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Renewal structure and local time for diffusions in random environment

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Abstract. We study a one-dimensional diffusion X in a drifted Brownian potential W_κ, with $0 < \kappa < 1$, and focus on the behavior of the local times $(\mathcal{L}(t,x),x)$ of X before time $t > 0$. In particular we characterize the limit law of the supremum of the local time, as well as the position of the favorite site. These limits can be written explicitly from a two dimensional stable Lévy process. Our analysis is based on the study of an extension of the renewal structure which is deeply involved in the asymptotic behavior of X.

1. Introduction

1.1. *Presentation of the model.* Let $(X(t),\ t \geq 0)$ be a diffusion in a random càdlàg potential $(V(x),\ x \in \mathbb{R})$, defined informally by $X(0) = 0$ and

$$\mathrm{d}X(t) = \mathrm{d}\beta(t) - \frac{1}{2}V'(X(t))\mathrm{d}t,$$

where $(\beta(s),\ s \geq 0)$ is a Brownian motion independent of V. Rigorously, X is defined by its conditional generator given V,

$$\frac{1}{2}e^{V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-V(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

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We put ourselves in the case where V is a negatively drifted Brownian motion: $V(x) = W_\kappa(x) := W(x) - \frac{\kappa}{2}x,\ x \in \mathbb{R}$, with $0 < \kappa < 1$ and $(W(x),\ x \in \mathbb{R})$ is a two sided Brownian motion. We explain at the end of Section 1.2 what should be done to extend our results to a more general Lévy potential.

We denote by P the probability measure associated to $W_\kappa(.)$. The probability conditionally on the potential W_κ is denoted by $\mathbb{P}^{W_\kappa}$ and is called the *quenched probability*. We also define the *annealed probability* as

$$\mathbb{P}(.) := \int \mathbb{P}^{W_\kappa}(.) P(W_\kappa \in \mathrm{d}\omega).$$

We denote respectively by $\mathbb{E}^{W_\kappa}$, $\mathbb{E}$, and E the expectations with regard to $\mathbb{P}^{W_\kappa}$, $\mathbb{P}$ and P. In particular, X is a Markov process under $\mathbb{P}^{W_\kappa}$ but not under $\mathbb{P}$.

This diffusion X has been introduced by Schumacher (1985). It is generally considered as a continuous time analogue of random walks in random environment (RWRE). We refer e.g. to Zeitouni (2004) for general properties of RWRE.

In our case, since $\kappa > 0$, the diffusion X is a.s. transient and its asymptotic behavior was first studied by Kawazu and Tanaka: if $H(r)$ is the hitting time of $r \in \mathbb{R}$ by X,

$$H(r) := \inf\{s > 0,\ X(s) = r\}, \tag{1.1}$$

Kawazu and Tanaka (1997) proved that, for $0 < \kappa < 1$ under the annealed probability $\mathbb{P}$, $H(r)/r^{1/\kappa}$ converges in law as $r \to +\infty$ to a κ-stable distribution (see also Hu et al., 1999, and Tanaka, 1997). Here we are interested in the local time of X, which is the jointly continuous process $(\mathcal{L}(t,x),\ t > 0,\ x \in \mathbb{R})$ satisfying, for any positive measurable function f,

$$\int_0^t f(X(s))\mathrm{d}s = \int_{-\infty}^{+\infty} f(x)\mathcal{L}(t,x)\mathrm{d}x, \qquad t > 0.$$

One quantity of particular interest is the supremum of the local time of X at time t, defined as

$$\mathcal{L}^*(t) := \sup_{x\in\mathbb{R}} \mathcal{L}(t,x), \qquad t > 0.$$

For Brox's diffusion, that is, for the diffusion X in the recurrent case $\kappa = 0$, it is proved in Andreoletti and Diel (2011) that the local time process until time t re-centered at the localization coordinate b_t (see Brox, 1986) and renormalized by t converges in law under the annealed probability $\mathbb{P}$. This allows the authors of Andreoletti and Diel (2011) to derive the limit law of the supremum of the local time at time t as $t \to +\infty$. We recall their result below in order to compare it with the results of the present paper. To this aim, we introduce for every $\kappa \geq 0$,

$$\mathcal{R}_\kappa := \int_0^{+\infty} e^{-W_\kappa^\uparrow(x)}\mathrm{d}x + \int_0^{+\infty} e^{-\widetilde{W}_\kappa^\uparrow(x)}\mathrm{d}x, \tag{1.2}$$

where $\left(W_\kappa^\uparrow(x),\ x \geq 0\right)$ and $\left(\widetilde{W}_\kappa^\uparrow,\ x \geq 0\right)$ are two independent copies of the process $(W_\kappa(x),\ x \geq 0)$ Doob-conditioned to remain positive.

Theorem 1.1. *(Andreoletti and Diel, 2011) If $\kappa = 0$, then*

$$\frac{\mathcal{L}^*(t)}{t} \xrightarrow{\mathcal{L}} \frac{1}{\mathcal{R}_\kappa},$$

where $\xrightarrow{\mathcal{L}}$ denotes convergence in law under the annealed probability $\mathbb{P}$ as $t \to +\infty$.

Extending their approach, and following the results of Shi (1998), Diel (2011) obtains the non-trivial normalizations for the almost sure behavior of the lim sup and the lim inf of $\mathcal{L}^*(t)$ as $t \to +\infty$ when $\kappa = 0$. Notice that corresponding results have been previously established in Dembo et al. (2007) and Gantert et al. (2010) for the discrete analogue of X in the recurrent case $\kappa = 0$, the recurrent RWRE generally called Sinai's random walk.

One of our aims in this paper is to extend the study of the local time of X in the case $0 < \kappa < 1$, and deduce from that the weak asymptotic behavior of $\mathcal{L}^*(t)$ suitably renormalized as $t \to +\infty$.

Before going any further, let us recall to the reader what is known for the slow transient cases. For transient RWRE in the case $0 < \kappa \leq 1$ (see Kesten et al., 1975 for the seminal paper), a result of Gantert and Shi (2002) states the almost sure behavior for the lim sup of the supremum of the local time $\mathcal{L}_S^*(n)$ of these random walks (denoted by S) at time n: there exists a constant $c > 0$ such that $\limsup_{n\to+\infty} \mathcal{L}_S^*(n)/n = c > 0$ $\mathbb{P}$ almost surely. Contrarily to the recurrent case (Gantert et al., 2010) their method, based on a relationship between the RWRE S and a branching process in random environment, cannot be exploited to determine the limit law of $\mathcal{L}_S^*(n)/n$.

For the transient diffusion X considered here, the only paper dealing with $\mathcal{L}^*(t)$ is Devulder (2016+), in which it is proved, among other results, that when $0 < \kappa < 1$, $\limsup_{t\to+\infty} \mathcal{L}^*(t)/t = +\infty$ almost surely. But once again his method cannot be used to characterize the limit law of $\mathcal{L}^*(t)/t$ in the case $0 < \kappa < 1$.

Our motivation here is twofold, first we prove that our approach enables to characterize the limit law of $\mathcal{L}^*(t)/t$ and open a way to determine the correct almost sure behavior of $\mathcal{L}^*(t)$ as was done for Brox's diffusion by Shi (1998) and Diel (2011). Second we make a first step on a specific way to study the local time which could be used in estimation problems in random environment, see Adelman and Enriquez (2004), Andreoletti (2011), Andreoletti and Diel (2012), Andreoletti et al. (2015), Comets et al. (2016+), Comets et al. (2014), Falconnet et al. (2014).

The method we develop here is an improvement of the one used in Andreoletti and Devulder (2015) about the localization of $X(t)$ for large t.

Before recalling the main result of this paper Andreoletti and Devulder (2015), we need to introduce some new objects. We start with the notion of h-extrema, with $h > 0$, introduced by Neveu and Pitman (1989) and studied more specifically in our case of drifted Brownian motions by Faggionato (2009). For $h > 0$, we say that $x \in \mathbb{R}$ is an *h-minimum* for a given continuous function f, $\mathbb{R} \to \mathbb{R}$, if there exist $u < x < v$ such that $f(y) \geq f(x)$ for all $y \in [u, v]$, $f(u) \geq f(x) + h$ and $f(v) \geq f(x) + h$. Moreover, x is an *h-maximum* for f iff x is an h-minimum for $-f$. Finally, x is an *h-extremum* for f iff it is an h-maximum or an h-minimum for f.

As we are interested in the diffusion X until time t for large t, we only focus on the h_t-extrema of W_κ, where

$$h_t := \log t - \phi(t), \qquad \text{with } 0 < \phi(t) = o(\log t), \quad \log\log t = o(\phi(t)),$$

and $t \mapsto \phi(t)$ is an increasing function, as in Andreoletti and Devulder (2015). It is known (see Faggionato, 2009) that almost surely, the h_t-extrema of W_κ form a sequence indexed by $\mathbb{Z}$, unbounded from below and above, and that the h_t-minima and h_t-maxima alternate. We denote respectively by $(m_j,\ j \in \mathbb{Z})$ and $(M_j,\ j \in \mathbb{Z})$ the increasing sequences of h_t-minima and of h_t-maxima of W_κ, such that $m_0 \leq 0 < m_1$ and $m_j < M_j < m_{j+1}$ for every $j \in \mathbb{Z}$. Define

$$N_t := \max\Big\{k \in \mathbb{N},\ \sup_{0\leq s\leq t} X(s) \geq m_k\Big\}, \tag{1.3}$$

the number of (positive) h_t-minima on $\mathbb{R}_+$ visited by X until time t. We have the following result.

Theorem 1.2. *(Andreoletti and Devulder, 2015) Assume $0 < \kappa < 1$. There exists a constant $\mathcal{C}_1 > 0$, such that*

$$\lim_{t\to+\infty} \mathbb{P}\Big(\big|X(t) - m_{N_t}\big| \leq \mathcal{C}_1\phi(t)\Big) = 1.$$

This result proves that before time t, the diffusion X visits the N_t leftmost positive h_t-minima, and then gets stuck in a very small neighborhood of an ultimate one, which is m_{N_t}. An analogous result was proved for transient RWRE in the zero speed regime $0 < \kappa < 1$ by Enriquez et al. (2009a). This phenomenon is due to two facts: the first one is the appearance of a renewal structure which is composed of the times it takes the process to move from one h_t-minimum to the following one. The second is the fact that like in Brox's case $\kappa = 0$, the process is trapped a significant amount of time in the neighborhood of the local minimum m_{N_t}.

It is the extension of this renewal structure to the sequence of local times at the h_t-minima that we study here. We now detail our results.

