$ . We want to prove that the difference  $\bar{l} - \pi$  is greater for  $G'$  than for  $G$ . For that purpose let us denote  $d_1 = d(v, y_1)$  and  $d_2 = d(v, y_2)$ . Note that

$$\pi(G') \leq \pi'(u) = \pi(u) + \frac{d_1d_2}{n-1} = \pi(G) + \frac{d_1d_2}{n-1}$$

and

$$\bar{l}(G') = \bar{l}(G) + \frac{2}{n(n-1)} \cdot d_2(n-d_1-d_2-1)d_1.$$

From here we obtain

$$\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G) + \frac{d_1d_2}{n-1} \left( \frac{2(n-d_1-d_2-1)}{n} - 1 \right).$$

By Lemma 2.1 we have  $n - d_1 - d_2 - 1 \geq \frac{n}{2}$ , therefore  $\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G)$ .

CASE II:  $u = v$ . Obviously,  $v$  is the only branching vertex in  $G$ . Therefore  $G$  consists of paths with common end point  $v$ . Let  $P_1$  and  $P_2$  be two shortest such path. If  $V \setminus (P_1 \cup P_2 \cup \{v\})$  contains at least  $\frac{n}{2}$  vertices, then we make the same argument as in case I. Otherwise  $G$  is a tree consisting of four paths of equal length with a common end point and the claim follows by Lemma 3.2.

Applying the transformations from cases I and II repeatedly, one obtains the claim.  $\square$

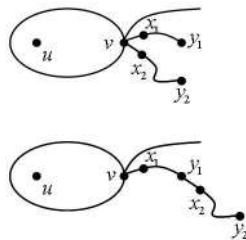


Figure 1: Tree transformation in the proof of Lemma 3.3.

**Lemma 3.4.** *Among trees with three leaves, the difference  $\bar{l} - \pi$  is maximal for the tree  $G$  on  $n$  vertices consisting of three paths of almost equal length with a common end vertex.*

*Proof.* Let  $G$  be a tree with three leaves. That implies  $G$  consists of three paths with a common end vertex. Let  $u$  be centroidal vertex of  $G$ , let  $v$  be the branching vertex furthest from  $u$ . If  $u \neq v$ , then by the same argument as in the case I of the proof of Lemma 3.3 we obtain that the difference  $\bar{l} - \pi$  is greater or equal for path  $P_n$  than for  $G$ . Now the claimed follows from Lemma 3.1. Else if  $u = v$ , then all three paths graph  $G$  consists of have less than  $\frac{n}{2}$  vertices. Let  $v_1$  be the leaf furthest from  $u$  and  $v_2$  the leaf closest to  $u$ . Let  $G'$

be a tree obtained from  $G$  by deleting the edge incident to  $v_1$  and adding the edge  $v_1v_2$ . We want to prove that the difference  $\bar{l} - \pi$  is greater or equal for  $G'$  than for  $G$ . Let  $d_1 = d(u, v_1)$  and  $d_2 = d(u, v_2)$ . We have

$$\pi'(u) = \pi(u) - \frac{d_1 - d_2 - 1}{n - 1}$$

and

$$\bar{l}(G') = \bar{l}(G) - \frac{2}{n(n - 1)}(n - d_1 - d_2 - 1)(d_1 - d_2 - 1).$$

From here we obtain

$$\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G) + \frac{d_1 - d_2 - 1}{n - 1} \left(1 - \frac{2}{n}(n - d_1 - d_2 - 1)\right).$$

Since all three paths of  $G$  have less than  $\frac{n}{2}$  vertices, we can conclude that  $n - d_1 - d_2 - 1 \leq \frac{n}{2}$  from which follows  $\bar{l}(G') - \pi(G') \geq \bar{l}(G) - \pi(G)$ . By repeating this tree transformation we obtain the claim.  $\square$

We can summarize the results of Lemmas 3.1, 3.2, 3.3 and 3.4 into following theorem.

