# Removal of internal multiples with the common-focus-point (CFP) approach: Part 1 - Explanation of the theory 

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#### Abstract

Removal of surface and internal multiples can be formulated by removing the influence of downwardscattering boundaries and downward-scattering layers. The involved algorithms can be applied in a modeldriven or a data-driven way. A unified description is proposed that relates both types of algorithms based on wave theory. The algorithm for the removal of surface multiples shows that muted shot records play the role of multichannel prediction filters. The algorithm for the removal of internal multiples shows that muted CFP gathers play the role of multichannel prediction filters. The internal multiple removal algorithm is illustrated with numerical examples. The conclusion is that the layerrelated version of the algorithm has significant practical advantages.


## INTRODUCTION

In an article from the late 1970s, Kennett (1979) describes a 1D forward model for surface-related multiples and proposes a 1D inversion scheme. However, his algorithm contains too many simplifications on both the data acquisition and subsurface properties to be successful on real data. Riley and Claerbout (1976) describe a 2D forward model for surfacerelated multiples, but they do not arrive at a proper inversion scheme. Berkhout (1982) has proposed a multidimensional inversion algorithm for the removal of surface-related as well as internal multiples. Essential in this formulation are that any subsurface model can be handled and, above all, that the data acquisition properties are taken into account. For instance, Berkhout's feedback model shows that the inverse of the source wavelet must be included in the removal process. The latter turns out to be an absolute necessity for success on field data. In his doctoral thesis, Verschuur (1991) successfully
demonstrates on field data that the inverse source wavelet can be estimated by making use of a least-squares subtraction process. Since then, many other authors have elaborated on this approach. In fact, Weglein et al. (1997) propose an algorithm to remove multiples with the aid of the inverse scattering theory.

In Berkhout and Verschuur (1997) the extension from surface to internal multiples is reformulated by replacing shot records with so-called common-focus-point (CFP) gathers. In Berkhout (1999) this concept is generalized by also considering the internal multiples generated by a complete layer instead of a single interface. In this paper, the theory for the boundary and layer approach to internal multiple removal is explained, and the advantages of using the CFP domain for the implementation are discussed. The layer approach can also be applied for surface-related multiples. This may have advantages in the situation of irregular sampling. The paper starts with summarizing the underlying theory of primary wavefield measurements, often referred to as the WRW model, followed by including the feedback path for multiples.

## SUMMARY OF THE WRW MODEL

The so-called WRW framework is an attractive starting point for the design of acquisition geometries and the derivation of seismic processing algorithms. In this 3D framework, the discrete version of the model for primary wavefields is formulated in the $(x, y, \omega)$ domain in terms of monochromatic vectors and matrices (Berkhout, 1982). For a source wavefield,

$$
\begin{equation*}
\mathbf{S}_{j}^{+}\left(z_{m}, z_{0}\right)=\mathbf{W}\left(z_{m}, z_{0}\right) \mathbf{S}_{j}\left(z_{0}\right) \tag{1a}
\end{equation*}
$$

where the plus sign denotes downgoing waves. For reflected wavefields,

$$
\begin{equation*}
\Delta \mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right)=\sum_{m=1}^{\infty} \mathbf{W}\left(z_{0}, z_{m}\right) \mathbf{R}\left(z_{m}, z_{m}\right) \mathbf{S}_{j}^{+}\left(z_{m}, z_{0}\right) \tag{1b}
\end{equation*}
$$

[^0]where the minus sign denotes upgoing waves. For wavefield measurements,
\[

$$
\begin{equation*}
\Delta \mathbf{P}_{j}\left(z_{0}, z_{0}\right)=\mathbf{D}\left(z_{0}\right) \Delta \mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right) \tag{1c}
\end{equation*}
$$

\]

where matrix $\mathbf{P}$ contains all seismic measurements, with $\Delta$ indicating that only primary reflection information is involved. In equations 1 , depth level $z_{m}$ may be generalized to depth surface $z_{m}(\mathbf{x})$, where $\mathbf{x}=(x, y)$.

In equations 1 , detector matrix $\mathbf{D}\left(z_{0}\right)$ represents the angledependent detector information for one shot record (one row defining the angle-dependent detection properties at one detector station), and source vector $\mathbf{S}_{j}\left(z_{0}\right)$ represents the angle-dependent emission information at source position $\left(\mathbf{x}_{j}, z_{0}\right)$; matrices $\mathbf{W}\left(z_{0}, z_{m}\right)$ and $\mathbf{W}\left(z_{m}, z_{0}\right)$ quantify the angle-dependent propagation properties among all individual gridpoints of surface $z_{0}$ and depth level $z_{m}$ (each column represents an upgoing and downgoing impulse response, respectively), and matrix $\mathbf{R}\left(z_{m}, z_{m}\right)$ quantifies the angle-dependent reflection properties for downward-traveling waves (each column representing one impulse response that transfers a downgoing into an upgoing wavefield) at depth level $z_{m}$ (see Figure 1). For a complex over burden, the columns of $\mathbf{W}$ define multiarrival events. In addition, $\mathbf{W}$ may contain frequency-dependent transmission effects (elastic absorption) from fine layering.
From equations 1 , it follows that the primary reflection measurements may also be written as

$$
\begin{equation*}
\Delta \mathbf{P}_{j}\left(z_{0}, z_{0}\right)=\mathbf{D}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{0}\right) \mathbf{S}_{j}\left(z_{0}\right) \tag{2a}
\end{equation*}
$$


b)


Figure 1. (a) WRW model for primary reflections $(m>0)$ in terms of vectors and matrices, where $\Delta \mathbf{P}_{j}=\mathbf{D} \Delta \mathbf{X} \mathbf{S}_{j}$ at $z_{0}$. Note that $\mathbf{W}$ and $\mathbf{R}$ may include frequency-dependent dispersion from fine layering interference. Note also that $\mathbf{W}$ may involve multiple raypaths. (b) One basic element of the WRW model for primary reflections, visualized in terms of a simple raypath $(m>0)$, where $\Delta \mathbf{X}=\sum \mathbf{W R W}$ at $z_{0}$.
where the transfer matrix $\Delta \mathbf{X}\left(z_{0}, z_{0}\right)$ defines the earth's multidimensional impulse responses for primary reflections in a half-space $z>z_{0}$ (each column represents one impulse response):

$$
\begin{equation*}
\Delta \mathbf{X}\left(z_{0}, z_{0}\right)=\sum_{m=1}^{\infty} \mathbf{W}\left(z_{0}, z_{m}\right) \mathbf{R}\left(z_{m}, z_{m}\right) \mathbf{W}\left(z_{m}, z_{0}\right) \tag{2b}
\end{equation*}
$$

If $\Delta \mathbf{G}$ defines the two-way Green's functions for primary reflections, then we may write

$$
\begin{equation*}
\Delta \mathbf{X}=\delta_{z} \Delta \mathbf{G} \tag{2c}
\end{equation*}
$$

the derivative being taken at the source positions. From a physics point of view, equation $2 c$ means that $\Delta \mathbf{G}$ and $\Delta \mathbf{X}$ refer to a monopole and a dipole response, respectively.

Equations 2a and 2b, developed in the late 1970s, are generally referred to as the WRW model (Berkhout, 1982). The WRW model facilitates a conceptual formulation of primary wavefield measurements that includes the influence of acquisition geometry and mode conversion. Note that for oceanbottom cable (OBC) data, the detector surface $z_{0}$ should be replaced by the sea bottom $z_{1}$.

The WRW model is formulated in terms of medium operators and not in terms of medium parameters. This means that equations $2 a$ and $2 b$ may be considered as inverse imaging equations. Much more than the classical forward-modeling equations, the inverse imaging equations give valuable insight in the design of stepwise inversion algorithms (Berkhout, 1989). It is also important to realize that reflection matrix $\mathbf{R}$ represents full angle-dependent elastic scattering (as it occurs in reality), meaning that it cannot be simplified to a diagonal matrix.

From equation 2 a it follows that one trace measured by a detector (array) at position $\mathbf{x}_{i}$ as a result of a seismic source (array) at position $\mathbf{x}_{j}$ is given by the scalar (Figure 1)

$$
\begin{equation*}
\Delta \mathbf{P}_{i j}\left(z_{0}, z_{0}\right)=\mathbf{D}_{i}^{\dagger}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{0}\right) \mathbf{S}_{j}\left(z_{0}\right) \tag{3a}
\end{equation*}
$$

where the dagger indicates we are dealing with a row vector, and row vector $\mathbf{D}_{i}^{\dagger}\left(z_{0}\right)$ represents the detector (array) at position $\mathbf{x}_{i}$.

If the source vectors are combined into one source matrix $\mathbf{S}$, and the corresponding primary responses at detector (array) position $\mathbf{x}_{i}$ are combined into row vector $\Delta \mathbf{P}_{i}^{\dagger}$, then equation 3a may be extended to

$$
\begin{equation*}
\Delta \mathbf{P}_{i}^{\dagger}\left(z_{0}, z_{0}\right)=\mathbf{D}_{i}^{\dagger}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{0}\right) \mathbf{S}\left(z_{0}\right) \tag{3b}
\end{equation*}
$$

Equation 3b formulates the expression of a detector gather for detector (array) position $i$. Note that column vector $\Delta \mathbf{P}_{j}\left(z_{0}, z_{0}\right)$ in equation 2 a and row vector $\Delta \mathbf{P}_{i}^{\dagger}\left(z_{0}, z_{0}\right)$ in equation 3 b define, respectively, one column and one row of the so-called data matrix $\Delta \mathbf{P}\left(z_{0}, z_{0}\right)$. If the responses of all sources under consideration are measured by the same detector distribution, then it follows from the foregoing that matrix $\Delta \mathbf{P}\left(z_{0}, z_{0}\right)$ can be written as

$$
\begin{equation*}
\Delta \mathbf{P}\left(z_{0}, z_{0}\right)=\mathbf{D}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{0}\right) \mathbf{S}\left(z_{0}\right) \tag{3c}
\end{equation*}
$$

where the primary transfer matrix $\Delta \mathbf{X}\left(z_{0}, z_{0}\right)$ is given by equation 2 b .