1.2. *Results.* Let us introduce some notation involved in the statement of our results. Assume that $0 < \kappa < 1$.

Denote by $\big(D([0,+\infty),\mathbb{R}^2), J_1\big)$ the space of càdlàg functions $[0,+\infty) \to \mathbb{R}^2$ with J_1-Skorokhod topology and denote by $\overset{\mathcal{L}_S}{\to}$ the convergence in law for this topology. On this space, define a 2-dimensional Lévy process $(\mathcal{Y}_1, \mathcal{Y}_2)$ taking values in $\mathbb{R}_+ \times \mathbb{R}_+$, which is a pure positive jump process with κ-stable Lévy measure ν given by

$$\forall x > 0,\ \forall y > 0, \qquad \nu\big([x,+\infty[\times[y,+\infty[\big) = \frac{\mathcal{C}_2}{y^\kappa}\mathbb{E}\left[(\mathcal{R}_\kappa)^\kappa \mathbb{1}_{\mathcal{R}_\kappa \leq \frac{y}{x}}\right] + \frac{\mathcal{C}_2}{x^\kappa}\mathbb{P}\left(\mathcal{R}_\kappa > \frac{y}{x}\right), \tag{1.4}$$

where $\mathcal{R}_\kappa$ is defined in (1.2) and $\mathcal{C}_2$ is a positive constant (see Lemma 4.1). The Laplace transform of $\mathcal{R}_\kappa$ is given by

$$E\big(e^{-\gamma\mathcal{R}_\kappa}\big) = \left(\frac{(2\gamma)^{\kappa/2}}{\kappa\Gamma(\kappa)I_\kappa(2\sqrt{2\gamma})}\right)^2 \qquad \gamma > 0,$$

as proved in Lemma 6.6 below, where I_κ is the modified Bessel function of the first kind of index κ. Moreover, $\mathcal{R}_\kappa$ admits moments of any positive order (see also Lemma 6.6). In particular $\mathbb{E}[(\mathcal{R}_\kappa)^\kappa]$ is finite and ν is well defined.

For a given càdlàg function f in $D([0,+\infty),\mathbb{R})$, define for any $s > 0, a > 0$:

$$f^\natural(s) := \sup_{0\leq r\leq s}(f(r) - f(r^-)), \qquad f^{-1}(a) := \inf\{x \geq 0,\ f(x) > a\},$$

where $f(r^-)$ denotes the left limit of f at r. In words, $f^\natural(s)$ is the largest jump of f before time s, whereas $f^{-1}(a)$ is the first time f is strictly larger than a. We also introduce the couple of random variables $(\mathcal{I}_1, \mathcal{I}_2)$ as follows,

$$\mathcal{I}_1 := \mathcal{Y}_1^{\natural}\big(\mathcal{Y}_2^{-1}(1)^-\big), \qquad \mathcal{I}_2 := \big(1 - \mathcal{Y}_2\big(\mathcal{Y}_2^{-1}(1)^-\big)\big) \times \frac{\mathcal{Y}_1\big(\mathcal{Y}_2^{-1}(1)\big) - \mathcal{Y}_1\big(\mathcal{Y}_2^{-1}(1)^-\big)}{\mathcal{Y}_2\big(\mathcal{Y}_2^{-1}(1)\big) - \mathcal{Y}_2\big(\mathcal{Y}_2^{-1}(1)^-\big)}. \tag{1.5}$$

We recall that $\xrightarrow{\mathfrak{L}}$ denotes convergence in law under the annealed probability $\mathbb{P}$ as $t \to +\infty$. We are now ready to state our first result.

Theorem 1.3. *Assume* $0 < \kappa < 1$. *We have,*

$$\frac{\mathcal{L}^*(t)}{t} \xrightarrow{\mathfrak{L}} \mathcal{I} =: \max(\mathcal{I}_1, \mathcal{I}_2).$$

Contrary to the recurrent case $\kappa = 0$, we have no scaling property for the potential, and the diffusion X cannot be localized in a single valley as we can see in Theorem 1.2. However in the transient case we can make appear and use a renewal structure.

We now give an intuitive interpretation of this theorem, explaining the appearance of the Lévy process $(\mathcal{Y}_1, \mathcal{Y}_2)$.

First for any $s > 0$, $\mathcal{Y}_1(s)$ is the limit of the sum of the first $\lfloor se^{\kappa\phi(t)} \rfloor$ normalized (by t) local times taken specifically at the $\lfloor se^{\kappa\phi(t)} \rfloor$ first h_t-minima (see Proposition 1.4 below). Similarly, $\mathcal{Y}_2(s)$ is the limit of the sum of the exit times of the $\lfloor se^{\kappa\phi(t)} \rfloor$ first h_t-valleys, normalized (by t), where an h_t-valley is a large neighborhood of an h_t-minimum. For a rigorous definition of these h_t-valleys, see Section 2.2 and Figure 2.1.

So, by definition, $\mathcal{I}_1$ is the largest jump of the process $\mathcal{Y}_1$ before the first time $\mathcal{Y}_2$ is larger than 1. It can be interpreted as the largest (re-normalized) local time among the local times at the h_t-minima visited by X until time t and from which X has already escaped. That is to say, $\mathcal{I}_1$ is the limit of the random variable $\sup_{k \leq N_t - 1} \mathcal{L}(m_k, t)/t$.

$\mathcal{I}_2$ is a product of two factors: the first one, $\big(1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)^-)\big)$, corresponds to the (re-normalized) amount of time left to the diffusion X before time t after it has reached the ultimate visited h_t-minimum m_{N_t}, that is, to $(t - H(m_{N_t}))/t$. The second factor corresponds to the local time of X at this ultimate h_t-minimum m_{N_t}, that is to say $\mathcal{I}_2$ is the limit of $\mathcal{L}(t, m_{N_t})/t$. Intuitively $\mathcal{Y}_2$ is built from $\mathcal{Y}_1$ by multiplying each of its jumps by an independent copy of the variable $\mathcal{R}_\kappa$. Therefore this second factor can be seen as an independent copy of $1/\mathcal{R}_\kappa$ taken at the instant of the overshoot of $\mathcal{Y}_2$ which makes it larger than 1. Notice that this variable $\mathcal{R}_\kappa$ plays a similar role as $\mathcal{R}_0$ of Theorem 1.1. Indeed as in the case $\kappa = 0$, the diffusion X is prisoner in the neighborhood of the last h_t-minimum visited before time t.

We prove Theorem 1.3 by showing first that portions of the trajectory of X re-centered at the local h_t-minima, until time t, are made (in probability) with independent parts. This has been partially proved in Andreoletti and Devulder (2015) but we have to improve their results and add simultaneously the study of the local time.

Second, we prove that the supremum of the local time is, mainly, a function of the sum of theses independent parts, which converges to a Lévy process. We now provide some details about this.

Recall that $(W_\kappa^\uparrow(s),\ s\geq 0)$ is defined as a continuous process, taking values in $\mathbb{R}_+$, with infinitesimal generator given for every $x>0$ by

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\kappa}{2}\coth\left(\frac{\kappa}{2}x\right)\frac{\mathrm{d}}{\mathrm{d}x}.$$

This process $W_\kappa^\uparrow$ can be thought of as a $(-\kappa/2)$-drifted Brownian motion W_κ Doob-conditioned to stay positive, with the terminology of Bertoin (1996), which is called Doob conditioned to reach $+\infty$ before 0 in Faggionato (2009) (for more details, see Section 2.1 in Andreoletti and Devulder (2015), where $W_\kappa^\uparrow$ is denoted by R). We call BES$(3,\kappa/2)$ the law of $(W_\kappa^\uparrow(s),\ s\geq 0)$. That is, $(W_\kappa^\uparrow(s),\ s\geq 0)$ is a 3-dimensional $(\kappa/2)$-drifted Bessel process starting from 0. For any process $(U(t),\quad t\in\mathbb{R}_+)$, we denote by

$$\tau^U(a):=\inf\{t>0,\ U(t)=a\},$$

the first time this process hits a, with the convention $\inf\emptyset=+\infty$. For $a<b$, $\big(W_\kappa^b(s),\ 0\leq s\leq \tau^{W_\kappa^b}(a)\big)$ is defined as a $(-\kappa/2)$-drifted Brownian motion starting from b and killed when it first hits a. We now introduce some functionals of W_κ and $W_\kappa^\uparrow$, which already appeared in Andreoletti and Devulder (2015, Section 4.1):

$$F^\pm(x) \quad := \quad \int_0^{\tau^{W_\kappa^\uparrow}(x)} \exp(\pm W_\kappa^\uparrow(s))\mathrm{d}s, \qquad x>0, \tag{1.6}$$

$$G^\pm(a,b) \quad := \quad \int_0^{\tau^{W_\kappa^b}(a)} \exp\big(\pm W_\kappa^b(s)\big)\mathrm{d}s, \qquad a<b. \tag{1.7}$$

Let $0<\delta<1$, define

$$n_t:=\big\lfloor e^{\kappa\phi(t)(1+\delta)}\big\rfloor, \qquad t>0,$$

which is, with large probability, an upper bound for N_t as stated in Lemma 3.1.

Let $(S_j,R_j,\mathbf{e}_j,j\leq n_t)$ be a sequence of i.i.d. random variables depending on t, with S_j, R_j and $\mathbf{e_j}$ independent, $S_1\overset{\mathcal{L}}{=}F^+(h_t)+G^+(h_t/2,h_t)$, $R_1\overset{\mathcal{L}}{=}F^-(h_t/2)+\tilde{F}^-(h_t/2)$ and $\mathbf{e}_1\overset{\mathcal{L}}{=}\mathcal{E}(1/2)$ (an exponential random variable with parameter 1/2), where $\tilde{F}^-$ is an independent copy of F^- and F^+ is independent of G^+, and $\overset{\mathcal{L}}{=}$ denotes equality in law. Define $\ell_j:=\mathbf{e}_jS_j$ and $\mathcal{H}_j:=\ell_jR_j$. Note that to simplify the notation, we do not make appear the dependence in t in the sequel. Intuitively, ℓ_j plays the role of the local time at the j-th positive h_t-minimum m_j if X escapes from the j-th h_t-valley before time t, that is, if $j<N_t$. Similarly, $\mathcal{H}_j$ plays the role of the time X spends in the j-th h_t-valley before escaping from it.