**Theorem 3.5.** *Among all trees on  $n \geq 4$  ( $n \neq 5$ ) vertices with average distance  $\bar{l}$  and proximity  $\pi$ , the difference  $\bar{l} - \pi$  is maximal for a tree  $G$  composed of three paths of almost equal lengths with a common end vertex.*

Therefore, we have proved Conjecture 1.1 for trees on  $n \geq 4$  ( $n \neq 5$ ) vertices. Note that for  $n = 3$  there is only one tree with  $n$  vertices and that is  $P_3$ . For  $n = 5$  the claim does not hold since for a star  $S_5$  (i.e. a graph consisting of one vertex of degree 4 and 4 vertices of degree 1) holds

$$\bar{l}(S_5) - \pi(S_5) = \frac{16}{10} - \frac{4}{4} = \frac{3}{5},$$

while for a tree  $G$  on  $n = 5$  vertices composed of three paths of almost equal lengths with a common end vertex holds

$$\bar{l}(G) - \pi(G) = \frac{18}{10} - \frac{5}{4} = \frac{11}{20}.$$

Therefore obviously  $\bar{l}(S_5) - \pi(S_5) > \bar{l}(G) - \pi(G)$ .

Now we want to prove Conjecture 1.1 for general graphs. If for every graph we find a tree for which difference  $\bar{l} - \pi$  is greater or equal, the Conjecture 1.1 for general graphs will follow from Theorem 3.5.

**Theorem 3.6.** *Among all connected graphs  $G$  on  $n \geq 4$  ( $n \neq 5$ ) vertices with average distance  $\bar{l}$  and proximity  $\pi$ , the difference  $\bar{l} - \pi$  is maximal for a graph  $G$  composed of three paths of almost equal lengths with a common end vertex.*

*Proof.* Let  $G$  be a connected graph on  $n \geq 3$  vertices and let  $u \in V(G)$  be a vertex in  $G$  such that  $\pi(u) = \pi(G)$ . Let  $G'$  be a breadth-first search tree of  $G$  rooted at  $u$ . Obviously,  $\pi(G) = \pi(u) = \pi'(u) \geq \pi(G')$ . As for  $\bar{l}$ , by deleting edges from  $G$  distances between vertices can only increase, therefore  $\bar{l}(G) \leq \bar{l}(G')$ . Now we have  $\bar{l}(G) - \pi(G) \leq \bar{l}(G') - \pi(G')$  and the claim follows from Theorem 3.5.  $\square$

#### 4. Average eccentricity and remoteness

Now, let us find maximal trees for  $ecc - \rho$ , proving thus that the Conjecture 1.2 holds for trees.

**Lemma 4.1.** *Let  $G$  be a tree on  $n$  vertices with diameter  $D$  and let  $P = v_0v_1 \dots v_D$  be a diametric path in  $G$ . If there is  $j \leq D/2$  such that the degree of  $v_k$  is at most 2 for  $k \geq j + 1$ , then the difference  $ecc - \rho$  is greater or equal for the path  $P_n$  than for  $G$ .*

*Proof.* Let  $w$  be a leaf in  $G$  distinct from  $v_0$  and  $v_D$ . Let  $G'$  be a tree obtained from  $G$  by deleting the edge incident to  $w$  and adding the edge  $v_Dw$ . Note that the diameter of  $G'$  equals  $D + 1$ . We want to prove that difference  $ecc - \rho$  did not decrease by this transformation. First note that eccentricity increased by 1 for at least  $n - \frac{D+1}{2}$  vertices. Therefore,  $ecc' \geq ecc + \frac{2n-D-1}{2}$ . As for remoteness, first note that  $\rho(G) = \pi(v_D)$  and  $\rho(G') = \pi'(w)$ . Now, let  $d_w$  be the distance between vertices  $w$  and  $v_D$  in  $G$ , i.e.  $d_w = d(w, v_D)$ . Obviously  $d_w \geq \frac{D+2}{2}$ . Now, we have

$$\pi'(w) = \pi(v_D) + \frac{n - d_w - 1}{n - 1} \leq \pi(v_D) + \frac{2n - D - 4}{2(n - 1)}.$$

Therefore,

$$ecc' - \rho' \geq ecc - \rho + \frac{2n - D - 1}{2n} - \frac{2n - D - 4}{2(n - 1)} \geq ecc - \rho.$$