If we include any type of multiple, equation 3 c can be generalized to

$$
\begin{equation*}
\mathbf{P}\left(z_{0}, z_{0}\right)=\mathbf{D}\left(z_{0}\right) \mathbf{X}\left(z_{0}, z_{0}\right) \mathbf{S}\left(z_{0}\right) \tag{4a}
\end{equation*}
$$

with $z=z_{0}$ representing a stress-free surface. If we include only internal multiples, equation 3 c can be extended to

$$
\begin{equation*}
\left\{\mathbf{P}\left(z_{0}, z_{0}\right)\right\}_{0}=\mathbf{D}\left(z_{0}\right)\left\{\mathbf{X}\left(z_{0}, z_{0}\right)\right\}_{0} \mathbf{S}\left(z_{0}\right), \tag{4b}
\end{equation*}
$$

where the subscript $\left\}_{0}\right.$ denotes that the multiples related to $z_{0}$ have been removed. Similarly, we can define

$$
\begin{equation*}
\left\{\mathbf{P}\left(z_{0}, z_{0}\right)\right\}_{n}=\mathbf{D}\left(z_{0}\right)\left\{\mathbf{X}\left(z_{0}, z_{0}\right)\right\}_{n} \mathbf{S}\left(z_{0}\right) \tag{4b}
\end{equation*}
$$

where the subscript $\left\}_{n}\right.$ denotes that all multiples related to $z \leq z_{n}$ have been removed and that only internal multiples related to levels $z>z_{n}$ are still present in the data.
We show in this paper that the expression for removing all multiples is

$$
\begin{equation*}
\Delta \mathbf{P}\left(z_{0}, z_{0}\right)=\mathbf{P}\left(z_{0}, z_{0}\right)-\mathbf{M}\left(z_{0}, z_{0}\right) \tag{5a}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathbf{M}\left(z_{0}, z_{0}\right)=\sum_{n} \mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{n}\right)\left\{\overline{\mathbf{P}}\left(z_{n}, z_{0}\right)\right\}_{n-1}, \tag{5b}
\end{equation*}
$$

and that the removal of multiples related to boundary $z_{n}$ only can be written as

$$
\begin{equation*}
\left\{\mathbf{P}\left(z_{0}, z_{0}\right)\right\}_{n}=\left\{\mathbf{P}\left(z_{0}, z_{0}\right)\right\}_{n-1}-\left\{\delta \mathbf{M}\left(z_{0}, z_{0}\right)\right\}_{n}, \tag{5c}
\end{equation*}
$$

with

$$
\begin{equation*}
\left\{\delta \mathbf{M}\left(z_{0}, z_{0}\right)\right\}_{n}=\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{n}\right)\left\{\overline{\mathbf{P}}\left(z_{n}, z_{0}\right)\right\}_{n-1} . \tag{5d}
\end{equation*}
$$

Note that $\mathbf{F}_{\mathbf{p r}}$ represents a multidimensional prediction filter. The overbar above $\mathbf{P}$ indicates a mute of all events up to and including the deepest indicated depth level (see also Appendix A). Equations 5a-5d include both the predicted surface $(n=0)$ and internal $(n>0)$ multiples. Each column of $\left\{\overline{\mathbf{P}}\left(z_{n}, z_{0}\right)\right\}_{n-1}$ represents a downward-extrapolated shot record with upward-traveling reflections at level $z_{n}$; all multiples related to levels $z \leq z_{n-1}$ and all primary reflections related to levels $z \leq z_{n}$ have been removed (see also Figure A-2). Each row of $\left\{\overline{\mathbf{P}}\left(z_{n}, z_{0}\right)\right\}_{n-1}$ represents a CFP gather with similar reflections (primaries for $z>z_{n}+$ internal multiples for $z>z_{n-1}$. Note that for $n=0\left\{\overline{\mathbf{P}}\left(z_{n}, z_{0}\right)\right\}_{n-1}=\mathbf{P}\left(z_{0}, z_{0}\right)$, being the input data with all primaries and multiples.

In the remainder of this paper we show that multiples are always predicted by applying a multiple-prediction operator to the seismic data with multiples related to the boundary or layer of consideration still included (the input data). The multiple-prediction operator should only contain reflections from below the multiple-generating boundary or layer, without including multiples related to this boundary or layer. In practice, in the model-driven implementation, this operator is approximated by the primaries-only response of the subsurface; in the data-driven implementation, the data with multiples are used in the first iteration.

In Appendix A, the operator formulation for seismic wave theory is summarized, and wavefield operators are visualized in the figures. For multiple scattering we need to distinguish between scattering operators that transfer downwardtraveling waves into upward-traveling waves, $\mathbf{R}\left(z_{m}, z_{m}\right)$ and
$\mathbf{X}\left(z_{m}, z_{m}\right)$, and scattering operators that transfer upwardtraveling waves into downward-traveling waves, $\mathbf{R}^{\wedge}\left(z_{m}, z_{m}\right)$ and $\mathbf{X}^{\wedge}\left(z_{m}, z_{m}\right)$.

## SURFACE-RELATED MULTIPLES

Using the feedback model, the expression for primary wavefields can be extended to include surface-related multiples. The resulting extended expression is very suitable for the derivation of effective multiple-removal algorithms, model driven as well as data driven.

## Feedback model for surface-related multiple reflections

The WRW model for primary reflections can be easily extended to include surface-related multiples by adding a feedback path at the surface (Berkhout, 1982). For this situation, we may write for a shot record with its source at position $\mathbf{x}_{j}$

$$
\begin{align*}
\mathbf{P}_{j}\left(z_{0}, z_{0}\right)= & \mathbf{D}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{0}\right) \\
& \times\left[\mathbf{S}_{j}\left(z_{0}\right)+\mathbf{R}^{\wedge}\left(z_{0}, z_{0}\right) \mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right)\right] \tag{6a}
\end{align*}
$$

or

$$
\begin{align*}
\mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right)= & \Delta \mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right)+\Delta \mathbf{X}\left(z_{0}, z_{0}\right) \\
& \times\left[\mathbf{R}^{\wedge}\left(z_{0}, z_{0}\right) \mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right)\right] \tag{6b}
\end{align*}
$$

In equations 6, boundary operator $\mathbf{R}^{\wedge}$ transforms upgoing into downgoing wavefields (from upgoing to downgoing wavefields), and half-space operator $\Delta \mathbf{X}$ transforms downgoing into upgoing wavefields (from downgoing to upgoing wavefields). Considering equations 6 , it follows that the expression for the surface-related multiples is given by

$$
\begin{equation*}
\left\{\delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{0}=\mathbf{D}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{0}\right)\left[\mathbf{R}^{\wedge}\left(z_{0}, z_{0}\right) \mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right)\right] \tag{7a}
\end{equation*}
$$

$\Delta \mathbf{X}\left(z_{0}, z_{0}\right)$ being given by equation 2 b . Figure 2 a displays the diagram of the feedback model (equation 6a), showing the extra multiple-generating feedback loop with respect to Figure 1a. Figure 2 b visualizes one element of equation 7 a in terms of raypaths. Note that multiplication of $\mathbf{P}_{j}^{-}$by $\Delta \mathbf{X} \mathbf{R}^{\wedge}$ causes an extra round trip in the subsurface, transforming primaries into first-order multiples, first-order multiples into second-order multiples, etc.

Using equation 3 c , we can also write equation 7 a as

$$
\begin{equation*}
\left\{\delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{0}=\Delta \mathbf{P}\left(z_{0}, z_{0}\right) \mathbf{A}\left(z_{0}, z_{0}\right) \mathbf{P}_{j}\left(z_{0}, z_{0}\right), \tag{7b}
\end{equation*}
$$

where

$$
\begin{align*}
\mathbf{A}\left(z_{0}, z_{0}\right) & =\mathbf{S}^{-1}\left(z_{0}\right) \mathbf{R}^{\wedge}\left(z_{0}, z_{0}\right) \mathbf{D}^{-1}\left(z_{0}\right) \\
& \approx-\left[\mathbf{D}\left(z_{0}\right) \mathbf{S}\left(z_{0}\right)\right]^{-1} \tag{7c}
\end{align*}
$$

In our algorithm, $\mathbf{A}$ is estimated from the data and the inverse of DS is not computed.

In equations 6 and equations 7 a and 7 b , the influence of internal multiples in the extra round trip is neglected. If we include these multiples as well, $\Delta \mathbf{X}$ needs to be replaced by $\{\mathbf{X}\}_{0}$
and $\Delta \mathbf{P}$ needs to be replaced by $\{\mathbf{P}\}_{0}$ :

$$
\begin{align*}
\mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right)= & \left\{\mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right)\right\}_{0}+\left\{\mathbf{X}\left(z_{0}, z_{0}\right)\right\}_{0} \\
& \times\left[\mathbf{R}^{\wedge}\left(z_{0}, z_{0}\right) \mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right)\right],  \tag{8a}\\
\left\{\delta \mathbf{M}_{j}^{-}\left(z_{0}, z_{0}\right)\right\}_{0}= & \left\{\mathbf{P}^{-}\left(z_{0}, z_{0}\right)\right\}_{0} \mathbf{A}\left(z_{0}, z_{0}\right) \mathbf{P}_{j}\left(z_{0}, z_{0}\right),  \tag{8b}\\
\left\{\mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right)\right\}_{0}= & \mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right)-\left\{\delta \mathbf{M}_{j}^{-}\left(z_{0}, z_{0}\right)\right\}_{0} . \tag{8c}
\end{align*}
$$

Bear in mind that in equations $8,\{\mathbf{X}\}_{0}$ and $\{\mathbf{P}\}_{0}$ refer to responses of a lower half-space $\left(z>z_{0}\right)$ with a reflection-free upper half-space ( $z \leq z_{0}$ ). Hence, $\{\delta \mathbf{M}\}_{0}=\mathbf{P}-\{\mathbf{P}\}_{0}$ represents the surface-related multiples.
Note also that multiplication of $\mathbf{P}_{j}$ by $\left\{\mathbf{P}^{-}\right\}_{0} \mathbf{A}$ causes an extra round trip in the subsurface, where this bounce includes internal multiples (compare equations 7 b and 8 b ). The full model for surface-related multiples is shown in Figure 2c.


Figure 2. (a) Feedback model for primary reflections ( $m>0$ ) and surface-related multiples using the boundary formulation, the multiple-generating boundary being given by $z_{0}=z_{0}(x$, $y$ ) and the downward-reflection operators of this boundary being represented by the columns of the matrix $\mathbf{R}^{\wedge}\left(z_{0}, z_{0}\right)$. Here, $\mathbf{P}_{j}^{-}=\Delta \mathbf{P}_{j}^{-}+\Delta \mathbf{X} \mathbf{R}^{\wedge} \mathbf{P}_{j}^{-}$at $z_{0} ; \mathbf{P}_{j}=\mathbf{D} \mathbf{P}_{j}^{-}$. (b) One basic element of the prediction process for surface-related multiples, visualized in terms of simple raypaths ( $m>0$ ), where $\mathbf{M}_{j}^{-}=\Delta \mathbf{X} \mathbf{R}^{\wedge} \mathbf{P}_{j}^{-}$at $z_{0} ; \mathbf{M}_{j}=\mathbf{D} \mathbf{M}_{j}^{-}$. (c) Feedback model similar to (a), but now internal multiples are included as well. Here, $\mathbf{P}_{j}^{-}=\left\{\mathbf{P}_{j}^{-}\right\}_{0}+\{\mathbf{X}\}_{0} \mathbf{R}^{\wedge} \mathbf{P}_{j}^{-}$at $z_{0} ; \mathbf{P}_{j}=\mathbf{D} \mathbf{P}_{j}^{-}$.