Define the family of processes $(Y_1,Y_2)^t$ indexed by t, by

$$\forall s\geq 0, \qquad (Y_1,Y_2)^t_s=\big(Y_1^t(s),Y_2^t(s)\big):=\frac{1}{t}\sum_{j=1}^{\lfloor se^{\kappa\phi(t)}\rfloor}(\ell_j,\mathcal{H}_j).$$

Recall that $\overset{\mathcal{L}_S}{\to}$ denotes convergence in law under J_1-Skorokhod topology. Here is our next result.

Proposition 1.4. *Assume $0<\kappa<1$. We have under $\mathbb{P}$, as $t\to+\infty$,*

$$(Y_1,Y_2)^t\overset{\mathcal{L}_S}{\to}(\mathcal{Y}_1,\mathcal{Y}_2).$$

Once this is proved, we check that we can approximate, in law, the renormalized local time $\mathcal{L}^*(t)/t$ by a function of $(Y_1, Y_2)^t$. We obtain such an expression in Proposition 5.1. Then to obtain the limit claimed in Theorem 1.3, we prove the continuity (in J_1-topology) of the involved mapping and apply a continuous mapping Theorem (see Section 4.3).

It appears that with this method we can also obtain some other asymptotics. Indeed, we obtain in the following theorem the convergence in law of the supremum of the local time of X before X hits the last h_t-minimum m_{N_t} visited before time t, of the supremum of the local time of X before X leaves the last h_t-valley visited before time t (the one around m_{N_t}) approximately at time $H(m_{N_t+1})$, and of the position of the favorite site.

Theorem 1.5. *Assume $0 < \kappa < 1$. We have the following convergences in law under $\mathbb{P}$ as $t \to +\infty$,*

$$\frac{\mathcal{L}^*(H(m_{N_t+1}))}{t} \overset{\mathcal{L}}{\to} \mathcal{Y}_1^\natural(\mathcal{Y}_2^{-1}(1)), \tag{1.8}$$

$$\frac{\mathcal{L}^*(H(m_{N_t}))}{t} \overset{\mathcal{L}}{\to} \mathcal{Y}_1^\natural(\mathcal{Y}_2^{-1}(1)^-) = \mathcal{I}_1. \tag{1.9}$$

Let us call F_t^* the position of the first favorite site, that is, $F_t^* := \inf\{x \in \mathbb{R}, \mathcal{L}(t,x) = \mathcal{L}^*(t)\}$. Then,

$$\frac{F_t^*}{X(t)} \overset{\mathcal{L}}{\to} \mathcal{B}U_{[0,1]} + 1 - \mathcal{B}, \tag{1.10}$$

where $\mathcal{B}$ is a Bernoulli random variable with parameter $\mathbb{P}(\mathcal{I}_1 < \mathcal{I}_2)$, and $U_{[0,1]}$ is a uniform random variable on $[0,1]$, independent of $\mathcal{B}$.

We remark that with probability one there is at most one point x such that $\mathcal{L}(t,x) = \mathcal{L}^*(t)$ so F_t^* is actually the favorite site. Note that similar questions about favorite points for X have been studied in the recurrent case $\kappa = 0$ by Cheliotis (2008).

One question we may ask here is: what happens in the discrete case (that is, for RWRE), or with a more general Lévy potential?

For RWRE, we expect a very similar behavior because the renewal structures which appear in both cases (RWRE and our diffusion X) are very similar (see Enriquez et al., 2009a). The main difference comes essentially from the functional $\mathcal{R}_\kappa$, which should be replaced by a sum of exponentials of simple random walks conditioned to remain positive (see Enriquez et al., 2009b,a).

For a more general Lévy potential, we have in mind for example a spectrally negative Lévy process (diffusions in such potentials have been studied by Singh, 2008). More work needs to be done, especially for the potential. First, to obtain a specific decomposition of the Lévy's path (similar to what is done for the drifted Brownian motion in Faggionato, 2009), and also to study the more complicated functional $\mathcal{R}_\kappa$ which is less known than in the Brownian case. This is a work in preparation by Véchambre (2016).

The rest of the paper is organized as follows.

In Section 2, we recall the results of Faggionato on the path decomposition of the trajectories of W_κ. Also we recall from Andreoletti and Devulder (2015) the construction of specific h_t-minima which plays an important role in the appearance of independence, under $\mathbb{P}$, on the path of X before time t.

In Section 3, we study the joint process of the hitting times of the h_t-minima m_j, $1 \leq j \leq n_t$ and of local times at these m_j. We show that parts of the trajectory of X are not important for our study, that is, we prove that the time spent outside the h_t-valleys, and the supremum of the local time outside the h_t-valleys are negligible compared to t. We then prove the main result of this section: Proposition 3.5. It shows that the joint process (exit times, local times) can be approximated in probability by $i.i.d$ random variables (which are the $\mathcal{H}_j$ and ℓ_j). This part makes use of some technical results inspired from Andreoletti and Devulder (2015), they are summarized in Section 6.

In Section 4, we prove Proposition 1.4, and study the continuity of certain functionals of $(\mathcal{Y}_1, \mathcal{Y}_2)$ which appear in the expression of the limit law $\mathcal{I}$. This section is independent of the other ones, we essentially prove a basic functional limit theorem and prepare to the application of continuous mapping theorem.

Section 5 is where we make appear the renewal structure in the problem we want to solve. In particular we show how the distribution of the supremum of the local time can be approximated by the distribution of some function of the couple $(Y_1, Y_2)^t$, the main step being Proposition 5.1.

Section 6 is a reminder of some key results and their extensions extracted from Andreoletti and Devulder (2015). For some of these results, sketch of proofs or complementary proofs are added in order for this paper to be more self-contained.

Finally, Section 7 is a reminder of some estimates on Brownian motion, Bessel processes, and functionals of both of these processes.

1.3. *Notation.* In this section we introduce typical notation and tools for the study of diffusions in a random potential.

For any process $(U(t), \quad t \in \mathbb{R}_+)$ we denote by $\mathcal{L}_U$ a bicontinuous version of the local time of U when it exists. Notice that for our main process X we simply write $\mathcal{L}$ for its local time. The inverse of the local time for every $x \in \mathbb{R}$ is denoted by $\sigma_U(t,x) := \inf\{s > 0,\ \mathcal{L}_U(s,x) \geq t\}$ and in the same way $\sigma(t,x) := \sigma_X(t,x)$. We also denote by U^a the process U starting from a, and by P^a the law of U^a, with the notation $U = U^0$. Now, let us introduce the following functional of W_κ,

$$A(r) := \int_0^r e^{W_\kappa(x)} \mathrm{d}x, \qquad r \in \mathbb{R}.$$

We recall that since $\kappa > 0$, $A_\infty := \lim_{r \to +\infty} A(r) < \infty$ a.s. As in Brox (1986), there exists a Brownian motion $(B(s),\ s \geq 0)$, independent of W_κ, such that $X(t) = A^{-1}[B(T^{-1}(t))]$ for every $t \geq 0$, where

$$T(r) := \int_0^r \exp\{-2W_\kappa[A^{-1}(B(s))]\}\mathrm{d}s, \qquad 0 \leq r < \tau^B(A_\infty). \tag{1.11}$$

The local time of the diffusion X at location x and time t, simply denoted by $\mathcal{L}(t,x)$, can be written as (see Shi, 1998, eq. (2.5))

$$\mathcal{L}(t,x) = e^{-W_\kappa(x)} \mathcal{L}_B(T^{-1}(t), A(x)), \qquad t > 0, x \in \mathbb{R}. \tag{1.12}$$

With this notation, we recall the following expression of the hitting times of X,

$$H(r) = T\big[\tau^B(A(r))\big] = \int_{-\infty}^r e^{-W_\kappa(u)} \mathcal{L}_B[\tau^B(A(r)), A(u)]\mathrm{d}u, \qquad r \geq 0. \tag{1.13}$$

2. Path decomposition and Valleys

2.1. *Path decomposition in the neighborhood of the h_t-minima m_i.* We first recall some results for h_t-extrema of W_κ. Let

$$V^{(i)}(x) := W_\kappa(x) - W_\kappa(m_i), \qquad x \in \mathbb{R},\ i \in \mathbb{N}^*,$$

which is the potential W_κ translated so that it is 0 at the local minimum m_i. We also define

$$\tau_i^-(h) \ := \ \sup\{s < m_i,\ V^{(i)}(s) = h\}, \qquad h > 0, \tag{2.1}$$

$$\tau_i(h) \ := \ \inf\{s > m_i,\ V^{(i)}(s) = h\}, \qquad h > 0. \tag{2.2}$$

The following result has been proved by Faggionato (2009) [for **(i)** and **(ii)**], and the last fact comes from the strong Markov property (see also Andreoletti and Devulder, 2015, Fact 2.1, and its proof).

Fact 2.1. *(path decomposition of W_κ around the h_t-minima m_i)*
(i) *The truncated trajectories $\big(V^{(i)}(m_i - s),\ 0 \le s \le m_i - \tau_i^-(h_t)\big)$, $\big(V^{(i)}(m_i + s),\ 0 \le s \le \tau_i(h_t) - m_i\big)$, $i \ge 1$ are independent.*
(ii) *Let $(W_\kappa^\uparrow(s),\ s \ge 0)$ be a process with law $BES(3, \kappa/2)$. All the truncated trajectories $\big(V^{(i)}(m_i - s),\ 0 \le s \le m_i - \tau_i^-(h_t)\big)$ for $i \ge 2$ and $\big(V^{(j)}(m_j + s),\ 0 \le s \le \tau_j(h_t) - m_j\big)$ for $j \ge 1$ are equal in law to $\big(W_\kappa^\uparrow(s),\ 0 \le s \le \tau^{W_\kappa^\uparrow}(h_t)\big)$.*
(iii) *For $i \ge 1$, the truncated trajectory $\big(V^{(i)}(s + \tau_i(h_t)),\ s \ge 0\big)$ is independent of $\big(W_\kappa(s),\ s \le \tau_i(h_t)\big)$ and is equal in law to $\big(W_\kappa^{h_t}(s),\ s \ge 0\big)$, that is, to a $(-\kappa/2)$-drifted Brownian motion starting from h_t.*

2.2. *Definition of h_t-valleys and of standard h_t-minima $\tilde{m}_j$, $j \in \mathbb{N}^*$.*
We are interested in the potential around the h_t-minima m_i, $i \in \mathbb{N}^*$, in fact intervals containing at least $[\tau_i^-((1+\kappa)h_t), M_i]$. However, these valleys could intersect. In order to define valleys which are well separated and i.i.d., we introduce the following notation. This notation is used to define valleys of the potential around some $\tilde{m}_i$, which are thanks to Lemma 2.2 equal to the m_i for $1 \le i \le n_t$ with large probability.