We obtain the claim by repeating this transformation.  $\square$

**Theorem 4.2.** *Among trees on  $n \geq 3$  vertices, the difference  $ecc - \rho$  is maximal for path  $P_n$ .*

*Proof.* Let  $G$  be a tree on  $n$  vertices and with diameter  $D$ . Let  $P = v_0v_1 \dots v_D$  be a diametric path in  $G$ . Let  $G_i$  be the tree that is connected component of  $G \setminus P$  rooted in  $v_i$  and let  $V_i$  be the vertex set of  $G_i$ . If there is  $j \leq D/2$  such that the degree of  $v_k$  is at most 2 for  $k \geq j + 1$ , then the claim follows from Lemma 4.1. Else, let  $v_j$  and  $v_k$  be vertices on  $P$  of degree at least 3 such that  $j \leq \frac{D}{2} < k$  and  $k - j$  is minimum possible. Let  $w_j$  be a vertex outside of  $P$  adjacent to  $v_j$  and let  $w_k$  be a vertex outside of  $P$  adjacent to  $v_k$ . Let  $G'$  be a tree obtained from  $G$  so that:

- 1) for every vertex  $w$  adjacent to  $v_j$ , except  $w = w_j$  and  $w = v_{j+1}$ , edge  $wv_j$  is deleted and edge  $ww_j$  added,
- 2) for every vertex  $w$  adjacent to  $v_k$ , except  $w = w_k$  and  $w = v_{k-1}$ , edge  $wv_k$  is deleted and edge  $ww_k$  added.

This transformation is illustrated in Figure 2. Note that diameter of  $G'$  equals  $D + 2$ . We want to prove that  $ecc' - \rho' \geq ecc - \rho$ . For that purpose, let us denote

$$V'_j = \{v \in V_j : d(v, w_j) < d(v, v_j)\},$$

$$V'_k = \{v \in V_k : d(v, w_k) < d(v, v_k)\}.$$

Now, let us introduce following partition of set of vertices  $V$

$$X_1 = V_0 \cup \dots \cup V_{j-1} \cup (V_j \setminus (V'_j \cup \{v_j\})),$$

$$X_2 = V'_j,$$

$$X_3 = \{v_j\} \cup V_{j+1} \cup \dots \cup V_{k-1} \cup \{v_k\},$$

$$X_4 = V'_k,$$

$$X_5 = (V_k \setminus (V'_k \cup \{v_k\})) \cup V_{k+1} \cup \dots \cup V_D.$$

Let  $x_i = |X_i|$ . Now, let us compare  $e'(v)$  and  $e(v)$  for every vertex  $v \in V$ . Note that for  $v \in X_2 \cup X_3 \cup X_4$  it holds that  $e'(v) = e(v) + 1$ , while for  $v \in X_1 \cup X_5$  it holds that  $e'(v) = e(v) + 2$ . Therefore,

$$ecc' = ecc + \frac{2x_1 + x_2 + x_3 + x_4 + 2x_5}{n} = ecc + \Delta_1.$$

Now, we want to compare  $\pi'(v)$  and  $\pi(v)$  for every  $v \in V$ . We distinguish several cases depending whether  $v \in X_1, v \in X_2, v \in X_3, v \in X_4$  or  $v \in X_5$ . It is sufficient to consider cases  $v \in X_1, v \in X_2$  and  $v \in X_3$ , since  $v \in X_4$  is analogous to  $v \in X_2$  and  $v \in X_5$  is analogous to  $v \in X_1$ .

If  $v \in X_1$ , then the difference  $d'(v, u) - d(v, u)$  equals 0 for  $u \in X_1$ , equals  $-1$  for  $u \in X_2$ , equals 1 for  $u \in X_3 \cup X_4$  and equals 2 for  $u \in X_5$ . Therefore,

$$\pi'(v) = \pi(v) + \frac{-x_2 + x_3 + x_4 + 2x_5}{n - 1} = \pi(v) + \Delta_2.$$