## Removal of surface-related multiples: The boundary formulation

From equation 7a it follows that the surface-related multiples are given by

$$
\begin{equation*}
\left\{\delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{0}=\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{0}\right) \mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right), \tag{9a}
\end{equation*}
$$

with prediction filter
$\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{0}\right) \approx \mathbf{D}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{0}\right) \mathbf{R}^{\wedge}\left(z_{0}, z_{0}\right)$ (model driven).

Alternatively, the surface-related multiples, as expressed in equation 8 b , can be rewritten as

$$
\begin{equation*}
\left\{\delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{0}=\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{0}\right) \mathbf{P}_{j}\left(z_{0}, z_{0}\right), \tag{9c}
\end{equation*}
$$

with prediction filter

$$
\begin{equation*}
\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{0}\right)=\left\{\mathbf{P}\left(z_{0}, z_{0}\right)\right\}_{0} \mathbf{A}\left(z_{0}, z_{0}\right) \quad \text { (data driven) } \tag{9d}
\end{equation*}
$$

In equations $9 \mathrm{~b}-9 \mathrm{~d}, \Delta \mathbf{X}$ is given by equation $2 \mathrm{~b},\{\mathbf{P}\}_{0}$ represents the measurements without surface multiples, and $\mathbf{A}$ is given by equation 7c. Note that in the expression of the model-driven version of $\mathbf{F}_{\mathbf{p}}$, as given by equation $9 b$, the internal multiples have been neglected. If internal multiples can be modeled, then $\Delta \mathbf{X}$ must be replaced by $\{\mathbf{X}\}_{0}$ and equation 9 b becomes exact. From equations $2 b$ and $9 b$ we may conclude that the prediction filter for surface-related multiples is far from simple. First, it is not only a multichannel filter, but it is also a multirecord filter ( $\mathbf{F}_{\mathbf{p r}}$ is not Toeplitz). Second, each trace of the multirecord filter (one column of $\mathbf{F}_{\mathbf{p r}}$ ) represents a scaled response of the subsurface that includes all primaries and internal multiples.

Hence, for complex subsurfaces, the prediction operator may become very complex. From equations 9 c and 9 d , it follows that one multiple trace is given by

$$
\begin{equation*}
\left\{\delta \mathbf{M}_{i j}\left(z_{0}, z_{0}\right)\right\}_{0}=\mathbf{F}_{\mathbf{p r}_{i}}^{\dagger}\left(z_{0}, z_{0}\right) \mathbf{P}_{j}\left(z_{0}, z_{0}\right) \tag{10a}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathbf{F}_{\mathbf{p r}_{i}}^{\dagger}\left(z_{0}, z_{0}\right)=\left\{\mathbf{P}_{i}\left(z_{0}, z_{0}\right)\right\}_{0}^{\dagger} \mathbf{A}\left(z_{0}, z_{0}\right) \tag{10b}
\end{equation*}
$$

In equations 10 the source (array) is located at $\mathbf{x}_{j}$ and the receiver (array) is at $\mathbf{x}_{i}$. The inner product of the two vectors in equation 10a describes a weighted summation of the operator wavefield and the total wavefield along the downwardreflecting surface locations (Figure 2b). It means that the source locations of the prediction operator are combined with receiver locations of the data.

If we want to compensate for small errors in the prediction result, the straightforward subtraction,

$$
\begin{equation*}
\left\{\mathbf{P}_{i j}\left(z_{0}, z_{0}\right)\right\}_{0}=\mathbf{P}_{i j}\left(z_{0}, z_{0}\right)-\left\{\delta \mathbf{M}_{i j}\left(z_{0}, z_{0}\right)\right\}_{0} \tag{11a}
\end{equation*}
$$

must be replaced by an adaptive subtraction,

$$
\begin{equation*}
\left\{\mathbf{P}_{i j}\left(z_{0}, z_{0}\right)\right\}_{0}=\mathbf{P}_{i j}\left(z_{0}, z_{0}\right)-\mathbf{F}_{\mathbf{l s}_{i}}^{\dagger}\left(z_{0}, z_{0}\right)\left\{\delta \mathbf{M}_{i j}\left(z_{0}, z_{0}\right)\right\}_{0} \tag{11b}
\end{equation*}
$$

$\mathbf{F}_{\mathbf{s}_{i}}^{\dagger}\left(z_{0}, z_{0}\right)$ being a gentle least-squares filter that minimizes the difference in the subtraction (Verschuur, 1991). Note that the extension to adaptive subtraction, as formulated in equation 11 b , can be used in any prediction-error process.

Equation 9b shows that in the model-driven version, a subsurface model must be specified, and $\Delta \mathbf{X}\left(z_{0}, z_{0}\right)$ must be computed according to equation 2 b . This model may be simplified by including the strong reflectors only. Also, the spatial sampling interval of $\Delta \mathbf{X}$ is user controlled. This is an important advantage of the model-driven version for 3D data. Equation 9d shows that, in the data-driven version, response $\left\{\mathbf{P}\left(z_{0}, z_{0}\right)\right\}_{0}$ must be available. It has been shown that $\left\{\mathbf{P}\left(z_{0}, z_{0}\right)\right\}_{0}$ may be replaced by the total response $\mathbf{P}\left(z_{0}, z_{0}\right)$, leading to an iterative application (Berkhout and Verschuur, 1997). It is our experience that, in practice, three iterations at most are needed. In the last decade, many successful field-data examples have been shown on surface-related multiple removal. For recent illustrations see Hadidi et al. (1995, 2002), Verschuur and Berkhout (1997), Dragoset and Jeričević (1998), Guitton and Cambois (1999), and Verschuur and Prein (1999).

## Removal of surface-related multiples: The layer formulation

The distribution of sources and detectors in 3D seismic acquisition is not well suited for data-driven, surface-related multiple removal as given by equation 8 b: source and/or detector distributions are relatively sparse, and source and detector locations generally do not coincide. This makes it difficult in equation 8 b to estimate transfer operator $\mathbf{A}\left(z_{0}, z_{0}\right)$ in a meaningful way. An interesting solution is obtained by replacing surface boundary $z_{0}$ by surface layer $\left(z_{0}, z_{1}\right)$, meaning that the downward-scattering operators $\mathbf{R}^{\wedge}\left(z_{0}, z_{0}\right)$ must be replaced by the downward-scattering operators $\Delta \mathbf{X}^{\wedge}\left(z_{1}, z_{1}\right)$ from boundary operator to layer operator. If $z_{1}$ is situated in the water layer, then $\Delta \mathbf{X}^{\wedge}\left(z_{1}, z_{1}\right)$ simplifies to

$$
\begin{equation*}
\Delta \mathbf{X}^{\wedge}\left(z_{1}, z_{1}\right)=\mathbf{W}\left(z_{1}, z_{0}\right) \mathbf{R}^{\wedge}\left(z_{0}, z_{0}\right) \mathbf{W}\left(z_{0}, z_{1}\right) \tag{12a}
\end{equation*}
$$

Equation 12a formulates upward propagation from $z_{1}$ to $z_{0}$, reflection against the surface $\left(z_{0}\right)$, and downward propagation from $z_{0}$ to $z_{1}$. These simple responses should be provided by the user. Figure 3a diagrams this extended feedback model. The 4D operator $\Delta \mathbf{X}^{\wedge}\left(z_{1}, z_{1}\right)$ connects the source and detector positions of the source and detector gathers at depth level $z_{1}$ (Figure 3b):

$$
\begin{equation*}
\left\{\delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{0}=\left\{\mathbf{P}\left(z_{0}, z_{1}\right)\right\}_{0} \Delta \mathbf{X}^{\wedge}\left(z_{1}, z_{1}\right) \mathbf{P}_{j}^{-}\left(z_{1}, z_{0}\right), \tag{12b}
\end{equation*}
$$

$\Delta \mathbf{X}^{\wedge}\left(z_{1}, z_{1}\right)$ being given by equation 12 a . In equation 12 b , $\mathbf{P}_{j}^{-}\left(z_{1}, z_{0}\right)$ is the shot record with its source position at $\left(x_{j}, z_{0}\right)$ and its detectors at $z_{1}$ :

$$
\begin{equation*}
\mathbf{P}_{j}^{-}\left(z_{1}, z_{0}\right)=\boldsymbol{\Gamma}\left(z_{1}, z_{0}\right) \mathbf{P}_{j}^{-}\left(z_{0}, z_{0}\right) \tag{13a}
\end{equation*}
$$

with $\Gamma\left(z_{1}, z_{0}\right)$ containing the downward extrapolation operators on the detector side. Similarly, $\left\{\mathbf{P}\left(z_{0}, z_{1}\right)\right\}_{0}$ is the data with detectors at $z_{0}$ and sources at $z_{1}$ :

$$
\begin{equation*}
\left\{\mathbf{P}\left(z_{0}, z_{1}\right)\right\}_{0}=\left\{\mathbf{P}\left(z_{0}, z_{0}\right)\right\}_{0} \boldsymbol{\Gamma}\left(z_{0}, z_{1}\right) \tag{13b}
\end{equation*}
$$

with $\Gamma\left(z_{0}, z_{1}\right)$ containing the downward extrapolation operators on the source side.

## SURFACE-RELATED AND INTERNAL MULTIPLES

The expressions obtained for the process of surface-relatedmultiple removal can be extended to include the case of internal-multiple removal. For the input of the internal-multiple-removal procedure, we assume that all multiples related to levels $z \leq z_{n-1}$ are removed in a previous multiple removal step, resulting in the data $\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{n-1}$. Let $\left\{\mathbf{P}_{j}\left(z_{n}, z_{0}\right)\right\}_{n-1}$ represents a downward-extrapolated shot record of such input data,

$$
\begin{equation*}
\left\{\mathbf{P}_{j}\left(z_{n}, z_{0}\right)\right\}_{n-1}=\boldsymbol{\Gamma}\left(z_{n}, z_{0}\right)\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{n-1}, \tag{14a}
\end{equation*}
$$

with its source position at $\left(\mathbf{x}_{j}, z_{0}\right)$ and its detector positions at $z_{n}$. If we define, in accordance with equation 2 b ,

$$
\begin{equation*}
\Delta \overline{\mathbf{X}}\left(z_{0}, z_{n}\right)=\sum_{m=n+1}^{\infty} \mathbf{W}\left(z_{0}, z_{m}\right) \mathbf{R}\left(z_{m}, z_{m}\right) \mathbf{W}\left(z_{m}, z_{n}\right) \tag{14b}
\end{equation*}
$$

then equation 7 a can be generalized to (model-driven version):

$$
\begin{align*}
\left\{\delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{n} & \approx \mathbf{D}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{n}\right) \\
& \times\left[\mathbf{R}^{\wedge}\left(z_{n}, z_{n}\right)\left\{\overline{\mathbf{P}}_{j}^{-}\left(z_{n}, z_{0}\right)\right\}_{n-1}\right] \tag{14c}
\end{align*}
$$

and equation (8b) can be generalized to (data-driven version)

$$
\begin{equation*}
\left\{\delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{n}=\left\{\overline{\mathbf{P}}\left(z_{0}, z_{n}\right)\right\}_{n} \mathbf{A}\left(z_{n}, z_{n}\right)\left\{\overline{\mathbf{P}}_{j}\left(z_{n}, z_{0}\right)\right\}_{n-1}, \tag{14d}
\end{equation*}
$$

where $n=0,1,2, \ldots \infty$. Note that $n=0$ represents the surfacerelated case.
a)

b)


Figure 3. (a) Feedback model for primary reflections ( $m>1$ ) and surface-related multiples using the layer formulation, the multiple-generating layer being given by $z_{0} \leq z \leq z_{1}$, and the downward-scattering operator of this layer being given by $\Delta \mathbf{X}^{\wedge}\left(z_{1}, z_{1}\right)$. Here, $\mathbf{P}_{j}^{-}=\Delta \mathbf{P}_{j}^{-}+\Delta \mathbf{X} \Delta \mathbf{X}^{\wedge} \mathbf{P}_{j}^{-}$at $z_{1} ; \mathbf{P}_{j}=$ $\mathbf{D W P}_{j}^{-}$. (b) One basic element of the prediction process for surface-related multiples, visualized in terms of simple raypaths $\left(z_{m}>z_{1}\right)$. Here, $\Delta \mathbf{M}_{j}^{-}=\Delta \mathbf{X} \Delta \mathbf{X}^{\wedge} \mathbf{P}_{j}^{-}$at $z_{1} ; \Delta \mathbf{M}_{j}=$ DW $\Delta \mathbf{M}_{j}^{-}$.