Let

$$h_t^+ := (1 + \kappa + 2\delta) h_t.$$

As in Andreoletti and Devulder (2015), we define $\tilde{L}_0^+ := 0$, $\tilde{m}_0 := 0$, and recursively for $i \ge 1$ (see Figure 2.1),

$$\begin{aligned} \tilde{L}_i^\sharp &:= \inf\{x > \tilde{L}_{i-1}^+,\ W_\kappa(x) \le W_\kappa(\tilde{L}_{i-1}^+) - h_t^+\}, \\ \tilde{\tau}_i(h_t) &:= \inf\big\{x \ge \tilde{L}_i^\sharp,\ W_\kappa(x) - \inf\nolimits_{[\tilde{L}_i^\sharp, x]} W_\kappa \ge h_t\big\}, \\ \tilde{m}_i &:= \inf\big\{x \ge \tilde{L}_i^\sharp,\ W_\kappa(x) = \inf\nolimits_{[\tilde{L}_i^\sharp, \tilde{\tau}_i(h_t)]} W_\kappa\big\}, \\ \tilde{L}_i^+ &:= \inf\{x > \tilde{\tau}_i(h_t),\ W_\kappa(x) \le W_\kappa(\tilde{\tau}_i(h_t)) - h_t - h_t^+\}. \end{aligned} \tag{2.3}$$

We also introduce the following random variables for $i \in \mathbb{N}^*$:

$$\tilde{M}_i := \inf\{s > \tilde{m}_i,\ W_\kappa(s) = \max_{\tilde{m}_i \le u \le \tilde{L}_i^+} W_\kappa(u)\},$$
$$\tilde{L}_i^* := \inf\{x > \tilde{\tau}_i(h_t),\ W_\kappa(x) - W_\kappa(\tilde{m}_i) = 3h_t/4\},$$
$$\tilde{L}_i := \inf\{x > \tilde{\tau}_i(h_t),\ W_\kappa(x) - W_\kappa(\tilde{m}_i) = h_t/2\}, \tag{2.4}$$
$$\tilde{\tau}_i(h) := \inf\{s > \tilde{m}_i,\ W_\kappa(x) - W_\kappa(\tilde{m}_i) = h\}, \qquad h > 0, \tag{2.5}$$
$$\tilde{\tau}_i^-(h) := \sup\{s < \tilde{m}_i,\ W_\kappa(x) - W_\kappa(\tilde{m}_i) = h\}, \qquad h > 0, \tag{2.6}$$
$$\tilde{L}_i^- := \tilde{\tau}_i^-(h_t^+).$$

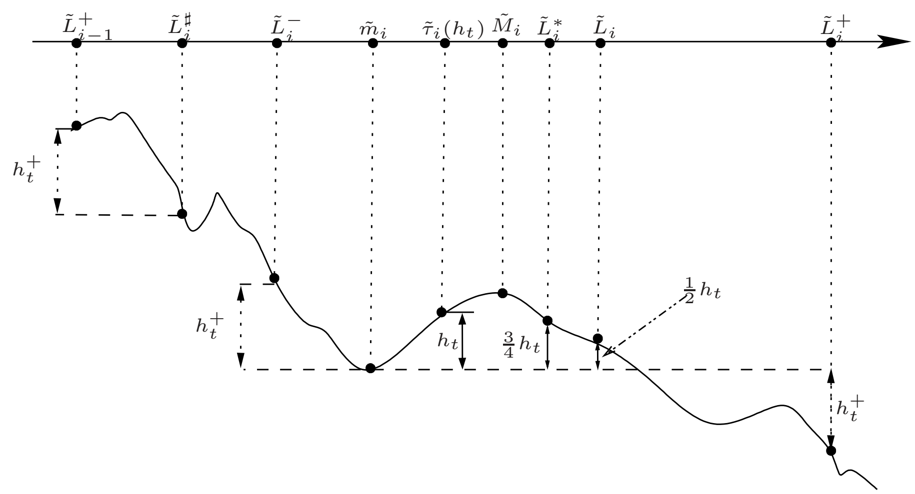

FIGURE 2.1. Schema of the potential between $\tilde{L}_{i-1}^+$ and $\tilde{L}_i^+$, in the case $\tilde{L}_i^\sharp < \tilde{L}_i^-$.

We stress that these random variables depend on t, which we do not write as a subscript to simplify the notation. Notice also that $\tilde{\tau}_i(h_t)$ is the same in definitions (2.3) and (2.5) with $h = h_t$. Moreover by continuity of W_κ, $W_\kappa(\tilde{\tau}_i(h_t)) = W_\kappa(\tilde{m}_i) + h_t$. Thus, the $\tilde{m}_i$, $i \in \mathbb{N}^*$, are h_t-minima, since $W_\kappa(\tilde{m}_i) = \inf_{[\tilde{L}_{i-1}^+, \tilde{\tau}_i(h_t)]} W_\kappa$, $W_\kappa(\tilde{\tau}_i(h_t)) = W_\kappa(\tilde{m}_i) + h_t$ and $W_\kappa(\tilde{L}_{i-1}^+) \ge W_\kappa(\tilde{m}_i) + h_t$. In addition,

$$\tilde{L}_{i-1}^+ < \tilde{L}_i^\sharp \le \tilde{m}_i < \tilde{\tau}_i(h_t) < \tilde{L}_i^* < \tilde{L}_i < \tilde{L}_i^+, \qquad i \in \mathbb{N}^*, \tag{2.7}$$
$$\tilde{L}_{i-1}^+ \le \tilde{L}_i^- < \tilde{m}_i < \tilde{\tau}_i(h_t) < \tilde{M}_i < \tilde{L}_i^+, \qquad i \in \mathbb{N}^*. \tag{2.8}$$

Also by induction, the random variables $\tilde{L}_i^\sharp$, $\tilde{\tau}_i(h_t)$ and $\tilde{L}_i^+$, $i \in \mathbb{N}^*$ are stopping times for the natural filtration of $(W_\kappa(x),\ x \ge 0)$, and so $\tilde{L}_i$, $\tilde{L}_i^*$, $i \in \mathbb{N}^*$, are also stopping times. Moreover by induction,

$$\begin{aligned} W_\kappa(\tilde{L}_i^\sharp) = \inf_{[0,\tilde{L}_i^\sharp]} W_\kappa, \qquad W_\kappa(\tilde{m}_i) = \inf_{[0,\tilde{\tau}_i(h_t)]} W_\kappa, \\ W_\kappa(\tilde{L}_i^+) = \inf_{[0,\tilde{L}_i^+]} W_\kappa = W_\kappa(\tilde{m}_i) - h_t^+, \end{aligned} \tag{2.9}$$

for $i \in \mathbb{N}^*$. We also introduce the analogue of $V^{(i)}$ for $\tilde{m}_i$ as follows:

$$\tilde{V}^{(i)}(x) := W_\kappa(x) - W_\kappa(\tilde{m}_i), \qquad x \in \mathbb{R},\ i \in \mathbb{N}^*.$$

We call i *th* h_t*-valley* the translated truncated potential $\big(\tilde{V}^{(i)}(x),\ \tilde{L}_i^- \leq x \leq \tilde{L}_i\big)$, for $i \geq 1$.

The following lemma states that, with a very large probability, the first n_t+1 positive h_t-minima m_i, $1 \leq i \leq n_t+1$, coincide with the random variables $\tilde{m}_i$, $1 \leq i \leq n_t+1$. We introduce the corresponding event $\mathcal{V}_t := \cap_{i=1}^{n_t+1}\{m_i = \tilde{m}_i\}$.

Lemma 2.2. *Assume $0 < \delta < 1$. There exists a constant $C_1 > 0$ such that for t large enough,*

$$P\left(\overline{\mathcal{V}}_t\right) \leq C_1 n_t e^{-\kappa h_t/2} = e^{[-\kappa/2+o(1)]h_t}.$$

Moreover, the sequence $\Big(\big(\tilde{V}^{(i)}(x+\tilde{L}_{i-1}^+),\ 0 \leq x \leq \tilde{L}_i^+ - \tilde{L}_{i-1}^+\big),\ i \geq 1\Big)$, is i.i.d.

Proof: This lemma is proved in Andreoletti and Devulder (2015): Lemma 2.3. □

The following remark is used several times in the rest of the paper.

Remark 2.3. On $\mathcal{V}_t$, we have for every $1 \leq i \leq n_t$, $m_i = \tilde{m}_i$, and as a consequence, $\tilde{V}^{(i)}(x) = V^{(i)}(x)$, $x \in \mathbb{R}$, $\tau_i^-(h) = \tilde{\tau}_i^-(h)$ and $\tau_i(h) = \tilde{\tau}_i(h)$ for $h > 0$. Moreover, $\tilde{M}_i = M_i$. Indeed, $\tilde{M}_i$ is an h_t-maximum for W_κ, which belongs to $[\tilde{m}_i, \tilde{m}_{i+1}] = [m_i, m_{i+1}]$ on $\mathcal{V}_t$, and there is exactly one h_t-maximum in this interval since the h_t-maxima and minima alternate, which we defined as M_i, so $\tilde{M}_i = M_i$. So in the following, on $\mathcal{V}_t$, we can write these random variables with or without tilde.