If  $v \in X_2$ , then the difference  $d'(v, u) - d(v, u)$  equals  $-1$  for  $u \in X_1$ , equals  $0$  for  $u \in X_2 \cup X_3 \cup X_4$  and equals  $1$  for  $u \in X_5$ . Therefore,

$$\pi'(v) = \pi(v) + \frac{-x_1 + x_5}{n - 1} = \pi(v) + \Delta_3.$$

If  $v \in X_3$ , then the difference  $d'(v, u) - d(v, u)$  equals  $1$  for  $u \in X_1 \cup X_5$  and equals  $0$  for  $u \in X_2 \cup X_3 \cup X_4$ . Therefore,

$$\pi'(v) = \pi(v) + \frac{x_1 + x_5}{n - 1} = \pi(v) + \Delta_4.$$

It is easily verified that  $\Delta_1 - \Delta_2 \geq 0$ ,  $\Delta_1 - \Delta_3 \geq 0$  and  $\Delta_1 - \Delta_4 \geq 0$ , so for every  $v \in V$  we obtain  $ecc' - \pi'(v) \geq ecc - \pi(v)$ .

Now, let  $u \in V$  be a vertex for which  $\pi'(u) = \rho(G')$ . We have

$$\begin{aligned} ecc(G') - \rho(G') &= ecc(G') - \pi'(u) \geq ecc(G) - \pi(u) \geq \\ &\geq ecc(G) - \max \{ \pi(v) : v \in V \} = ecc(G) - \rho(G). \end{aligned}$$

□

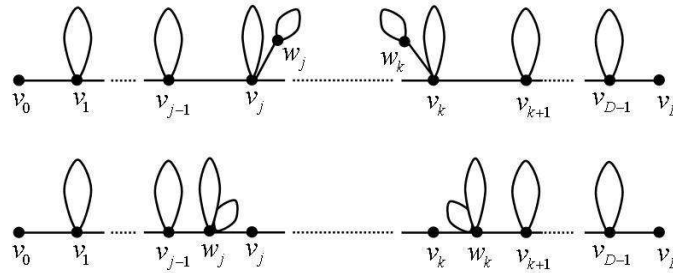


Figure 2: Tree transformation in the proof of Theorem 4.2.

Therefore, we have proved that  $P_n$  is the tree which maximizes the difference  $ecc - \rho$ . Now, from

$$ecc(P_n) - \rho(P_n) = \begin{cases} \frac{n-2}{4} & \text{for even } n, \\ \frac{n}{4} - \frac{2n+1}{4n} & \text{for odd } n. \end{cases}$$

easily follows that Conjecture 1.2 holds for trees.

### 5. Remoteness and radius

First, we want to find minimal trees for  $\rho - r$ . For that purpose, the first step is to reduce the problem to caterpillar trees.

**Lemma 5.1.** *Let  $G$  be a tree on  $n$  vertices. There is a caterpillar tree  $G'$  on  $n$  vertices for which the difference  $\rho - r$  is less or equal than for  $G$ .*

*Proof.* Let  $P = v_0v_1 \dots v_D$  be a diametric path in  $G$ . Let  $G_i$  be the tree that is connected component of  $G \setminus P$  rooted in  $v_i$  and let  $V_i$  be the vertex set of  $G_i$ . Let  $G'$  be the caterpillar tree obtained from  $G$  in a following manner. In a tree  $G_i$  let  $v$  be the non-leaf vertex furthest from  $v_i$ , let  $w_1, \dots, w_k$  be all leaves adjacent to  $v$ , and let  $u$  be the only remaining vertex adjacent to  $v$ . Now, for every  $j = 1, \dots, k$  edge  $w_jv$  is deleted and edge  $w_ju$  is added. This transformation is illustrated in Figure 3. The procedure is done repeatedly in every  $G_i$

( $2 \leq i \leq D - 2$ ) until the caterpillar tree  $G'$  is obtained. Note that  $G'$  has the same diameter (and therefore radius) as  $G$ . What remains to be proved is that remoteness in  $G'$  is less or equal than in  $G$ . It is sufficient to prove that the described transformation does not increase remoteness. Obviously,  $\pi'(u) \leq \pi(u)$  for every  $u \in V \setminus \{v\}$ . Number  $\pi'(v)$  can be greater than  $\pi(v)$ , but note that  $\pi'(v) = \pi'(w_i) \leq \pi(w_i) \leq \rho$ . Therefore  $\rho' \leq \rho$ .  $\square$

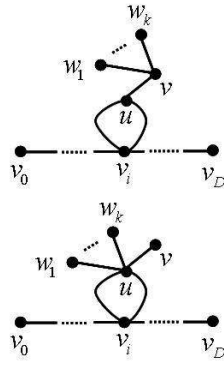


Figure 3: Tree transformation in the proof of Lemma 5.1.