In equation 14d, $\mathbf{A}\left(z_{n}, z_{n}\right)$ is a scaled version of $\mathbf{R}^{\wedge}\left(z_{n}, z_{n}\right)$, $\left\{\overline{\mathbf{P}}_{j}^{-}\left(z_{n}, z_{0}\right)\right\}_{n-1}$ is a muted version of $\left\{\mathbf{P}_{j}^{-}\left(z_{n}, z_{0}\right)\right\}_{n-1}$, and $\left\{\overline{\mathbf{P}}\left(z_{0}, z_{n}\right)\right\}_{n}$ is the muted version of $\left\{\mathbf{P}\left(z_{0}, z_{n}\right)\right\}_{n}$. Muting in both cases means removing the reflections for $z \leq z_{n}$. This is consistent with the notation introduced in equation $4 a$ and further explained in Appendix A.

Figure 4 a displays the diagram for this version of the feedback model, showing the multiple-generating feedback loop. Figure 4 b shows one element of equation 14 c in terms of raypaths.
Considering equations 14 c and 14 d , the expression for all types of multiples can be easily established:

$$
\begin{align*}
\mathbf{M}_{j}\left(z_{0}, z_{0}\right) \approx & \mathbf{D}\left(z_{0}\right) \sum_{n=0}^{\infty} \Delta \mathbf{X}\left(z_{0}, z_{n}\right) \\
& \times \mathbf{R}^{\wedge}\left(z_{n}, z_{n}\right)\left\{\overline{\mathbf{P}}_{j}^{-}\left(z_{n}, z_{0}\right)\right\}_{n-1} \tag{15a}
\end{align*}
$$

or

$$
\begin{equation*}
\mathbf{M}_{j}\left(z_{0}, z_{0}\right)=\sum_{n=0}^{\infty}\left\{\overline{\mathbf{P}}\left(z_{0}, z_{n}\right)\right\}_{n} \mathbf{A}\left(z_{n}, z_{n}\right)\left\{\overline{\mathbf{P}}_{j}\left(z_{n}, z_{0}\right)\right\}_{n-1}, \tag{15b}
\end{equation*}
$$

where $\left\{\overline{\mathbf{P}}\left(z_{0}, z_{n}\right)\right\}_{n}$ and $\left\{\overline{\mathbf{P}}_{j}\left(z_{n}, z_{0}\right)\right\}_{n-1}$ are computed by downward extrapolation followed by muting. Note that for $n=0$, $\left\{\overline{\mathbf{P}}_{j}\left(z_{n}, z_{0}\right)\right\}_{n-1}=\mathbf{P}_{j}\left(z_{0}, z_{0}\right)$.
If we substitute equation 14 b into equation 15 a and interchange the two summations, we obtain:

$$
\begin{equation*}
\mathbf{M}_{j}\left(z_{0}, z_{0}\right)=\mathbf{D}\left(z_{0}\right) \sum_{m=1}^{\infty} \mathbf{W}\left(z_{0}, z_{m}\right) \mathbf{R}\left(z_{m}, z_{m}\right) \mathbf{P}_{j}^{+}\left(z_{m}, z_{0}\right) \tag{16a}
\end{equation*}
$$


b)


Figure 4. (a) Feedback model for primary reflections ( $m>$ $n$ ) and internal multiples using the boundary formulation, the multiple-generating boundary being $z_{n}=z_{n}(x, y)$ with $n>0$, and the downward reflecting operators at this boundary being given by the columns of the matrix $\mathbf{R}^{\wedge}\left(z_{n}, z_{n}\right)$. Here, $\mathbf{P}_{j}^{-}=\Delta \mathbf{P}_{j}^{-}+\Delta \mathbf{X} \mathbf{R}^{\wedge} \mathbf{P}_{j}^{-}$at $z_{n} ; \mathbf{P}_{j}=\mathbf{D W} \mathbf{P}_{j}^{-}$. (b) One basic element of the prediction process for boundary-related internal multiples ( $n>0$ ), visualized in terms of simple raypaths ( $m>$ $n)$. Here, $\delta \mathbf{M}_{j}^{-}=\Delta \mathbf{X} \mathbf{R}^{\wedge} \mathbf{P}_{j}^{-}$at $z_{n} ; \delta \mathbf{M}_{j}=\mathbf{D W} \delta \mathbf{M}_{j}^{-}$.
with

$$
\begin{equation*}
\mathbf{P}_{j}^{+}\left(z_{m}, z_{0}\right)=\sum_{n=0}^{m-1} \mathbf{W}\left(z_{m}, z_{n}\right) \mathbf{R}^{\wedge}\left(z_{n}, z_{n}\right) \mathbf{X}\left(z_{n}, z_{0}\right) \mathbf{S}_{j}\left(z_{0}\right) \tag{16b}
\end{equation*}
$$

for $m=1,2, \ldots, M$. Note that equation 16a can be directly obtained from the WRW model for primary reflections if we replace in equation 1 b the primary-source wavefield $\mathbf{S}_{j}^{+}\left(z_{m}, z_{0}\right)$ by a multisource wavefield $\mathbf{P}_{j}^{+}\left(z_{m}, z_{0}\right)$. Therefore, equations 16 represent the WRW model for multiple reflections. Note also that our multiple removal algorithms are not based on multiple scattering equations 16 but on feedback equations 15 .

## Removal of internal multiples: The boundary formulation

For the removal of internal multiples, we assume that the relatively strong surface-related multiples have been removed already. If we replace in equation 9a the multiple-generating surface $z_{0}$ by the internal multiple-generating surface $z_{n}$, then it follows from equations 14 c and 14 d that the algorithm for the internal multiples can be formulated as
$\left\{\delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{n}=\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{n}\right)\left\{\overline{\mathbf{P}}_{j}^{-}\left(z_{n}, z_{0}\right)\right\}_{n-1} \quad$ for $\quad n>0$,
with prediction filter

$$
\begin{equation*}
\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{n}\right) \approx \mathbf{D}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{n}\right) \mathbf{R}^{\wedge}\left(z_{n}, z_{n}\right) \tag{17b}
\end{equation*}
$$

(model driven),
or as

$$
\begin{equation*}
\left\{\delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{n}=\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{n}\right)\left\{\overline{\mathbf{P}}_{j}\left(z_{n}, z_{0}\right)\right\}_{n-1} \quad \text { for } \quad n>0, \tag{17c}
\end{equation*}
$$

with prediction filter

$$
\begin{equation*}
\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{n}\right)=\left\{\overline{\mathbf{P}}\left(z_{0}, z_{n}\right)\right\}_{n} \mathbf{A}\left(z_{n}, z_{n}\right) \quad \text { (data driven) } \tag{17d}
\end{equation*}
$$

In equations $9 \mathrm{~b}-9 \mathrm{~d}, \Delta \mathbf{X}\left(z_{0}, z_{n}\right)$ is given by equation 14 b , and $\mathbf{A}\left(z_{n}, z_{n}\right)$ represents the scaled version of $\mathbf{R}^{\wedge}\left(z_{n}, z_{n}\right)$. Note that the algorithm for surface-related multiples, based on equations 9 a and 9 b , is very similar to the algorithm for internal multiples, based on equations 17 a and 17 b : shot records with detectors at the surface must be replaced by shot records with detectors at depth level $z_{n}$. Similarly, comparing equations 9c and 9 d with equations 17 c and 17 d , shot records with sources at the surface must be replaced by shot records with sources at depth level $z_{n}$. In marine data, internal multiples are generally weaker than surface multiples. This means that in the data-driven algorithm, based on equations 17 c and 17 d , one iteration is often sufficient.

The first synthetic example of internal multiple removal based on equation 17d is shown in Berkhout (1982). A first field-data study was reported by Hadidi and Verschuur (1997). Based on the inverse-scattering approach, Matson et al. (1999) also demonstrated the internal-multiple removal on a fielddata example.

## Removal of internal multiples: The layer formulation

Similar to what was proposed for the surface-related multiples from boundary to layer, see equations $12 a$ and $12 b$, we also extend internal-multiple removal to the layer approach, replacing boundary operator $\mathbf{R}^{\wedge}\left(z_{n}, z_{n}\right)$ in equation 17 b with layer operator $\Delta \mathbf{X}^{\wedge}\left(z_{n}, z_{n}\right)$. To show this, we extend equation 17 b in three steps. First, we define a single downwardscattering boundary at $z_{n}(n>0)$ :

$$
\begin{equation*}
\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{n}\right)=\mathbf{D}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{n}\right) \mathbf{R}^{\wedge}\left(z_{n}, z_{n}\right) \tag{18a}
\end{equation*}
$$

Then we define a single downward-scattering boundary at $z_{k}<z_{n}(n>0)$ :

$$
\begin{align*}
\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{n}\right)= & \mathbf{D}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{n}\right) \\
& \times\left[\mathbf{W}\left(z_{n}, z_{k}\right) \mathbf{R}^{\wedge}\left(z_{k}, z_{k}\right) \mathbf{W}\left(z_{k}, z_{n}\right)\right] \tag{18b}
\end{align*}
$$

Finally, we define multiple downward-scattering boundaries between $z_{l}$ and $z_{n}(l<n)$ :

$$
\begin{align*}
& \mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{n}\right)=\mathbf{D}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{n}\right) \\
& \quad \times \sum_{k=l}^{n}\left[\mathbf{W}\left(z_{n}, z_{k}\right) \mathbf{R}^{\wedge}\left(z_{k}, z_{k}\right) \mathbf{W}\left(z_{k}, z_{n}\right)\right] . \tag{18c}
\end{align*}
$$