3. Contributions for hitting and local times

3.1. *Negligible parts for hitting times.*

In the following lemma we recall results of Andreoletti and Devulder (2015) which say, roughly speaking, that the time spent by the diffusion X outside the h_t-valleys is negligible compared to the amount of time spent by X inside the h_t-valleys. This lemma also gives an upper bound for the number of h_t-valleys visited before time t. Finally, it tells us that with large probability, up to time t, after first hitting the bottom $\tilde{m}_j$ of each h_t-valley $[\tilde{L}_j^-, \tilde{L}_j]$, X leaves this h_t-valley on its right, that is on $\tilde{L}_j$, and that X never backtracks in a previously visited h_t-valley. We define $H_{x\to y} := \inf\{s > H(x),\ X(s) = y\} - H(x)$ for any $x \geq 0$ and $y \geq 0$, which is equal to $H(y) - H(x)$ if $x < y$. Let

$$U_0 := 0, \qquad U_i := H(\tilde{L}_i) - H(\tilde{m}_i) = H_{\tilde{m}_i \to \tilde{L}_i}, \quad i \geq 1,$$

$$\mathcal{B}_1(m) := \bigcap_{k=1}^{m}\left\{0 \leq H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t\right\}, \qquad m \geq 1,$$

where $\tilde{v}_t := 2t/\log h_t$ and $\sum_{i=1}^{0} U_i = 0$ by convention. Finally, we introduce

$$\mathcal{B}_2(m) := \bigcap_{j=1}^{m}\left\{H_{\tilde{m}_j \to \tilde{L}_j} < H_{\tilde{m}_j \to \tilde{L}_j^-},\ H_{\tilde{L}_j \to \tilde{m}_{j+1}} < H_{\tilde{L}_j \to \tilde{L}_j^*}\right\}, \qquad m \geq 1.$$

Lemma 3.1. *For any $\delta > 0$ small enough, we have for all large t,*

$$\mathbb{P}\big[H(\tilde{m}_1) \leq \tilde{v}_t\big] \geq \mathbb{P}\big[\mathcal{B}_1(n_t)\big] \geq 1 - C_2 v_t, \tag{3.1}$$

with $v_t := n_t \cdot (\log h_t) e^{-\phi(t)} = o(1)$ *and* $C_2 > 0$. *Moreover, there exists* $C_3 > 0$ *such that for large* t,

$$\begin{aligned} \mathbb{P}\left(\mathcal{B}_2(n_t)\right) &\geq 1 - C_3 n_t e^{-\delta\kappa h_t}, \qquad (3.2)\\ \mathbb{P}(N_t < n_t) &\geq 1 - e^{-\phi(t)}. \qquad (3.3)\end{aligned}$$

Proof: The first statement is Lemma 3.7 in Andreoletti and Devulder (2015). The second one follows directly from Lemmata 3.2 and 3.3 in Andreoletti and Devulder (2015). For the proof of (3.3) see Lemma 6.1. □

3.2. *Negligible parts for local times.*
We now provide estimates for the local time of X at time t. We first prove that the local time of X outside the first n_t h_t-valleys is negligible compared to t. Second, we prove that for every $1 \leq j \leq n_t$ the local time of X inside the h_t-valley $[\tilde{L}_j^-, \tilde{L}_j]$ but outside a small neighborhood of $\tilde{m}_j$ is also negligible compared to t.

3.2.1. *Supremum of the local time outside the valleys.*
The aim of this subsection is to prove that at time t, the maximum of the local time outside the h_t-valleys is negligible compared to t. More precisely, let $f(t) := te^{[\kappa(1+3\delta)-1]\phi(t)}$ and, for $m \geq 1$,

$$\begin{aligned}
\mathcal{B}_3^1(m) &:= \left\{ \sup_{x\in[0,\tilde{m}_1]} \mathcal{L}(H(\tilde{m}_1), x) \leq f(t) \right\} \\
&\qquad \cap \bigcap_{j=1}^{m-1} \left\{ \sup_{x\in[\tilde{L}_j, \tilde{m}_{j+1}]} \mathcal{L}(H(\tilde{m}_{j+1}), x) \leq f(t) \right\}, \\
\mathcal{B}_3^2(m) &:= \bigcap_{j=1}^{m-1} \left\{ \sup\nolimits_{x\leq \tilde{L}_j} \Big(\mathcal{L}(H(\tilde{m}_{j+1}), x) - \mathcal{L}\big(H\big(\tilde{L}j\big), x\big) \Big) \leq f(t) \right\}, \\
\mathcal{B}_3(m) &:= \mathcal{B}_3^1(m) \cap \mathcal{B}_3^2(m).
\end{aligned}$$

This section is devoted to the proof of the following lemma.

Lemma 3.2. *Assume that* δ *is small enough such that* $\kappa(1+3\delta) < 1$. *There exists* $C_5 > 0$ *such that for any large* t

$$\mathbb{P}\left(\mathcal{B}_3(n_t)\right) \geq 1 - C_5 w_t,$$

with $w_t := e^{-\kappa\delta\phi(t)}$.

Its proof is based on Lemma 3.3 below, for which we introduce the following notation, depending only on the potential W_κ:

$$\begin{aligned}
\tau_1^*(h) &:= \inf\{u \geq 0,\ W_\kappa(u) - \inf\nolimits_{[0,u]} W_\kappa \geq h\}, \qquad h > 0,\\
m_1^*(h) &:= \inf\{y \geq 0,\ W_\kappa(y) = \inf\nolimits_{[0,\tau_1^*(h)]} W_\kappa\}, \qquad h > 0.
\end{aligned}$$

Throughout the paper, C_+ (resp. c_-) denotes a positive constant that may grow (resp. decrease) from line to line.

Lemma 3.3. *Assume that* $\kappa(1+3\delta) < 1$. *For large* t,

$$\mathbb{P}\left(\sup\nolimits_{x\in[0,m_1^*(h_t)]} \mathcal{L}[H(\tau_1^*(h_t)), x] > te^{[\kappa(1+3\delta)-1]\phi(t)}\right) \leq \frac{C_+}{n_t e^{\kappa\delta\phi(t)}}. \qquad (3.4)$$

Proof of Lemma 3.3: Thanks to (1.12) and (1.13) there exists a Brownian motion $(B(s),\ s\geq 0)$, independent of W_κ, such that

$$\mathcal{L}[H(\tau_1^*(h_t)),x]=e^{-W_\kappa(x)}\mathcal{L}_B[\tau^B(A(\tau_1^*(h_t))),A(x)],\qquad x\in\mathbb{R}. \tag{3.5}$$

By the first Ray–Knight theorem (see e.g. Revuz and Yor, 1999, chap. XI), for every $\alpha>0$, there exists a Bessel processes Q_2 of dimension 2 starting from 0, such that $\mathcal{L}_B(\tau^B(\alpha),x)$ is equal to $Q_2^2(\alpha-x)$ for every $x\in[0,\alpha]$. Consequently, using (3.5) and the independence of B and W_κ, there exists a 2-dimensional Bessel process Q_2 such that

$$\mathcal{L}[H(\tau_1^*(h_t)),x]=e^{-W_\kappa(x)}Q_2^2\big[A(\tau_1^*(h_t))-A(x)\big]\qquad 0\leq x\leq \tau_1^*(h_t). \tag{3.6}$$

In order to evaluate this quantity, the idea is to say that loosely speaking, Q_2^2 grows almost linearly. More formally, we consider the functions $k(t):=e^{2\kappa^{-1}\phi(t)}$, $a(t):=4\phi(t)$ and $b(t):=6\kappa^{-1}\phi(t)e^{\kappa h_t}$, and define the following events

$$\begin{aligned}
\mathcal{A}_0 &:= \left\{A_\infty:=\int_0^{+\infty}e^{W_\kappa(u)}\mathrm{d}u\leq k(t)\right\},\\
\mathcal{A}_1 &:= \big\{\forall u\in(0,k(t)],\ Q_2^2(u)\leq 2eu\big[a(t)+4\log\log[ek(t)/u]\big]\big\},\\
\mathcal{A}_2 &:= \big\{\inf\nolimits_{[0,\tau_1^*(h_t)]}W_\kappa\geq -b(t)\big\}.
\end{aligned}$$

We know that $P(A_\infty\geq y)\leq C_+y^{-\kappa}$ for $y>0$ since $2/A_\infty$ is a gamma variable of parameter $(\kappa,1)$ (see Dufresne, 2000, or Borodin and Salminen, 2002 IV.48 p. 78), having a density equal to $e^{-x}x^{\kappa-1}\mathbb{1}_{\mathbb{R}_+}(x)/\Gamma(\kappa)$, so $P\big(\overline{\mathcal{A}_0}\big)\leq C_+k(t)^{-\kappa}=C_+e^{-2\phi(t)}$. Moreover, $\mathbb{P}\big(\overline{\mathcal{A}_1}\big)\leq C_+\exp[-a(t)/2]=C_+e^{-2\phi(t)}$ by Lemma 7.5. Also we know that $-\inf_{[0,\tau_1^*(h)]}W_\kappa$, denoted by $-\beta$ in Faggionato (2009, eq. (2.2)), is exponentially distributed with mean $2\kappa^{-1}\sinh(\kappa h/2)e^{\kappa h/2}$ (Faggionato, 2009, eq. (2.4)). So for large t,

$$\begin{aligned}
P\big(\overline{\mathcal{A}_2}\big) &= P[-\inf\nolimits_{[0,\tau_1^*(h_t)]}W_\kappa>b(t)]\\
&= \exp\big[-b(t)\kappa/(2\sinh(\kappa h_t/2)e^{\kappa h_t/2})\big]\\
&\leq e^{-2\phi(t)}.
\end{aligned}$$

Now, assume we are on $\mathcal{A}_0\cap\mathcal{A}_1\cap\mathcal{A}_2$. Due to (3.6), we have for every $0\leq x<\tau_1^*(h_t)$, since $0<A(\tau_1^*(h_t))-A(x)\leq A_\infty\leq k(t)$,

$$\begin{aligned}
&\mathcal{L}[H(\tau_1^*(h_t)),x]\\
&\leq e^{-W_\kappa(x)}2e[A(\tau_1^*(h_t))-A(x)]\big\{a(t)+4\log\log\big[ek(t)/[A(\tau_1^*(h_t))-A(x)]\big]\big\}.
\end{aligned} \tag{3.7}$$

We now introduce

$$f_i:=\inf\{u\geq 0,\ W_\kappa(u)\leq -i\}=\tau^{W_\kappa}(-i),\qquad i\in\mathbb{N},$$

and let $0\leq x<\tau_1^*(h_t)$. There exists $i\in\mathbb{N}$ such that $f_i\leq x<f_{i+1}$. Moreover, we are on $\mathcal{A}_2$, so $i\leq b(t)$. Furthermore, $x<f_{i+1}$, so $W_\kappa(x)\geq -(i+1)$ and then $e^{-W_\kappa(x)}\leq e^{i+1}=e^{-W_\kappa(f_i)+1}$. All this leads to

$$\begin{aligned}
e^{-W_\kappa(x)}[A(\tau_1^*(h_t))-A(x)] &= e^{-W_\kappa(x)}\int_x^{\tau_1^*(h_t)}e^{W_\kappa(u)}\mathrm{d}u\\
&\leq e\int_{f_i}^{\tau_1^*(h_t)}e^{W_\kappa(u)-W_\kappa(f_i)}\mathrm{d}u.
\end{aligned} \tag{3.8}$$

To bound this, we introduce the event

$$\mathcal{A}_3 := \bigcap_{i=0}^{\lfloor b(t)\rfloor} \left\{ \int_{f_i}^{\tau_1^*(h_t)} e^{W_\kappa(u)-W_\kappa(f_i)}\mathrm{d}u \leq e^{(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)} \right\}.$$