Now that we reduced the problem to the caterpillar trees, let us prove some auxiliary results for such trees. First note that because of Lemma 2.1, a leaf in a tree can not be centroidal vertex. Therefore, in a caterpillar tree a centroidal vertex must be on diametric path  $P$ .

**Lemma 5.2.** *Let  $G \neq P_n$  be a caterpillar tree on  $n$  vertices with diameter  $D$ , remoteness  $\rho$  and only one centroidal vertex. Let  $P = v_0v_1 \dots v_D$  be the diametric path in  $G$  such that  $v_j \in P$  is the only centroidal vertex in  $G$  and every of the vertices  $v_{j+1}, \dots, v_D$  is of the degree at most 2. Then there is a caterpillar tree  $G'$  on  $n$  vertices of the diameter  $D + 1$  and the remoteness at most  $\rho + \frac{1}{2}$ .*

*Proof.* If  $v_j$  is of degree 2, then by Lemma 2.1 follows that  $j \leq \frac{D}{2}$ , so  $\rho = \pi(v_D)$ . Let  $w$  be any leaf in  $G$  distinct from  $v_0$  and  $v_D$ . Let  $G'$  be a graph obtained from  $G$  by first deleting edge incident to  $w$ , then deleting edge  $v_{j-1}v_j$  and adding path  $v_{j-1}wv_j$  instead. This transformation is illustrated in Figure 4. Note that the diameter of  $G'$  is  $D + 1$ , while the remoteness is still obtained for  $v_D$ . Note that the distance from  $v_D$  has increased by 1 for at most  $\frac{n}{2} - 1$  vertices. Therefore,  $\pi'(v_D) \leq \pi(v_D) + \frac{n-2}{2(n-1)}$  from which follows  $\rho' \leq \rho + \frac{1}{2}$  and the claim is proved in this case.

If the degree of  $v_j$  is greater than 2, then  $v_j$  must have at least one neighbor that is a leaf. Let us denote that leaf neighboring to  $v_j$  by  $w$ . Let  $V_L = V_1 \cup \dots \cup V_{j-1}$  and  $V_R = V_{j+1} \cup \dots \cup V_D$ . Since  $v_j$  is a centroidal vertex, from Lemma 2.1 follows that  $V_L$  and  $V_R$  have at most  $\frac{n}{2}$  vertices. If any of them had exactly  $\frac{n}{2}$  vertices, then  $G$  would have two centroidal vertices by Lemma 2.1, which would be contradiction with  $v_j$  being the only centroidal vertex. Therefore, we conclude  $|V_L| \leq \frac{n-1}{2}$  and  $|V_R| \leq \frac{n-1}{2}$ . Now it is possible to divide the set of vertices  $V_j \setminus \{v_j\}$  into two subsets  $V'_j$  and  $V''_j$  such that  $|V_L \cup V'_j| \leq \frac{n-1}{2}$  and  $|V_R \cup V''_j| \leq \frac{n-1}{2}$ . Let  $G'$  be a graph obtained from  $G$  by first deleting the edge incident to  $w$ , then deleting the edge  $v_jv_{j+1}$  and adding a path  $v_jwv_{j+1}$  instead, and finally for every vertex  $v \in V''_j$  the edge  $vv_j$  is deleted and the edge  $vw$  added. This transformation is illustrated in Figure 4. Note that the diameter of  $G'$  is  $D + 1$ . Now, if  $v \in V_L \cup V'_j \cup \{v_j, w\}$  the distance  $d(v, u)$  has increased by 1 only if  $u \in V_R \cup V''_j$ , therefore  $\pi'(v) \leq \pi(v) + \frac{1}{2}$ . If  $v \in V_R \cup V''_j$  the distance  $d(v, u)$  has increased by 1 only if  $u \in V_L \cup V'_j$ , therefore  $\pi'(v) \leq \pi(v) + \frac{1}{2}$ . We conclude  $\rho' \leq \rho + \frac{1}{2}$ , and the claim is proved in this case too.  $\square$