If we define

$$
\begin{equation*}
\left\{\Delta \mathbf{X}^{\wedge}\left(z_{n}, z_{n}\right)\right\}_{n}^{l}=\sum_{k=l}^{n}\left[\mathbf{W}\left(z_{n}, z_{k}\right) \mathbf{R}^{\wedge}\left(z_{k}, z_{k}\right) \mathbf{W}\left(z_{k}, z_{n}\right)\right] \tag{19a}
\end{equation*}
$$

then equation 18 c can be rewritten as

$$
\begin{equation*}
\mathbf{F}_{\mathbf{p r}}\left(z_{0}, z_{n}\right)=\mathbf{D}\left(z_{0}\right) \Delta \mathbf{X}\left(z_{0}, z_{n}\right)\left\{\Delta \mathbf{X}^{\wedge}\left(z_{n}, z_{n}\right)\right\}_{n}^{l} \tag{19b}
\end{equation*}
$$

leading to the expression of the internal multiples generated by a downward-scattering layer:

$$
\begin{align*}
& \left\{\Delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{n}^{l} \approx \mathbf{D}\left(z_{0}\right)\left[\Delta \mathbf{X}\left(z_{0}, z_{n}\right)\left\{\Delta \mathbf{X}^{\wedge}\left(z_{n}, z_{n}\right)\right\}_{n}^{l}\right] \\
& \quad \times\left\{\overline{\mathbf{P}}_{j}^{-}\left(z_{n}, z_{0}\right)\right\}_{n-1}, \quad \text { (model driven) } \tag{19c}
\end{align*}
$$

with $\Delta \mathbf{X}\left(z_{0}, z_{n}\right)$ given by equation $14 \mathrm{~b},\left\{\Delta \mathbf{X}^{\wedge}\left(z_{n}, z_{n}\right)\right\}_{n}^{l}$ given by equation 19a, and $n>0$. Note that in equation $19 b$, prediction filter $\mathbf{F}_{\mathbf{p r}}$ is model driven and is $O\left(r^{2}\right)$, where $r$ is the average reflection strength in the data. This means the predicted internal multiples are $O\left(r^{3}\right)$.

Figure 5a displays the diagram for this version of the feedback model. Note the similarity between Figures 5a and 3a. Note also that $\left\{\Delta \mathbf{X}^{\wedge}\left(z_{n}, z_{n}\right)\right\}_{n}^{l}$ contains the primary reflections from the reflectors at $z_{l} \leq z<z_{n}$, measured at depth level $z_{n}$ and seen from below. If a model of the subsurface is available (only main reflectors are required), $\left\{\Delta \mathbf{X}^{\wedge}\left(z_{n}, z_{n}\right)\right\}_{n}^{l}$ can easily be computed by equation 19a. If this model is not available, the following data-driven version is proposed.
First, start with $\left\{\mathbf{P}\left(z_{n}, z_{0}\right)\right\}_{l-1}$, meaning that sources are situated at the surface and detectors at depth level $z_{n}$, from which multiples to depth level $z_{l-1}$ have already been removed, and downward extrapolate the sources to depth level $z_{n}$, i.e.,

$$
\begin{equation*}
\left\{\mathbf{P}\left(z_{n}, z_{n}\right)\right\}_{l-1}=\left\{\mathbf{P}\left(z_{n}, z_{0}\right)\right\}_{l-1} \boldsymbol{\Gamma}\left(z_{0}, z_{n}\right) \tag{20a}
\end{equation*}
$$

Next, use the causal part of the time-reversed version of $\left\{\mathbf{P}\left(z_{n}, z_{n}\right)\right\}_{l-1}$ as an estimate for the downward-scattering op-
erator, i.e.,

$$
\begin{equation*}
\left\{\Delta \mathbf{X}^{\wedge}\left(z_{n}, z_{n}\right)\right\}_{n}^{l} \approx-\left\{\Delta \mathbf{Q}\left(z_{n}, z_{n}\right)\right\}_{n}^{l} \tag{20b}
\end{equation*}
$$

with $\left\{\Delta \mathbf{Q}\left(z_{n}, z_{n}\right)\right\}_{n}^{l}$ containing only primary reflection events between levels $z_{l} \leq z<z_{n}$, as seen from depth level $z_{n}$. This leads to the expression of the internal multiples generated by the downward-scattering layer $\left(z_{l}, z_{n}\right)$ :

$$
\begin{align*}
& \left\{\Delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{n}^{l} \approx-\left\{\overline{\mathbf{P}}\left(z_{0}, z_{n}\right)\right\}_{n}\left\{\Delta \mathbf{Q}\left(z_{n}, z_{n}\right)\right\}_{n}^{l} \\
& \quad \times\left\{\overline{\mathbf{P}}_{j}\left(z_{n}, z_{0}\right)\right\}_{n-1} \quad(\text { data driven }) \tag{20c}
\end{align*}
$$

for $n>0$.
Note that in equations 20 b and 20 c , the internal multiples in layer $\left(z_{l}, z_{n}\right)$ are not included in the prediction operator. This means that, in practice, this layer is always chosen around one major reflecting boundary (see Berkhout and Verschuur, 2005; hereafter referred to as part 2). In practice, it is also advisable to choose for $z_{l}=z_{l}(x, y)$ and $z_{n}=z_{n}(x, y)$ virtual boundaries in areas of weak reflectivity to avoid sensitivity for errors in the causal-noncausal separation process, caused by the kinematic errors in the downward-extrapolation operators.
The velocity independence of internal-multiple prediction has been stated by Araujo et al. (1994) using an inverse-scattering-series formulation (see also Weglein et al., 1997). In Appendix B, we confirm this statement but indicate some limitations of this property in complex geology.
a)

b)


Figure 5. (a) Feedback model for primaries and internal multiples, the multiples being generated by upward scattering in the lower half-space $\left(z>z_{n}\right)$, quantified by the feed-forward operators WRW and the downward scattering in the upper half-space ( $z_{1} \leq z<z_{n}$ ), quantified by the feedback operators
$\mathbf{W R}^{\wedge} \mathbf{W}$. Here, $\mathbf{P}_{j}^{-}=\Delta \mathbf{P}_{j}^{-}+\Delta \mathbf{X} \Delta \mathbf{X}^{\wedge} \mathbf{P}_{j}^{-}$at $z_{n} ; \mathbf{P}_{j}=\mathbf{D W P} \mathbf{P}_{j}^{-}$. (b) One basic element of the data-driven prediction process for layer-related internal multiples ( $n>0$ ), visualized in terms of simple raypaths $(m>n)$. Here, $\Delta \mathbf{M}_{j}^{-}=\Delta \mathbf{X} \Delta \mathbf{X}^{\wedge} \mathbf{P}_{j}^{-}$at $z_{n}$; $\Delta \mathbf{M}_{j}=\mathbf{D W} \Delta \mathbf{M}_{j}^{-}$.

## Iterative application

Similar to the surface-multiple removal algorithm, internal multiples can be removed in an iterative way. This allows us to replace $\left\{\overline{\mathbf{P}}\left(z_{0}, z_{n}\right)\right\}_{n}$ by $\left\{\mathbf{P}^{-}\left(z_{n}, z_{0}\right)\right\}_{n-1}^{T}$ in equation 20 c in the first iteration, with ${ }^{T}$ indicating the adjoint. Generally, one iteration is sufficient.

## IMPLEMENTATION OF THE ALGORITHM

Looking at the expressions that describe the data-driven prediction of surface-related multiples in the layer formulation (equation 12b) and the prediction of internal multiples in the boundary formulation (equations 17 c and 17 d ), as well as the layer formulation (equations 20), we observe that seismic data are required with sources at the surface and receivers in the subsurface, or vice versa. Except for ocean-bottom seismic data, such gathers are not directly available in practice. Therefore, these gathers must be constructed from the measured surface data. Looking at the construction of layer-related internal multiples (referring to Figure 5 b for the model-driven case and equation 20c for the data-driven implementation) for one source-receiver location, this can be written as (omitting the multiple removal level indications)

$$
\begin{equation*}
\Delta \mathbf{M}_{i j}\left(z_{0}, z_{0}\right)=-\overline{\mathbf{P}}_{i}^{\dagger}\left(z_{0}, z_{n}\right) \Delta \mathbf{Q}\left(z_{n}, z_{n}\right) \overline{\mathbf{P}}_{j}\left(z_{n}, z_{0}\right) \tag{21a}
\end{equation*}
$$

This requires the time-reversed, muted double-focused gathers $\Delta \mathbf{Q}\left(z_{n}, z_{n}\right)$; a downward-extrapolated common-receiver gather $\overline{\mathbf{P}}_{i}^{\dagger}\left(z_{0}, z_{n}\right)$ with its receiver at surface position $x_{i}$ and


Figure 6. (a) The data-driven prediction process shown for layer-related internal multiples, organized for one pair of gridpoints at the surface ( $i$ and $j$ are fixed) and many gridpoints in the subsurface ( $k$ and $l$ are variable). This leads to an algorithm that handles downward-extrapolated shot records. (b) The data-driven prediction process shown for layer-related internal multiples, organized for one pair of gridpoints in the subsurface ( $k$ and $l$ are fixed) and many gridpoints at the surface ( $i$ and $j$ are variable). This leads to an algorithm that handles CFP gathers.
the source locations at depth level $z_{n}$; and a downwardextrapolated common-source gather $\overline{\mathbf{P}}_{j}\left(z_{n}, z_{0}\right)$ with its source at surface position $x_{j}$ and the detector locations at depth level $z_{n}$. Equation 21a describes two spatial convolutions at level $z_{n}$, which can be rewritten more explicitly as

$$
\begin{equation*}
\Delta \mathbf{M}_{i j}\left(z_{0}, z_{0}\right)=-\sum_{k, l} \overline{\mathbf{P}}_{i k}\left(z_{0}, z_{n}\right) \Delta \mathbf{Q}_{k l}\left(z_{n}, z_{n}\right) \overline{\mathbf{P}}_{l j}\left(z_{n}, z_{0}\right) \tag{21b}
\end{equation*}
$$

showing how the layer-related internal multiples for one output trace are generated. For boundary-related internal-multiple prediction, the layer-related reflectivity is replaced by a boundary-related reflectivity. The latter is generally simplified to a scalar.

Thus far, we have organized the algorithm per surface gridpoint, i.e., in equation 21 b , surface locations $i$ and $j$ are fixed and subsurface locations $k$ and $l$ vary. However, the algorithm can also be organized per subsurface gridpoint, meaning that subsurface locations $k$ and $l$ from equation 21b are now fixed and surface locations $i$ and $j$ vary. This is illustrated in Figure 6. For the second implementation, the downward-extrapolated surface data are re-sorted into gathers with one point in the subsurface and many points at the surface. The partial contributions per shot record are thus replaced by partial contributions per so-called CFP gather (see also part 2). Berkhout (1997a) describes this by a focusing-in-detection process to obtain gathers with sources at the surface and one common receiver in the subsurface and by a focusing-in-emission process to obtain gathers with receivers at the surface and one common source in the subsurface. The resulting seismic gathers are referred to as CFP gathers.