We now consider $\tau_1^*(u,h_t) := \inf\{y \geq u,\ W_\kappa(y) - \inf_{[u,y]} W_\kappa \geq h_t\} \geq \tau_1^*(h_t)$ for $u \geq 0$. We have

$$E\bigg(\int_{f_i}^{\tau_1^*(h_t)} e^{W_\kappa(u)-W_\kappa(f_i)}\mathrm{d}u\bigg) \leq E\bigg(\int_{f_i}^{\tau_1^*(f_i,h_t)} e^{W_\kappa(u)-W_\kappa(f_i)}\mathrm{d}u\bigg) = \beta_0(h_t),$$

by the strong Markov property applied at stopping time f_i, where we define $\beta_0(h) := E\left(\int_0^{\tau_1^*(h)} e^{W_\kappa(u)}\mathrm{d}u\right)$. By (6.15), $\beta_0(h) \leq C_+ e^{(1-\kappa)h}$ for large h. Hence for large t by Markov inequality,

$$\begin{aligned} P(\overline{\mathcal{A}_3}) &\leq \sum_{i=0}^{\lfloor b(t)\rfloor} P\bigg(\int_{f_i}^{\tau_1^*(h_t)} e^{W_\kappa(u)-W_\kappa(f_i)}\mathrm{d}u > e^{(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)}\bigg) \\ &\leq \frac{[b(t)+1]\beta_0(h_t)}{e^{(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)}} \leq \frac{C_+}{n_t e^{\kappa\delta\phi(t)}}. \end{aligned}$$

Now, on $\cap_{j=0}^3 \mathcal{A}_j$, (3.7) and (3.8) lead to

$$\begin{aligned} &\mathcal{L}[H(\tau_1^*(h_t)), x] \\ \leq\ & 2e^{2+(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)} \big\{a(t) + 4\log\log\big[ek(t)/[A(\tau_1^*(h_t)) - A(x)]\big]\big\}. \end{aligned} \tag{3.9}$$

We now consider only $0 \leq x \leq m_1^*(h_t)$. By definition of $\mathcal{A}_2$, $\inf_{[0,\tau_1^*(h_t)]} W_\kappa \geq -b(t)$, such that

$$\begin{aligned} A(\tau_1^*(h_t)) - A(x) &= \int_x^{\tau_1^*(h_t)} e^{W_\kappa(u)}\mathrm{d}u \\ &\geq \int_{m_1^*(h_t)}^{\tau_1^*(h_t)} e^{W_\kappa(u)}\mathrm{d}u \\ &\geq e^{-b(t)}[\tau_1^*(h_t) - m_1^*(h_t)] \\ &\geq e^{-b(t)} \end{aligned}$$

on the event $\cap_{i=0}^4 \mathcal{A}_i$ with $\mathcal{A}_4 := \{\tau_1^*(h_t) - m_1^*(h_t) \geq 1\}$. Since $m_1 = m_1^*(h_t)$ and $\tau_1(h_t) = \tau_1^*(h_t)$ on $\{M_0 \leq 0\}$ by definition of h_t-extrema, we have

$$\begin{aligned} P(\overline{\mathcal{A}_4}) &\leq P(0 < M_0 < m_1) + P[\tau_1(h_t) - m_1 < 1] \\ &\leq C_+ h_t e^{-\kappa h_t} + P\big[\tau^{W_\kappa^\uparrow}(h_t) - \tau^{W_\kappa^\uparrow}(h_t/2) < 1\big] \\ &\leq C_+ h_t e^{-\kappa h_t} + C_+ \exp[-(c_-)h_t^2] \end{aligned}$$

due to Andreoletti and Devulder (2015, eq. (2.8)), coming from Faggionato (2009), Fact 2.1 **(ii)** and (7.4).

Now, we have $ek(t)/[A(\tau_1^*(h_t)) - A(x)] \leq ek(t)e^{b(t)}$ on $\cap_{i=0}^4 \mathcal{A}_i$, and then, on this event, (3.9) leads to

$$\begin{aligned} \mathcal{L}[H(\tau_1^*(h_t)), x] &\leq 2e^{2+(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)} \big\{a(t) + 4\log\log\big[ek(t)e^{b(t)}\big]\big\}. \\ &\leq C_+ t\phi(t) e^{[\kappa(1+\delta)-1]\phi(t)} e^{\kappa\delta\phi(t)} h_t, \end{aligned}$$

since $\phi(t) = o(\log t)$, $h_t = \log t - \phi(t)$ and $n_t = \lfloor e^{\kappa(1+\delta)\phi(t)} \rfloor$. We notice that for large t, $C_+\phi(t)h_t \leq e^{\kappa\delta\phi(t)}$ since $\log\log t = o(\phi(t))$. Hence, for large t,

$$\mathcal{L}[H(\tau_1^*(h_t)), x] \leq te^{[\kappa(1+3\delta)-1]\phi(t)},$$

on $\cap_{i=0}^4 \mathcal{A}_i$ for every $0 \leq x \leq m_1^*(h_t)$. This gives for large t,

$$\mathbb{P}\left(\sup_{x\in[0,m_1^*(h_t)]} \mathcal{L}[H(\tau_1^*(h_t)), x] \leq te^{[\kappa(1+3\delta)-1]\phi(t)}\right) \geq \mathbb{P}\left(\cap_{i=0}^4 \mathcal{A}_i\right) \geq 1 - \frac{C_+}{n_t e^{\kappa\delta\phi(t)}},$$

due to the previous bounds for $\mathbb{P}(\mathcal{A}_i)$, $0 \leq i \leq 4$. This proves the lemma. $\square$

With the help of the previous lemma, we can now prove Lemma 3.2.

Proof of Lemma 3.2: The method is to do a coupling, similarly as in the proof of Lemma 3.7 of Andreoletti and Devulder (2015). Recall the definition of $\tilde{L}_i^* < \tilde{L}_i < \tilde{L}_{i+1}^\sharp$ just above (2.5). Also, let

$$\begin{aligned}
\tilde{\tau}_{i+1}^*(h_t) &:= \inf\left\{u \geq \tilde{L}_i^*,\ W_\kappa(u) - \inf_{[\tilde{L}_i^*,u]} W_\kappa \geq h_t\right\} \leq \tilde{\tau}_{i+1}(h_t), \qquad i \geq 1,\\
\tilde{m}_{i+1}^*(h_t) &:= \inf\left\{u \geq \tilde{L}_i^*,\ W_\kappa(u) = \inf_{[\tilde{L}_i^*,\tilde{\tau}_{i+1}^*(h_t)]} W_\kappa\right\}, \qquad i \geq 1,\\
\mathcal{A}_5 &:= \cap_{i=1}^{n_t-1}\left\{\tilde{\tau}_{i+1}^*(h_t) = \tilde{\tau}_{i+1}(h_t)\right\},\\
X_i(u) &:= X\left(u + H\left(\tilde{L}_i\right)\right), \qquad X_i^*(u) := X\left(u + H\left(\tilde{L}_i^*\right)\right), \qquad u \geq 0,\ i \geq 1.
\end{aligned} \tag{3.10}$$

Let $i \geq 1$. By the strong Markov property, X_i and X_i^* are diffusions in the potential W_κ, starting respectively from $\tilde{L}_i$ and $\tilde{L}_i^*$. We denote respectively by $\mathcal{L}_{X_i}$, $\mathcal{L}_{X_i^*}$, H_{X_i} and $H_{X_i^*}$ the local times and hitting times of X_i and X_i^*. We have for every $x \geq \tilde{L}_i^*$,

$$\begin{aligned}
\mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i), x) &\leq \mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i^*), x)\\
&= \mathcal{L}_{X_i^*}\left(H_{X_i^*}(\tilde{m}_{i+1}), x\right).
\end{aligned}$$

Consequently, on $\mathcal{A}_5 \cap \mathcal{A}_6$ with $\mathcal{A}_6 := \cap_{j=1}^{n_t-1}\left\{H_{X_j}(\tilde{m}_{j+1}) < H_{X_j}\left(\tilde{L}_j^*\right)\right\}$, for $1 \leq i \leq n_t - 1$,

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left(\mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}\left(H\left(\tilde{L}_i\right), x\right)\right)\\
&= \sup_{\tilde{L}_i^* \leq x \leq \tilde{m}_{i+1}} \left(\mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}\left(H\left(\tilde{L}_i\right), x\right)\right)\\
&\leq \sup_{\tilde{L}_i^* \leq x \leq \tilde{m}_{i+1}} \mathcal{L}_{X_i^*}\left(H_{X_i^*}(\tilde{m}_{i+1}), x\right)\\
&\leq \sup_{\tilde{L}_i^* \leq x \leq \tilde{m}_{i+1}^*} \mathcal{L}_{X_i^*}\left(H_{X_i^*}(\tilde{\tau}_{i+1}^*(h_t)), x\right),
\end{aligned} \tag{3.11}$$

since $\tilde{m}_{i+1}^* = \tilde{m}_{i+1} \leq \tilde{\tau}_{i+1}(h_t) = \tilde{\tau}_{i+1}^*(h_t)$ on $\mathcal{A}_5$. Now, notice that the right hand side of (3.11) is the supremum of the local times of $X_i^* - \tilde{L}_i^*$, up to its first hitting time of $\tilde{\tau}_{i+1}^*(h_t) - \tilde{L}_i^*$, over all locations in $[0, \tilde{m}_{i+1}^* - \tilde{L}_i^*]$. Since $X_i^* - \tilde{L}_i^*$ is a diffusion in the potential $\left(W_\kappa(\tilde{L}_i^* + x) - W_\kappa(\tilde{L}_i^*),\ x \in \mathbb{R}\right)$, which has on $[0, +\infty)$ the same law as $(W_\kappa(x),\ x \geq 0)$ because $\tilde{L}_i^*$ is a stopping time for W_κ, the right hand side of (3.11) has the same law, under the annealed probability $\mathbb{P}$, as $\sup_{x\in[0,m_1^*(h_t)]}\mathcal{L}[H(\tau_1^*(h_t)),x]$. Consequently,

$$\begin{aligned}
&\mathbb{P}\Big(\bigcup_{i=1}^{n_t-1}\Big\{\sup_{x\in\mathbb{R}}\Big(\mathcal{L}(H(\tilde{m}_{i+1}),x)-\mathcal{L}(H(\tilde{L}_i),x)\Big)>te^{[\kappa(1+3\delta)-1]\phi(t)}\Big\}\Big)\\
\leq\quad & n_t\Big[\mathbb{P}\Big(\sup_{x\in[0,m_1^*(h_t)]}\mathcal{L}\big[H(\tau_1^*(h_t)),x\big]>te^{[\kappa(1+3\delta)-1]\phi(t)}\Big)+\mathbb{P}\big(\overline{\mathcal{A}_5}\big)+\mathbb{P}\big(\overline{\mathcal{A}_6}\big)\Big]\\
\leq\quad & C_+e^{-\kappa\delta\phi(t)} \qquad (3.12)
\end{aligned}$$