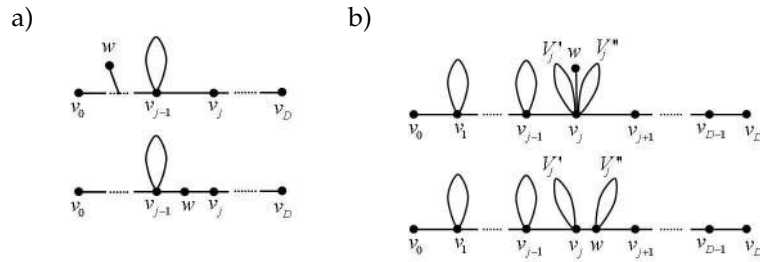


Figure 4: Tree transformations in the proof of Lemma 5.2: a)  $v_j$  is of degree 2, b)  $v_j$  is of degree at least 3.

**Lemma 5.3.** Let  $G \neq P_n$  be a caterpillar tree on  $n$  vertices with diameter  $D$ , remoteness  $\rho$  and exactly two centroidal vertices. Let  $P = v_0v_1 \dots v_D$  be a diametric path in  $G$  such that  $v_j, v_{j+1} \in P$  are centroidal vertices and every of the vertices  $v_{j+1}, \dots, v_D$  is of degree at most 2. Then there is a caterpillar tree  $G'$  on  $n$  vertices of the diameter  $D + 1$  and the remoteness at most  $\rho + \frac{1}{2}$ .

*Proof.* Since  $v_{j+1}$  is centroidal vertex, from Lemma 2.1 follows that  $j \leq \frac{D}{2}$ , so  $\rho = \pi(v_D)$ . Let  $w$  be any leaf in  $G$  distinct from  $v_0$  and  $v_D$ . Let  $G'$  be a graph obtained from  $G$  by first deleting the edge incident to  $w$ , then deleting the edge  $v_jv_{j+1}$  and adding the path  $v_jwv_{j+1}$  instead. The diameter of  $G'$  is  $D + 1$  and the remoteness is still obtained for  $v_D$ . Note that distances from  $v_D$  increased by 1 for at most  $\frac{n}{2} - 1$  vertices, so  $\pi'(v_D) \leq \pi(v_D) + \frac{n-2}{2(n-1)}$ . Therefore,  $\rho' \leq \rho + \frac{1}{2}$ .  $\square$

**Lemma 5.4.** Let  $G \neq P_n$  be a caterpillar tree on  $n$  vertices with diameter  $D$ , remoteness  $\rho$  and exactly two centroidal vertices of different degrees. Let  $P = v_0v_1 \dots v_D$  be a diametric path in  $G$  such that  $v_j, v_{j+1} \in P$  are centroidal vertices and every of the vertices  $v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_D$  is of degree at most 2. Then there is a caterpillar tree  $G'$  on  $n$  vertices of the diameter  $D + 1$  and the remoteness at most  $\rho + \frac{1}{2}$ .

*Proof.* Let  $d_1 = d(v_0, v_j)$  and  $d_2 = d(v_{j+1}, v_D)$ . Without loss of generality we may assume that  $d_1 \leq d_2$ . Since the degrees of  $v_j$  and  $v_{j+1}$  differ, from Lemma 2.1 we conclude  $d_1 \neq d_2$ . Therefore,  $d_1 < d_2$ . From this follows  $j + 1 \leq \frac{D}{2}$ , so  $\rho = \pi(v_D)$ . Let  $G'$  be a graph obtained from  $G$  so that for every leaf  $w$  incident to  $v_j$  (distinct from  $v_0$ ) we delete the edge  $wv_j$  and add the edge  $wv_{j+1}$ . The diameter of  $G'$  is still  $D$ , while the remoteness  $\rho'$  is less or equal than  $\rho$ . Note that  $G'$  has diametric path  $P = v_0v_1 \dots v_D$  and only one centroidal vertex which is  $v_{j+1}$ . All other vertices on  $P$  are of degree at most 2. Therefore, we can apply Lemma 5.2 on  $G'$  and the claim follows.  $\square$

**Lemma 5.5.** Let  $G \neq P_n$  be a caterpillar tree on  $n$  vertices with diameter  $D$ , remoteness  $\rho$  and exactly two centroidal vertices of equal degrees. Let  $P = v_0v_1 \dots v_D$  be a diametric path in  $G$  such that  $v_j, v_{j+1} \in P$  are centroidal vertices and every of the vertices  $v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_D$  is of degree at most 2. Then the difference  $\rho - r$  is less or equal for path  $P_n$  than for  $G$ .