By considering the multiple-prediction process in terms of CFP gathers, muting becomes a straightforward process because the involved focusing operator acts as the mute line (see next section). Furthermore, in the case of boundary-related multiple removal, operator updating can easily be applied to correct for operator errors (see part 2). The actual implementation has therefore been done in terms of CFP gathers, i.e., according to Figure 6b. This means that all CFP gathers are created along the desired level first, followed by convolving the CFP gathers for each subsurface gridpoint pair. Hence, the total contribution of each subsurface gridpoint pair to the final predicted internal multiples is computed for all sourcereceiver combinations at the surface.

## Muting in the CFP domain

Our algorithm shows that, for the removal of internal multiples, a muting process must be applied to the focused seismic data to remove all reflections that have their downward bounce above and including depth level $z_{n}-$ from $\mathbf{P}_{j}\left(z_{n}, z_{0}\right)$ to $\overline{\mathbf{P}}_{j}\left(z_{n}, z_{0}\right)$. Earlier we argued that, in the CFP domain, muting becomes a rather straightforward process because the time-reversed focusing operator can be used to separate the events from below and above this level. This can be understood easily by considering the principle of equal traveltime as stated in Berkhout (1997a): If the focusing operator is correct, the time-reversed focusing operator will coincide in time with the corresponding reflection event in the CFP gather. Therefore, this time-reversed focusing operator defines the mute curve for removing events from above the depth level of
interest. This important feature will be demonstrated in the examples in the Numerical Illustrations section.

## Focusing operator updating for boundary-related multiple prediction

In the situation of boundary-related internal-multiple removal, it is important that the CFP gathers be constructed with correct focusing operators. Erroneous CFP gathers lead to an erroneous multiple-prediction result. Berkhout (1997a) shows that correct operators can be achieved in a velocityindependent manner using the focusing operator updating procedure. Examples of this updating process are shown by Morton (1996), Thorbecke (1997), Bolte et al. (1999), and Berkhout and Verschuur (2001). In part 2, the subject of focusing operator updating is treated more extensively.

## Construction of gridpoint gathers for layer-related multiple prediction

For the layer formulation of multiple prediction, the downward-scattering operators of the chosen layer need to be specified. For surface-related multiples, this is surface layer $\left(z_{0}, z_{1}\right)$; the involved operators $\Delta \mathbf{X}^{\wedge}\left(z_{1}, z_{1}\right)$ describe the downward reflection against the surface with sources and receivers positioned at depth level $z_{1}$. Assuming that level $z_{1}$ is located in the water layer, these scattering operators are easily constructed in a model-driven manner (see equation 12a). In the case of internal multiples, the downward-scattering operators $\Delta \mathbf{X}^{\wedge}\left(z_{n}, z_{n}\right)$ can also be constructed in a model-driven way, assuming that a model of the multiple-generating reflectors is available. Equations 20a and 20 b show how $\Delta \mathbf{X}^{\wedge}\left(z_{n}, z_{n}\right)$ can be obtained in a data-driven manner. It requires application of a second focusing step to the CFP gathers, resulting in fully redatumed shot records, each record having its source and receivers at depth level $z_{n}$. In Berkhout (1997b) these gathers are called gridpoint gathers. Each gridpoint gather contains the angle-averaged reflection information of its gridpoint at zero time and zero offset. The angle-dependent reflecting properties can be obtained by applying a linear Radon transformation and looking at zero intercept time ( $\tau=0$ ). Furthermore, in this domain, the reflections from above and below depth level $z_{n}$ can be separated easily by considering the negative and positive $\tau$-values, respectively. Therefore, the downward-scattering from above level $z_{n}$ can be obtained by reversing the $\tau$-axis and muting the data for $\tau<0$.

As stated in Appendix B, the layer-related approach to multiple prediction does not require accurate focusing operators as long as events from above and below the bottom of the layer can be separated by muting. This is illustrated by the numerical examples in the Numerical Illustrations section.

## Recursive application of internal-multiple removal

Both the boundary-related and layer-related approaches to internal-multiple removal can be applied in a recursive manner, such that all relevant multiple-generating structures are included. For the boundary-related approach, the subsurface is scanned boundary by boundary. Equations 17 c and 17 d can
be repeated, each recursion using the output of the previous process as input for the next. Optionally, this can be implemented as part of the migration process, as formulated by Berkhout (1982). In the case of the layer-related approach, the subsurface model is scanned layer by layer. The output of the previous recursion is again used as input for the next process, according to equations 20.

## NUMERICAL ILLUSTRATIONS

In the following, the theory of the boundary-related and layer-related approaches to internal-multiple removal is illustrated with numerical examples. For this purpose, two synthetic data sets are used from 2D subsurface models with one and two (closely spaced) internal-multiple-generating reflectors, respectively. Data are simulated using ray tracing. The source and receiver spacing is 25 m , and the maximum frequency within the source signal is 30 Hz .

## Example with one internal-multiple-generating reflector

The first subsurface model under consideration is depicted in Figure 7 ; it consists of three reflectors, the upper two being dipping ( $+3^{\circ}$, and $-5^{\circ}$, respectively). In this example we concentrate on the internal multiples generated by downward reflection at the first reflector. In Figure 8, the zero-offset section is displayed. There are no surface multiples. Internal


Figure 7. Subsurface model with three dipping reflectors.


Figure 8. Zero-offset data, showing primaries and internal multiples for the subsurface model of Figure 7.


Figure 9. Shot record $\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{0}$ with its source at $x=2100$ m , showing primaries and internal multiples for the subsurface model of Figure 7.


Figure 10. CFP gather for one focus point at reflector 1. (a) CFP gather before muting $\left\{\mathbf{P}_{i}\left(z_{1}, z_{0}\right)\right\}_{0}^{\dagger}$; (b) time-reversed focusing operator $\left[\boldsymbol{\Gamma}_{i}^{*}\left(z_{1}, z_{0}\right)\right]^{\dagger}$; (c) CFP gather after muting $\left\{\overline{\mathbf{P}}_{i}\left(z_{1}, z_{0}\right)\right\}_{0}^{\dagger}$. The arrow points to a muting problem.
a)

b)

c)


Figure 11. Internal-multiple-removal result for the boundary-related approach, the boundary being reflector 1 . (a) Input shot record $\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{0}$; (b) predicted internal multiples $\left\{\delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{1}$; (c) result after adaptive subtraction $\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{1}$. The upper arrow points to a remaining multiple at far offset, caused by the muting problem indicated in Figure 10c. Of course, the internal multiples generated by reflector 2 are still present (see lower arrow).
multiples, the majority of them being related to the first reflector, are clearly visible. They have been indicated in the figure by a sequence of bounces (upward-downward-upward, etc.). With this data set, the two data-driven internal-multipleremoval approaches are illustrated: first, the boundary-related formulation is used; then, the layer-related formulation. The results are shown for a shot located at $x=2100 \mathrm{~m}$. Figure 9 shows this shot record with internal multiples. The two peg-leg internal multiples (events 3-1-2 and 2-1-3) coincide in the zerooffset section (Figure 8), but they are two separated events in the shot record (Figure 9).

For all boundary-related examples, we consider the use of focusing operators without errors. The aspects of erroneous operators, together with operator updating, are treated in part 2.

## Boundary-related approach with correct operators

For the boundary-related approach, the CFP gathers with focus points along the boundary of interest (reflector 1 for our example) must be calculated. The CFP gather for a selected focus point at the first reflector $(x=2100 \mathrm{~m})$ is shown in Figure 10a. Figure 10b displays the timereversed focusing operator that was used to calculate this CFP gather. Since the depth of the first reflector and the velocity above this reflector were correct, this operator indeed coincides with the first reflection event in the CFP gather (principle of equal traveltime). Therefore, the operator times can be used to apply an automatic mute to this CFP gather (see Figure 10c). After automatically muting all CFP gathers, a prestack multigather convolution is applied according to equations 17 c and 17 d . The result for the shot of Figure 9 (repeated in Figure 11a) is displayed in Figure 11b, showing the predicted (unscaled) internal multiples. An adaptive least-squares subtraction from the input yields the multiple-free estimate (Figure 11c).

Note some remaining multiples for large offsets below the third reflection (at the upper arrow in Figure 11c). They occur because during the muting process (see Figure 10a,b), part of the second primary event was removed. This results in an incomplete predicted multiple with respect to the multiple events that have bounced at the second reflector. This interference problem could be accommodated by replacing muting along the operator by a least-squares subtraction of the operator from the data. As expected, the internal multiple related to the second reflector (event 3-2-3 in Figure 9) has not been predicted and consequently is still
visible around 3.0 s in the multiple-free estimate. Note also that the second-order multiple has not been fully removed because the first iteration does not predict secondorder multiples (event 2-1-2-1-2 in Figure 9) with correct amplitudes. To fully remove this second-order multiple, a second iteration of equations 17 c and 17 d is required, using the output of the first iteration as the multiple prediction operator, similar to the algorithm for surface multiples (see Berkhout and Verschuur, 1997).

## Layer-related approach with correct operators

The same multiple-removal result can be obtained if an arbitrary depth level between reflectors 1 and 2 is chosen for constructing the focusing operators. However, as described by equation 18 b , not the downward-reflection operators (in the previous example assumed to be a scalar) but the full downward-scattering operators must included now. For the subsurface model under consideration, a horizontal level $A$ at 800 m depth (i.e., between the first two reflectors) is chosen. First, CFP gathers related to this datum level are calculated (as shown in Figure 12a). In this display, the operator (Figure 12b) has traveltimes between the first two reflections, as expected. The muted version of these CFP gathers (Figure 12c) is used in the multiple prediction. Note that Figure 12c uses a different time axis for display, by which deeper events become visible.
The computed CFP gathers describe the wavefield for sources at the datum level and receivers still at the surface (first focusing step). Next, a second focusing step is applied to these CFP gathers (without muting): both sources and receivers are now at the datum level. The resulting data volume consists of so-called gridpoint gathers: fully redatumed shot records at datum level $A$. For the lateral location under consideration ( $x=2100 \mathrm{~m}$ ), the gridpoint gather is shown in Figure 13a. The anticausal part of this gridpoint gather contains the scattering from above; the causal part contains the scattering from below. By selecting the anticausal part and reversing the time axis (see Figure 13b), the operators are retrieved that form the required scaleddownward scattering matrix $\Delta \mathbf{Q}\left(z_{n}, z_{n}\right)$. According to equations 18 a and 18 b , the CFP gathers and the downward-scattering matrix are combined in a lateral convolution process to predict the internal multiples, as displayed in Figure 14b. After adaptive subtraction, a multiple-free shot record is obtained (Figure 14c). Note the strong resemblance (as expected) to the result obtained by considering reflector 1 as the datum level (Figure 11c). Note also that the layer-related result is better for the


Figure 12. CFP gather for one focus point at depth level $A$ (see Figure 7). (a) CFP gather before muting $\left\{\mathbf{P}_{i}\left(z_{A}, z_{0}\right)\right\}_{0}^{\dagger}$; (b) time-reversed focusing operator $\left[\boldsymbol{\Gamma}_{i}^{*}\left(z_{A}, z_{0}\right)\right]^{\dagger}$; (c) CFP gather after muting $\left\{\overline{\mathbf{P}}_{i}\left(z_{A}, z_{0}\right)\right\}_{0}^{\dagger}$. Note the different time axes used for display in (a) and (b) versus (c).