by Lemma 3.3, since $\mathbb{P}\big(\overline{\mathcal{A}_5}\big)\leq C_+n_th_te^{-\kappa h_t}$ by (6.9), $\mathbb{P}\big(\overline{\mathcal{A}_6}\big)\leq\mathbb{P}\big(\overline{\mathcal{B}_2}(n_t)\big)\leq C_3n_te^{-\delta\kappa h_t}$ by (3.2) and since $\phi(t)=o(\log t)$. Notice that, as before, $\tilde{m}_1=m_1=m_1^*(h_t)$ on $\mathcal{V}_t\cap\{M_0\leq 0\}$. Finally,

$$\begin{aligned}
\mathbb{P}\Big(\sup_{x\in[0,\tilde{m}_1]}\mathcal{L}(H(\tilde{m}_1),x)>te^{[\kappa(1+3\delta)-1]\phi(t)}\Big)&\leq\frac{C_+}{e^{\kappa\delta\phi(t)}}+P\big(\overline{\mathcal{V}_t}\big)+P(0<M_0<m_1)\\
&\leq\frac{C_+}{e^{\kappa\delta\phi(t)}}
\end{aligned}$$

also by Lemma 3.3, Lemma 2.2, and since $P(0<M_0<m_1)\leq C_+h_te^{-\kappa h_t}$ due to (6.8). This and (3.12) prove the lemma. $\square$

3.2.2. *Local time inside the valley* $\big[\tilde{L}_j^-,\tilde{L}_j\big]$ *but far from* $\tilde{m}_j$.
We introduce for $t>0$ and $j\geq 1$,

$$r_t:=C_0\phi(t),\qquad \mathcal{D}_j:=[\tilde{m}_j-r_t,\tilde{m}_j+r_t], \tag{3.13}$$

where $C_0>0$ is a constant that can be chosen as large as needed. We also define

$$\mathcal{B}_4(m):=\bigcap_{j=1}^{m}\Big\{\sup_{x\in\overline{\mathcal{D}_j}\cap[\tilde{\tau}_j^-(h_t^+),\tilde{L}_j]}\Big(\mathcal{L}\big(H(\tilde{L}_j),x\big)-\mathcal{L}\big(H(\tilde{m}_j),x\big)\Big)<te^{-2\phi(t)}\Big\}$$

for $m\geq 1$, where $\overline{\mathcal{D}_j}$ is the complementary of $\mathcal{D}_j$. Moreover, we recall that $\tilde{L}_j^-=\tilde{\tau}_j^-(h_t^+)$.

Lemma 3.4. *There exists $C_6>0$ such that if C_0 is large enough, for large t,*

$$\mathbb{P}\big[\mathcal{B}_4(n_t)\big]\geq 1-C_6n_te^{-2\phi(t)}.$$

Proof: Let $j\in[1,n_t]$. Throughout the rest of the paper, for $y\in\mathbb{R}$, we denote by $\mathbb{P}_y^{W_\kappa}$ the law of X starting from y instead of 0, conditionally on W_κ. As we are interested in the local time at x after X reaches $\tilde{m}_j$ we work under $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$. So first, thanks to (1.12) and (1.13), under $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$, there exists a Brownian motion $(B(s),\ s\geq 0)$, independent of $\tilde{V}^{(j)}$, such that

$$\mathcal{L}\big[H(\tilde{L}_j),x\big]=e^{-\tilde{V}^{(j)}(x)}\mathcal{L}_B[\tau^B(A^j(\tilde{L}_j)),A^j(x)],\qquad x\in\mathbb{R},$$

where $A^j(x):=\int_{\tilde{m}_j}^x e^{\tilde{V}^{(j)}(s)}\mathrm{d}s$. Let $\tilde{B}^j(.):=B^j((A^j(\tilde{L}_j))^2.)/A^j(\tilde{L}_j)$. By scaling, and because B is independent from W_κ, we notice that conditionally to W_κ, $\tilde{B}^j$ is a standard Brownian motion. Therefore, even if W_κ appears in the expression of $\tilde{B}^j$, $\tilde{B}^j$ is (probabilistically) independent of W_κ. We still denote it by B in the sequel to simplify the notation. With this notation, we have

$$\mathcal{L}\big[H(\tilde{L}_j),x\big]=e^{-\tilde{V}^{(j)}(x)}A^j(\tilde{L}_j)\mathcal{L}_B[\tau^B(1),A^j(x)/A^j(\tilde{L}_j)],\qquad x\in\mathbb{R}. \tag{3.14}$$

In order to bound the factors $\mathcal{L}_B\big[\tau^B(1),.\big]$ and $A^j\big(\tilde{L}_j\big)$ in (3.14), we first introduce

$$\mathcal{A}_1 := \big\{\sup_{u\in\mathbb{R}}\mathcal{L}_B[\tau^B(1),u] \le e^{2\phi(t)}\big\}, \qquad \mathcal{A}_2 := \big\{A^j\big(\tilde{L}_j\big) \le 2e^{h_t+2\phi(t)/\kappa}\big\}. \tag{3.15}$$

We have $\mathbb{P}\big(\overline{\mathcal{A}_1}\big) \le 5e^{-2\phi(t)}$ for large t by Lemma 7.4 eq. (7.12) and (7.13). Moreover on $\mathcal{V}_t$, we have by Remark 2.3 and Fact 2.1 **(ii)** and **(iii)**,

$$\begin{aligned} A^j\big(\tilde{L}_j\big) &\le \big[\tilde{\tau}_j(h_t)-\tilde{m}_j\big]e^{h_t} + \int_{\tilde{\tau}_j(h_t)}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(s)}\mathrm{d}s \\ &= \big[\tau_j(h_t)-m_j\big]e^{h_t} + \int_{\tau_j(h_t)}^{L_j} e^{V^{(j)}(s)}\mathrm{d}s \\ &\overset{\mathcal{L}}{=} e^{h_t}\tau^{W_\kappa^\uparrow}(h_t) + G^+(h_t/2,h_t), \end{aligned}$$

where $W_\kappa^\uparrow$ has law $BES(3,\kappa/2)$ and is independent of $G^+(h_t/2,h_t)$, which is defined in (1.7), and with $\tilde{L}_j = \inf\{s>\tilde{\tau}_j(h_t), \tilde{V}^{(j)}(s) = h_t/2\}$ as defined in (2.4), and $L_j := \inf\{s>\tau_j(h_t), V^{(j)}(s)=h_t/2\}$. Consequently,

$$\begin{aligned} P\big(\overline{\mathcal{A}_2}\big) &\le P\big(\tau^{W_\kappa^\uparrow}(h_t) > e^{2\phi(t)/\kappa}\big) + P\big(G^+(h_t/2,h_t) > e^{h_t+2\phi(t)/\kappa}\big) + P\big(\overline{\mathcal{V}_t}\big) \\ &\le C_+e^{-2\phi(t)} \end{aligned}$$

for large t by Lemma 7.2 eq. (7.5), Lemma 7.3 eq. (7.10) and Lemma 2.2, and since $\phi(t) = o(\log t)$ and $\log\log t = o(\phi(t))$.

Now, we would like to bound the factor $e^{-\tilde{V}^{(j)}(x)}$ that appears in (3.14). To this aim, let

$$\begin{aligned} \mathcal{A}_3 &:= \big\{\tilde{\tau}_j[\kappa C_0\phi(t)/8] \le \tilde{m}_j + C_0\phi(t)\big\}, \\ \mathcal{A}_4 &:= \Big\{\inf_{[\tau_j[\kappa C_0\phi(t)/8],\tau_j(h_t)]} V^{(j)} \ge \kappa C_0\phi(t)/16\Big\}, \end{aligned}$$

with $\tilde{\tau}_j$ and $\tilde{\tau}_j^-$ defined in (2.5) and (2.6), and τ_j and τ_j^- in (2.1) and (2.2). First, using (6.12), $P\big(\overline{\mathcal{A}_3}\big) \le C_+e^{-[\kappa^2C_0\phi(t)]/(16\sqrt{2})} \le e^{-2\phi(t)}$ if we choose C_0 large enough. Moreover Fact 2.1 together with (7.3) (applied with $h = C_0\phi(t)$, $\alpha=\kappa/8$, $\gamma=\kappa/16$ and $\omega = h_t/(C_0\phi(t))$, see also the remark at the end of Lemma 7.2) give $P\big(\overline{\mathcal{A}_4}\big) \le 2e^{-\kappa^2C_0\phi(t)/16} \le e^{-2\phi(t)}$ for large t.
We notice that $\inf_{[\tilde{m}_j+C_0\phi(t),\tilde{\tau}_j(h_t)]}\tilde{V}^{(j)} \ge \kappa C_0\phi(t)/16$ on $\mathcal{A}_3\cap\mathcal{A}_4\cap\mathcal{V}_t$, since $\tau_j = \tilde{\tau}_j$ and $V^{(j)} = \tilde{V}^{(j)}$ on $\mathcal{V}_t$ thanks to Remark 2.3. We prove similarly that

$$P\big(\overline{\mathcal{A}_5}\big) \le C_+e^{-\kappa^2C_0\phi(t)/(16\sqrt{2})} + P\big(\overline{\mathcal{V}_t}\big) \le 2e^{-2\phi(t)},$$

where

$$\begin{aligned} \mathcal{A}_5 &:= \Big\{\inf_{[\tilde{\tau}_j^-(h_t),\tilde{m}_j-C_0\phi(t)]}\tilde{V}^{(j)} \ge \kappa C_0\phi(t)/16\Big\}, \\ \mathcal{A}_6 &:= \Big\{\inf_{[\tilde{\tau}_j^-(h_t^+),\tilde{\tau}_j^-(h_t)]}\tilde{V}^{(j)} \ge h_t/2\Big\}. \end{aligned}$$

Also by (6.10), $P\big(\overline{\mathcal{A}_6}\big) \le e^{-\kappa h_t/8}$. We also know that $\tilde{V}^{(j)}(x) \ge h_t/2 \ge \kappa C_0\phi(t)/16$ for all $\tilde{\tau}_j(h_t) \le x \le \tilde{L}_j$ by definition of $\tilde{L}_j$, uniformly for large t. Consequently on $\cap_{i=3}^6\mathcal{A}_i\cap\mathcal{V}_t$, for all $x\in\overline{\mathcal{D}}_j\cap[\tilde{\tau}_j^-(h_t^+),\tilde{L}_j]$, we have $e^{-\tilde{V}^{(j)}(x)} \le e^{-\kappa C_0\phi(t)/16}$.