*Proof.* Let  $d_1 = d(v_0, v_j)$  and  $d_2 = d(v_{j+1}, v_D)$ . Since  $v_j$  and  $v_{j+1}$  have equal degrees, and every of the vertices  $v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_D$  is of degree at most 2, we conclude that  $d_1 = d_2$ . Now, we will transform the tree twice which is illustrated in Figure 5. First, since  $G$  is not a path, both  $v_j$  and  $v_{j+1}$  must have a pendent leaf. Denote those leaves with  $w_1$  and  $w_2$  respectively. Let  $G'$  be a graph obtained from  $G$  by first deleting edges incident to  $w_1$  and  $w_2$ , then deleting edge  $v_jv_{j+1}$  and adding path  $v_jw_1w_2v_{j+1}$  instead. Note that  $D' = D + 2$ . Therefore,  $r' = r + 1$ . Note that remoteness in both  $G$  and  $G'$  is obtained for  $v_0$  and  $v_D$ . Since distances from  $v_0$  have increased by 2 for at most  $\frac{n}{2} - 1$  vertices, we conclude  $\pi'(v_0) \leq \pi(v_0) + \frac{2(n-2)}{2(n-1)}$  from which follows  $\rho' \leq \rho + 1$ . Thus we obtain  $\rho' - r' \leq \rho - r$ . If  $G'$  is a path, then the claim is proved. Else, we transform  $G'$  so that for every leaf  $w$  in  $G'$  incident to  $v_j$  edge  $wv_j$  is deleted and edge  $wv_{j+1}$  is added. Also, for every leaf  $w$  in  $G'$  incident to  $v_{j+1}$  edge  $wv_{j+1}$  is deleted and edge  $wv_j$  is added. Note that this transformation changes neither radius neither remoteness. Thus we obtain the tree on which we can repeat the whole procedure. After repeating procedure finite number of times we obtain the path  $P_n$  and the claim is proved.  $\square$

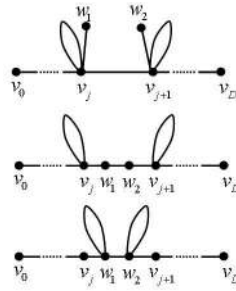


Figure 5: Tree transformations in the proof of Lemma 5.5.

Now that we have established auxiliary results for caterpillar trees, we can find minimal trees for  $\rho - r$  among caterpillar trees.

**Lemma 5.6.** *Let  $G$  be a caterpillar tree on  $n$  vertices. If  $n$  is odd, then the difference  $\rho - r$  is less or equal for path  $P_n$  than for  $G$ . If  $n$  is even, then the difference  $\rho - r$  is less or equal for path  $P_{n-1}$  with a leaf appended to a central vertex than for  $G$ .*

*Proof.* Let  $D$  be the diameter in  $G$  and let  $P = v_0v_1 \dots v_D$  be the diametric path in  $G$ . Suppose  $D \leq n - 3$ . That means  $G$  has at least two leaves outside  $P$ . Let  $v_j \in P$  be a centroidal vertex in  $G$ . If there are two vertices  $v_k$  and  $v_l$  on  $P$  ( $k < j < l$ ) with a pendent leaf on them (distinct from  $v_0$  and  $v_D$ ), then the caterpillar tree  $G'$  obtained from  $G$  by deleting a leaf from  $v_j$  and  $v_k$  and adding a leaf on  $v_{j+1}$  and  $v_{k-1}$  has the same radius and the remoteness which is less or equal than in  $G$ . By repeating this procedure, we obtain a caterpillar tree  $G'$  of the same diameter as  $G$  with diametric path  $P = v_0v_1 \dots v_D$  such that:

1.  $G'$  has exactly one centroidal vertex  $v_j \in P$  and every of the vertices  $v_{j+1}, \dots, v_D$  is of degree at most 2,
2.  $G'$  has two centroidal vertices  $v_j, v_{j+1} \in P$  and every of the vertices  $v_{j+1}, \dots, v_D$  is of degree at most 2,
3.  $G'$  has two centroidal vertices  $v_j, v_{j+1} \in P$  and every of the vertices  $v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_D$  is of degree at most 2.