Figure 13. Gridpoint gather for one focus point at depth level $A$ (see Figure 7). (a) Gridpoint gather without mute $\left\{\mathbf{P}_{j}\left(z_{A}, z_{A}\right)\right\}_{0}$; (b) gridpoint gather after time reversing and muting at $t=0$, i.e., causal response $\left\{\Delta \mathbf{Q}_{j}\left(z_{A}, z_{A}\right)\right\}_{A}$.


Figure 14. Result of internal-multiple removal for the layer-related approach at level $A$ (see Figure 7). (a) Input shot record $\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{0}$; (b) predicted internal multiples $\left\{\Delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{A}$; (c) result after adaptive subtraction of the internal multiples $\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{A}$. Compare with Figure 11.
rightmost offsets (compare Figure 14c with Figure 11c). The reason for this is the improved CFP muting process in the layer-related approach. It shows the importance of the muting step in internal-multiple removal.


Figure 15. Homogeneous subsurface model for testing layerrelated internal-multiple removal with incorrect operators.


Figure 16. Erroneous CFP gather for one focus point at depth level $B$ (see Figure 15), using the data of the model in Figure 7. (a) CFP gather $\left\{\mathbf{P}_{i}\left(z_{B}, z_{0}\right)\right\}_{0}^{\dagger}$ for $x=2100 \mathrm{~m}$; (b) time-reversed focusing operator $\left[\boldsymbol{\Gamma}_{i}^{*}\left(z_{B}, z_{0}\right)\right]^{\dagger}$; (c) CFP gather after muting, i.e., $\left\{\overline{\mathbf{P}}_{i}\left(z_{B}, z_{0}\right)\right\}_{0}^{\dagger}$. Note the different time axes used for display in (a) and (b) versus (c).
a)



Figure 17. Erroneous gridpoint gather for one focus point at depth level $B$ (see Figure 15). (a) Gridpoint gather $\left\{\mathbf{P}_{j}\left(z_{B}, z_{B}\right)\right\}_{0}$ for $x=2100 \mathrm{~m}$; (b) gridpoint gather after time reversing and muting at $t=0$, i.e., causal response $\left\{\Delta \mathbf{Q}_{j}\left(z_{B}, z_{B}\right)\right\}_{B}$.

## Layer-related approach with incorrect operators

From our theory, it follows that the exact subsurface model does not need to be known. Any subsurface model is allowed, as long as a suitable time level can be chosen such that muting can be applied in the CFP gathers and gridpoint gathers to distinguish between events from above and below the multiple-generating boundary. In practice, the chosen effective medium should not have higher velocities than the true medium, as this will limit the angles in the predicted multiples. Therefore, we now consider a homogeneous medium with a velocity of $1800 \mathrm{~m} / \mathrm{s}$ and define a depth level $B$ at $z=1000 \mathrm{~m}$ (see Figure 15). This means that the operators used to create the CFP gathers are simple hyperbolic operators with a zero-offset, one-way time of 0.56 s . With these operators, CFP gathers can be constructed for all focus points located along the virtual boundary B.

Figure 16a shows one CFP gather for lateral location $x=2100 \mathrm{~m}$. Figure 16b displays the hyperbolic traveltimes of the time-reversed focusing operator. As expected, the operator does not coincide with any of the reflection events because the operator was created using a wrong model. The CFP gathers are muted along the operator times (Figure $16 \mathrm{c})$. Next, gridpoint gathers are constructed for all points along level $B$. One example of such a gridpoint gather is shown in Figure 17a. Note that the erroneous CFP gathers and erroneous gridpoint gathers can no longer be interpreted as actual physical experiments. However, the operators in the homogeneous model have the property that muting causes the desired time separation of events. From the gridpoint gathers, only the time-reversed, noncausal part is needed (Figure 17b). Next, a convolution of the three data sets is applied: the muted CFP gathers with the timereversed, muted gridpoint gathers and again with the muted CFP gathers, in accordance with equations 18 a and 18 b . The resulting predicted multiples are shown in Figure 18b. Note the resemblance to the previous multipleprediction results, showing that the layer-related approach can accommodate velocity errors.

## Example with a multiple-generating reflector pair

All of the foregoing experiments have been repeated for a more complex subsurface model, as depicted in Figure 19. The difference from the previous model is that the first reflector is replaced by two closely spaced reflectors. The internal multiples under consideration are now generated by this complex reflector. In Figure 20, the zero-offset section is displayed. Again, surface-related multiples were not included in the modeling. Furthermore, the thin-layer multiples between the reflector pair are neglected. Compared to Figure 8, internal multiples now appear in pairs. They are indicated in Figure 20 by the sequence of bounces (upward-downward-upward, etc.), where the top two reflectors are called $1 a$ and $1 b$.

Again, the boundary-related and layer-related approaches of internal-multiple removal will be demonstrated.

## Boundary-related approach with correct operators

For the boundary-related approach, reflector 1 b is chosen, i.e., the lower of the reflector pairs. First, the CFP gathers related to focus points along this boundary are calculated. The CFP gather for one lateral position $(x=$ 2100 m ) is shown in Figure 21a. The time-reversed focusing operator used to calculate this CFP gather is displayed in Figure 21b. The principle of equal traveltime is fulfilled for the second event in the data. Next, the first two events are muted along the operator times (see Figure 21c). After muting all CFP gathers, a multigather convolution is applied according to equations 17 a and 17 c . The result for the lateral position under consideration ( $x=2100 \mathrm{~m}$ ) is shown in Figure 22b, defining the predicted (unscaled) internal multiples. An adaptive least-squares subtraction from the input yields the multiple-free estimate (Figure 22c). Note that only internal multiples related to boundary 1 b are predicted.

## Layer-related approach with incorrect operators

Theory shows that the layer-related approach should fully remove the complete effect of the two multiple-generating reflectors. Results can be obtained with incorrect operators as a result of velocity errors (see Appendix B). To demonstrate this, we use operators based on the virtual boundary at level $B$ in the homogeneous medium of Figure 15. With these erroneous operators, CFP gathers can be constructed for all focus points located along the virtual boundary at level $B$ (see Figure 15). The muted CFP gathers (displayed in Figure 23b) will be used in the multiple-prediction step.
To construct the gridpoint gathers, a second focusing step is applied to the CFP gathers (without muting): both sources and receivers are now at the datum level. By selecting the anticausal part and reversing the time axis, the required scaled, downward-scattering matrix $\Delta \mathbf{Q}\left(z_{n}, z_{n}\right)$ is obtained. For the lateral position under consideration ( $x=2100 \mathrm{~m}$ ), the timereversed, muted gridpoint gather is shown in Figure 23 b . The convolution of three data sets the muted CFP gathers with the time-reversed, muted gridpoint gathers and again with the muted CFP gathers - produces the predicted multiples. Figure 24 b shows the result for the shot under consideration. After adaptive subtraction (Figure 24c), an excellent multiple-removal result is obtained: all multiples against the two upper reflectors have been properly removed. The result confirms that the layer-related approach can handle complex multiple-generating boundaries. In addition, the results also confirm that the approach can accommodate velocity errors. These two properties are very important for practical use.


Figure 18. Internal-multiple-removal result for the layer-related approach using the erroneous CFP gathers and erroneous gridpoint gathers. (a) Input shot record $\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{0} ;$ (b) predicted internal multiples $\left\{\Delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{B}$; (c) result after adaptive subtraction of the internal multiples $\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{B}$. Compare with Figure 14.


Figure 19. Subsurface model with a complex internal-multiplegenerating reflector pair.


Figure 20. Zero-offset data with internal multiples simulated in the subsurface model of Figure 19.


Figure 21. CFP gather for one focus point at reflector $1 b$ in the model of Figure 19. (a) CFP gather $\left\{\mathbf{P}_{i}\left(z_{1 b}, z_{0}\right)\right\}_{0}^{\dagger}$ at $x=2100 \mathrm{~m}$; (b) time-reversed focusing operator $\left[\boldsymbol{\Gamma}_{i}^{*}\left(z_{0}, z_{1 b}\right)\right]^{\dagger}$; (c) CFP gather after muting $\left\{\overline{\mathbf{P}}_{i}\left(z_{1 b}, z_{0}\right)\right\}_{0}^{\dagger}$. The arrow points to a muting problem.


Figure 22. Internal-multiple removal result for the boundary-related approach at reflector 1b. (a) Input shot record $\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{0}$; (b) predicted internal multiples $\left\{\delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{1 b}$; (c) result after adaptive subtraction of the internal multiples. Note the remaining internal multiples related to reflector 1 a .


Figure 23. CFP gather and time-reversed, muted gridpoint gather for one focus point at depth level $B$ (see Figure 15). (a) CFP gather after muting $\left\{\overline{\mathbf{P}}_{i}\left(z_{B}, z_{0}\right)\right\}_{0}^{\dagger}$; (b) gridpoint gather after time reversing and muting $\left\{\Delta \mathbf{Q}_{j}\left(z_{B}, z_{B}\right)\right\}_{B}$.


Figure 24. Result of internal-multiple removal for the layer-related approach using the erroneous focusing operators based on level $B$ (see Figure 15). (a) Input shot record $\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{0} ;$ (b) predicted internal multiples $\left\{\Delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{B}$; (c) result after adaptive subtraction $\left\{\mathbf{P}_{j}\left(z_{0}, z_{0}\right)\right\}_{B}$.

## CONCLUSIONS

Based on the feedback model for seismic wavefields, an expression for primary and multiple reflections has been derived. The feed-forward path of this model describes the contribution of upward reflection in the lower part of the subsurface $\left(z>z_{m}\right)$, and the feedback path describes the contribution of downward reflection in the upper part of the subsurface $\left(z \leq z_{m}\right)$. For the special situation of $m=0$, downward reflection describes the surface-related multiples.

The expression of multiple reflections can be interpreted as the output of a spacevariant, multitrace prediction filter. The expression of this prediction filter describes a spatial convolution process between the upgoing reflection response of the lower part of the subsurface $\left(z>z_{m}\right)$ and the downgoing reflection response of the upper part of the subsurface ( $z \leq$ $z_{m}$ ). This convolution generates one extra round trip between the lower and upper parts of the subsurface.

Computation of the prediction filter is model driven or data driven. Data-driven prediction of the relatively strong surface multiples requires an iterative process, typically two or three iterations. For the weaker internal multiples, one or two iterations appear sufficient.

In the algorithm for internal multiples, a separation process (muting) needs to be applied between events from above ( $z \leq$ $\left.z_{m}\right)$ and below $\left(z>z_{m}\right)$ the multiple-generating level. In the CFP domain, muting becomes a straightforward process.

In the boundary-related approach to internal-multiple removal, a fully data-driven procedure can be followed. This procedure includes updating the focal operators. This means that the need for explicit velocity-depth model information is avoided.

In the layer-related approach to internal-multiple removal, the multiple-generating properties of a complete layer can be addressed in one step. The involved focal operators do not have to be accurate: the only requirement is proper separation between events from above and below the chosen depth level.
The layer-related approach to multiple removal is prefered for internal multiples. We expect that this conclusion is also valid for surface multiples. This issue is currently under investigation.

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## APPENDIX A

## WAVEFIELD OPERATORS

The notation for the wavefield operators and data types used in this paper is explained in the following overview.

## APPENDIX B

## INSENSITIVITY TO EXTRAPOLATION ERRORS

The velocity independence of internal multiple prediction is stated by Araujo et al. (1994) using an inverse-scatteringseries formulation (see also Weglein et al., 1997). To show this, let us look at the errors in the downward-extrapolated data (single extrapolation):

$$
\begin{equation*}
\left\langle\left\{\mathbf{P}_{j}\left(z_{n}, z_{0}\right)\right\}_{n-1}\right\rangle=\Delta \mathbf{V}\left(z_{n}\right)\left\{\mathbf{P}_{j}\left(z_{n}, z_{0}\right)\right\}_{n-1} \tag{B-1a}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle\left\{\mathbf{P}\left(z_{0}, z_{n}\right)\right\}_{n}\right\rangle=\left\{\mathbf{P}\left(z_{0}, z_{n}\right)\right\}_{n} \Delta \mathbf{V}^{T}\left(z_{n}\right), \tag{B-1b}
\end{equation*}
$$

with $\Delta \mathbf{V}\left(z_{n}\right)$ representing the errors in the extrapolation operators resulting from erroneous velocities. Similarly, the error in the time-reversed, causal part of the double extrapolated data is given by

$$
\begin{equation*}
\left\langle\Delta \mathbf{Q}\left(z_{n}, z_{n}\right)\right\rangle=\Delta \mathbf{V}^{*}\left(z_{n}\right) \Delta \mathbf{Q}\left(z_{n}, z_{n}\right) \Delta \mathbf{V}^{H}\left(z_{n}\right) \tag{B-1c}
\end{equation*}
$$

If we substitute equation $\mathrm{B}-1 \mathrm{a}-\mathrm{B}-1 \mathrm{c}$ in the expression for the internal multiples,

$$
\begin{equation*}
\Delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)=-\left\{\overline{\mathbf{P}}\left(z_{0}, z_{n}\right)\right\}_{n} \Delta \mathbf{Q}\left(z_{n}, z_{n}\right)\left\{\overline{\mathbf{P}}_{j}\left(z_{n}, z_{0}\right)\right\}_{n-1}, \tag{B-1d}
\end{equation*}
$$

we conclude that the kinematics of the predicted internal multiples are insensitive to velocity errors to a limited extent. Our theory shows that errors may occur in the causal-noncausal separation process as a result of defocusing. Note that the least-squares subtraction process may help to compensate for amplitude and phase differences:

$$
\begin{align*}
\left\{\mathbf{P}_{i j}\left(z_{0}, z_{0}\right)\right\}_{n}= & \left\{\mathbf{P}_{i j}\left(z_{0}, z_{0}\right)\right\}_{l-1} \\
& -\sum \mathbf{F}_{\mathbf{1 s}_{i}}^{\dagger}\left(z_{0}, z_{0}\right)\left\{\Delta \mathbf{M}_{j}\left(z_{0}, z_{0}\right)\right\}_{n}^{l}, \tag{B-2}
\end{align*}
$$

Table 1. Notation for wavefield operators and data types.

| Operator/type | Explanation |
| :---: | :---: |
| $\Delta \mathbf{P}\left(z_{0}, z_{0}\right), \mathbf{P}\left(z_{0}, z_{0}\right)$ | Matrix representing a full 3D data set with primary reflections only $(\Delta \mathbf{P})$ or with primary reflections and all multiples ( $\mathbf{P}$ ) (see Figure A-1). |
| $\left.{ }^{\text {P }}\left(z_{0}, z_{0}\right)\right\}_{n}$ | Matrix representing a full 3D data set with all primary reflections and internal multiples for levels $z>$ $z_{n}$ only. The data are the output of multiple removal up to level $z_{n}$ and are used as the input for internal-multiple removal for depth level $z_{n+1}$ (see Figure A-1). For $n=0$, this represents the data after surface-related-multiple removal. |
|  | Seismic wavefields traveling downward ( $\mathbf{P}^{+}$) or upward ( $\mathbf{P}^{-}$) at the receiver side. |
| $\left\{\mathbf{P}_{j}\left(z_{n}, z_{0}\right)\right\}_{m}$ | Column vector representing a shot gather with its source positioned at $\left(\mathbf{x}_{j}, z_{0}\right)$ and the receivers positioned at depth level $z_{n}$, with all multiples up to level $z_{m}$ removed [see Figure A-2 for one element, $\left.\left\{\mathbf{P}_{i j}\left(z_{n}, z_{0}\right)\right\}_{m}\right]$. |
| $\overline{\mathbf{P}}_{j}\left(z_{n}, z_{0}\right)$ | Column vector representing a shot gather with its source positioned at $\left(\mathbf{x}_{j}, z_{0}\right)$ and the receivers positioned at depth level $z_{n}$, containing only upgoing events at level $z_{n}$. This can be achieved by muting all reflections from depth levels $\left(z \leq z_{n}\right)$ from wavefield $\mathbf{P}_{j}\left(z_{n}, z_{0}\right)$ [see Figure A-2 for one element, $\left.\overline{\mathbf{P}}_{i j}\left(z_{n}, z_{0}\right)\right]$. |
| $\overline{\mathbf{P}}_{i}^{\dagger}\left(z_{0}, z_{n}\right)$ | Row vector representing a detector gather with its detector position at $\left(\mathbf{x}_{i}, z_{0}\right)$ and the sources positioned at depth level $z_{n}$, with all reflections (primaries and multiples) up to level $z_{n}$ removed [see Figure A-2 for one element, $\overline{\mathbf{P}}_{i j}\left(z_{0}, z_{n}\right)$ ]. |
| $\left\{\mathbf{P}\left(z_{n}, z_{n}\right)\right\}_{m}$ | Matrix representing a full 3D data set with sources and receivers downward extrapolated toward depth level $z_{n}$ and all multiples up to level $z_{m}$ removed. |
| $\left\{\Delta \mathbf{Q}\left(z_{n}, z_{n}\right)\right\}_{m}^{l}$ | Matrix representing a full 3D data set with sources and receivers downward extrapolated toward depth level $z_{n}$, after time reversing and keeping only primary reflections related to depth levels $z_{l} \leq z<z_{m}$. |
| $\mathbf{R}\left(z_{m}, z_{m}\right), \mathbf{R}^{\wedge}\left(z_{m}, z_{m}\right)$ | Boundary-related reflection operator transferring a downgoing incident wavefield into an upgoing response $\mathbf{R}$ or an upgoing incident wavefield into a downgoing response $\mathbf{R}^{\wedge}$ (see Figure A-3a). |
| $\mathbf{W}\left(z_{n}, z_{m}\right)$ | Layer-related propagation operator transferring an incident wavefield at depth level $z_{m}$ into a transmitted wavefield at depth level $z_{n}$, excluding any multiple reflections but including fine layering effects. For $z_{n}<z_{m}, \mathbf{W}$ describes downward propagation; for $z_{n}>z_{m}, \mathbf{W}$ describes upward propagation. |
| $\Delta \mathbf{X}, \mathbf{X}$ | Half-space operator transferring an incident wavefield into a response with primary reflections only $(\Delta \mathbf{X})$ or with primary reflections and all multiples $(\mathbf{X})$. |
| $\{\mathbf{X}\}_{n}$ | Half-space operator transferring an incident wavefield into a response with primary reflections and multiples related to depth level $z>z_{n}$ only. |
| $\mathbf{X}\left(z_{m}, z_{m}\right)$ | Half-space operator transferring a downgoing incident wavefield into an upgoing response, where sources and detectors are situated just below $z_{m}$ (see Figure A-3b). |
| $\mathbf{X}^{\wedge}\left(z_{m}, z_{m}\right)$ | Half-space operator transferring an upgoing incident wavefield into a downgoing response, where sources and detectors are situated just above $z_{m}$ (see Figure A-3b). |
| $\Delta \mathbf{X}\left(z_{0}, z_{n}\right)$ | Half-space operator transferring an incident wavefield at depth level $z_{n}$ into a response at depth level $z_{0}$ that contains all primary reflections related to depth level $z>z_{n}$ only. |
| $\{\delta \mathbf{M}\}_{n},\{\Delta \mathbf{M}\}_{n}^{l}, \mathbf{M}$ | Matrix representing a full 3D data set with multiples related to one boundary at $z_{n}$ only $(\delta \mathbf{M})$, with multiples related to one layer between depth levels $z_{l} \leq z<z_{n}$ only ( $\Delta \mathbf{M}$ ), or with all multiples ( $\mathbf{M}$ ). |
| $\boldsymbol{\Gamma}\left(z_{n}, z_{m}\right)$ | Layer-related inverse extrapolation operator, removing the primary propagation effect between level $z_{n}$ and $z_{m}$. For $z_{n}<z_{m}, \boldsymbol{\Gamma}$ describes inverse extrapolation from $z_{n}$ to $z_{m}$ at the source side; for $z_{n}>z_{m}, \boldsymbol{\Gamma}$ describes inverse extrapolation from $z_{m}$ to $z_{n}$ at the receiver side. |



Figure A-1. (a) Primaries ( $\Delta \mathbf{P}$ ), (b) primaries plus all multiples $(\mathbf{P})$, (c) primaries plus internal multiples ( $\{\mathbf{P}\}_{0}$ ), and (d) primaries plus multiples related to level $z>z_{n}$ only $\left(\{\mathbf{P}\}_{n}\right)$.


Figure A-2. Downward extrapolation results for $n>0$; full extrapolation result $\left(\{\mathbf{P}\}_{n-1}\right)$; result with primaries from below $z_{n}$ and multiples from below $z_{n-1}$ only $\left(\{\overline{\mathbf{P}}\}_{n=1}\right)$; result with primaries and multiples from below $z_{n}$ only $\left(\{\overline{\mathbf{P}}\}_{n}\right)$.
a)

b)


Figure A-3. (a) Definition of reflection operators for a single boundary, angle dependent. (b) Definition of reflection operators for a complete layer, primaries only. Each operator is represented by a matrix, and one element of this matrix is shown here by simple raypaths.
with row vector $\mathbf{F}_{\mathbf{l s}_{i}}^{\dagger}\left(z_{0}, z_{0}\right)$ representing the least-squares filter for lateral position $i$ and the summation being carried out over all layers involved. If the thickness of the layers approaches zero, the layer formulation approaches the boundary formulation and $\Delta \mathbf{M}$ becomes $\delta \mathbf{M}$.

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