Therefore, on the obtained graph  $G'$  one of the Lemmas 5.2, 5.3, 5.4 or 5.5 can be applied. If Lemma 5.5 is applied, the claim is proved. Else if Lemma 5.2, 5.3 or 5.4 is applied, we obtain graph  $G'$  of diameter  $D + 1$  and remoteness  $\rho + \frac{1}{2}$ . Since for  $D + 1$  it holds that  $D + 1 \leq n - 2$ , we can apply the whole procedure with  $G = G'$  (as the second step) and thus obtain a caterpillar tree  $G'$  of diameter  $D + 2$  and remoteness  $\rho' \leq \rho + 1$ . Since for thus obtained  $G'$  it holds that  $D' = D + 2$ , we conclude  $r' = r + 1$ . Therefore,  $\rho' - r' \leq \rho - r$ .

Repeating this double step, we obtain a caterpillar tree  $G'$  of diameter  $D' = n - 2$  or  $D' = n - 1$  for which the difference  $\rho - r$  is less or equal than for  $G$ . Now we distinguish several cases with respect to  $D'$  and parity of  $n$ . Suppose first  $D' = n - 1$ . Then  $G' = P_n$ . If  $n$  is odd then the claim is proved. If  $n$  is even it is easily verified that the difference  $\rho - r$  is less for path  $P_{n-1}$  with a leaf appended to a central vertex than for  $G' = P_n$  and the claim is proved in this case too. Suppose now that  $D' = n - 2$ . That means  $G'$  is a path  $P_{n-1}$  with a leaf appended to one vertex of  $P_{n-1}$ . If  $n$  is odd, then deleting the only leaf in  $G'$  to extend it to  $P_n$  increases radius by 1 and remoteness by less than 1, so the claim holds. If  $n$  is even, then deleting the leaf in  $G'$  outside  $P_{n-1}$  and appending it to central vertex of  $P_{n-1}$  preserves the radius and decreases the remoteness. Therefore, the claim holds in this case too.  $\square$

We can summarize the results of these lemmas in the following theorem which gives minimal trees for  $\rho - r$ .

**Theorem 5.7.** *Let  $G$  be a tree on  $n$  vertices. If  $n$  is odd, then the difference  $\rho - r$  is less or equal for path  $P_n$  than for  $G$ . If  $n$  is even, then the difference  $\rho - r$  is less or equal for path  $P_{n-1}$  with a leaf appended to a central vertex than for  $G$ .*

*Proof.* Follows from Lemmas 5.1 and 5.6.  $\square$

For a path  $P_n$  on odd number of vertices  $n$  it holds that  $\rho - r = \frac{1}{2}$  which, together with Theorem 5.7, obviously implies that trees on odd number of vertices satisfy Conjecture 1.3. Now, let us consider graph  $G$  on even number of vertices  $n$  consisting of a path  $P_{n-1}$  with a leaf appended to a central vertex. For  $G$  it holds that  $\rho - r = \frac{n}{2(n-1)}$  which implies that trees on even number of vertices satisfy Conjecture 1.3 too.

## 6. Conclusion

We have established that the maximal tree for  $\bar{l} - \pi$  is a tree composed of three paths of almost equal lengths with a common end point. Thus, we proved that Conjecture 1.1 posed in [4] for general graph holds for trees. Using reduction of a graph to a corresponding subtree, this result enabled us to prove Conjecture 1.1 for general graphs too. Furthermore, we established that the maximal tree for  $ecc - \rho$  is the path  $P_n$  and that the minimal tree for  $\rho - r$  is the path  $P_n$  in case of odd  $n$  and the path  $P_{n-1}$  with a leaf appended to a central vertex in case of even  $n$ . Since for these extremal trees Conjectures 1.2 and 1.3 posed in [4] hold, it follows that those conjectures hold for trees.

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