Hence on $\cap_{i=1}^6 \mathcal{A}_i \cap \mathcal{V}_t$, we have under $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$, by (3.14) and (3.15),

$$\sup_{x \in \overline{\mathcal{D}_j} \cap [\tilde{\tau}_j^-(h_t^+), \tilde{L}_j]} \mathcal{L}\big[H(\tilde{L}_j), x\big] \leq 2te^{(1+2/\kappa)\phi(t)} e^{-\kappa C_0 \phi(t)/16} < te^{-2\phi(t)},$$

if we choose C_0 large enough. So, conditioning by W_κ and applying the strong Markov property at time $H(\tilde{m}_j)$, we get

$$\begin{aligned} &\mathbb{P}\Bigg[\sup_{x \in \overline{\mathcal{D}_j} \cap [\tilde{\tau}_j^-(h_t^+), \tilde{L}_j]} \big(\mathcal{L}\big[H(\tilde{L}_j), x\big] - \mathcal{L}\big[H(\tilde{m}_j), x\big]\big) < te^{-2\phi(t)}\Bigg] \\ \geq\ & \mathbb{E}\big(\mathbb{P}_{\tilde{m}_j}^{W_\kappa}\big(\cap_{i=1}^6 \mathcal{A}_i \cap \mathcal{V}_t\big)\big) \geq 1 - C_+ e^{-2\phi(t)} \end{aligned}$$

uniformly for large t due to the previous estimates and thanks to Lemma 2.2. This proves the lemma. □

3.3. *Approximation of the main contributions.*

In this section we give an approximation of the exit time of each h_t-valley $[\tilde{L}_j^-, \tilde{L}_j]$ and of the local time at the bottom $\tilde{m}_j$ of this h_t-valley for every $1 \leq j \leq n_t$. More precisely, we make a link between the family $\big((U_j := H(\tilde{L}_j) - H(\tilde{m}_j), \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j)), 1 \leq j \leq n_t\big)$, and the i.i.d. sequence $\big((\mathcal{H}_j, \ell_j), 1 \leq j \leq n_t\big)$ described in the introduction.

In the following, $F_1^+(h_t)$, $G^+(h_t/2, h_t)$, $F_2^-(h_t/2)$ and $F_3^-(h_t/2)$ denote independent r.v. with law respectively $F^+(h_t)$, $G^+(h_t/2, h_t)$, $F^-(h_t/2)$ and $F^-(h_t/2)$, defined in (1.6) and (1.7).

Proposition 3.5. *For $\delta > 0$ small enough (recall that δ appears in the definitions of n_t and h_t^+), there exist $d_1 = d_1(\delta, \kappa) > 0$ and $D_1(d_1) > 0$ such that for large t, possibly on an enlarged probability space, there exists a sequence $((S_j, R_j, \mathbf{e}_j),\ 1 \leq j \leq n_t)$ of i.i.d. random variables depending on t, with S_j, R_j and $\mathbf{e}_j$ independent for every j and $S_j \overset{\mathcal{L}}{=} F_1^+(h_t) + G^+(h_t/2, h_t)$, $R_j \overset{\mathcal{L}}{=} F_2^-(h_t/2) + F_3^-(h_t/2)$ and $\mathbf{e}_j \overset{\mathcal{L}}{=} \mathcal{E}(1/2)$ (exponential variable with mean 2) such that*

$$\mathbb{P}\left(\cap_{j=1}^{n_t}\Big\{\big|U_j - \mathcal{H}_j\big| \leq \varepsilon_t \mathcal{H}_j,\ \big|\mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) - \ell_j\big| \leq \varepsilon_t \ell_j\Big\}\right) \geq 1 - e^{-D_1 h_t}, \quad (3.16)$$

where $\ell_j := S_j \mathbf{e}_j$, $\mathcal{H}_j := R_j \ell_j$ and $\varepsilon_t := e^{-d_1 h_t}$.

The proof of the above proposition, which is in the spirit of the proofs of Propositions 3.4 and 4.4 in Andreoletti and Devulder (2015), makes use of the following lemma:

Lemma 3.6. *For $\delta > 0$ small enough, there exist constants $d_- > 0$ and $D_- > 0$, possibly depending on κ and δ, such that the two following statements are true for $t > 0$ large enough.*
(i) *There exists a sequence $(\mathbf{e}_j, 1 \leq j \leq n_t)$ of i.i.d. random variables with exponential law of mean 2 and independent of W_κ, such that*

$$\mathbb{P}\left(\bigcap_{j=1}^{n_t}\Big\{|U_j - \tilde{\mathbb{H}}_j| \leq e^{-(d_-)h_t}\tilde{\mathbb{H}}_j,\ \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) = \mathbb{L}_j\Big\}\right) \geq 1 - e^{-(D_-)h_t}, \quad (3.17)$$

where $\mathbb{L}_j := \mathbf{e}_j \int_{\tilde{m}_j}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(x)} \mathrm{d}x$, $\tilde{R}_j := \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j(h_t/2)} e^{-\tilde{V}^{(j)}(x)} \mathrm{d}x$ and $\tilde{\mathbb{H}}_j := \mathbb{L}_j \tilde{R}_j$ for all $1 \leq j \leq n_t$. Moreover the random variables $(\mathbb{L}_j, \tilde{\mathbb{H}}_j)$, $1 \leq j \leq n_t$, are i.i.d.

(ii) *Possibly on an enlarged probability space, there exist random variables R_j and S_j, $1 \le j \le n_t$, such that all the random variables R_j, S_j, $\mathbf{e}_j$, $1 \le j \le n_t$ are independent, with $S_j \overset{\mathcal{L}}{=} F_1^+(h_t) + G^+(h_t/2, h_t)$, and $R_j \overset{\mathcal{L}}{=} F_2^-(h_t/2) + F_3^-(h_t/2)$ for every $1 \le j \le n_t$, such that*

$$P\left(\bigcap_{j=1}^{n_t} \left\{ \left| \int_{\tilde{m}_j}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(x)} \mathrm{d}x - S_j \right| \le e^{-(d_-)h_t} S_j,\ \tilde{R}_j = R_j \right\} \right) \ge 1 - e^{-(D_-)h_t}, \quad (3.18)$$

Proof of Lemma 3.6: We start with **(i)**. Recall that $\tilde{m}_j < \tilde{L}_j < \tilde{m}_{j+1}$ for every $j \ge 1$, e.g. by (2.7). By the strong Markov property applied under $\mathbb{P}^{W_\kappa}$ at stopping times $H(\tilde{m}_j)$, the random variables $\big(U_j, \mathcal{L}[H(\tilde{L}_j), \tilde{m}_j]\big)$, $1 \le j \le n_t$, are independent under $\mathbb{P}^{W_\kappa}$. By the same Markov property and formulas (1.12) and (1.13), the sequence $(U_j, \mathcal{L}[H(\tilde{L}_j), \tilde{m}_j]$, $1 \le j \le n_t)$ is equal to the sequence $(H_j(\tilde{L}_j)$, $\mathcal{L}_j[H_j(\tilde{L}_j), \tilde{m}_j]$, $1 \le j \le n_t)$, where

$$\begin{aligned} H_j(\tilde{L}_j) &:= \int_{-\infty}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B^j}\big[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)\big] \mathrm{d}u, \\ \mathcal{L}_j[H_j(\tilde{L}_j), \tilde{m}_j] &= \mathcal{L}_{B^j}\big[\tau^{B^j}(A^j(\tilde{L}_j)), 0\big], \qquad A^j(u) := \int_{\tilde{m}_j}^{u} e^{\tilde{V}^{(j)}(x)} \mathrm{d}x, \quad u \in \mathbb{R}, \end{aligned} \quad (3.19)$$

with $(B^j, 1 \le j \le n_t)$ a sequence of independent standard Brownian motions independent of W_κ, such that B^j starts at $A^j(\tilde{m}_j) = 0$ and is killed when it first hits $A^j(\tilde{L}_j)$. Recall that $\mathcal{L}_{B^j}$ denotes the local time of B^j. Define $\mathcal{A}_j := \big\{ \max_{u < \tilde{L}_j^-} \mathcal{L}_{B^j}\big[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)\big] = 0 \big\}$, $1 \le j \le n_t$. By (6.6), there exists $c_- > 0$ (possibly depending on κ and δ) such that $\mathbb{P}\big(\cap_{j=1}^{n_t} \mathcal{A}_j \big) \ge 1 - e^{-(c_-)h_t}$ for large t. So for large t,

$$\mathbb{P}\left(\bigcap_{j=1}^{n_t} \left\{ H_j(\tilde{L}_j) = \tilde{h}_j \right\} \right) \ge 1 - e^{-(c_-)h_t}, \quad (3.20)$$

where

$$\tilde{h}_j := \int_{\tilde{L}_j^-}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B^j}\big[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)\big] \mathrm{d}u, \qquad 1 \le j \le n_t.$$

We also notice that for every $1 \le j \le n_t$, $\big(\tilde{h}_j, \mathcal{L}_j[H_j(\tilde{L}_j), \tilde{m}_j]\big)$ is measurable with respect to the σ-field generated by $(\tilde{V}^{(j)}(x + \tilde{L}_{j-1}^+),\ 0 \le x < \tilde{L}_j^+ - \tilde{L}_{j-1}^+)$ and B^j, where by (2.7) and (2.8), $\tilde{L}_{j-1}^+ < \tilde{L}_j^- < \tilde{m}_j < \tilde{L}_j < \tilde{L}_j^+$. Hence, the random variables $\big(\tilde{h}_j, \mathcal{L}_j[H_j(\tilde{L}_j), \tilde{m}_j]\big)$, $1 \le j \le n_t$ are i.i.d under $\mathbb{P}$ by the second fact of Lemma 2.2. For the same reason, $\big(\tilde{R}_j, A^j(\tilde{L}_j)\big)$, $1 \le j \le n_t$ are also i.i.d.
For $1 \le j \le n_t$, let $\tilde{B}^j(.) := B^j\big((A^j(\tilde{L}_j))^2.\big)/A^j(\tilde{L}_j)$. Notice that

$$\mathcal{L}_{B^j}\big[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)\big] = A^j(\tilde{L}_j) \mathcal{L}_{\tilde{B}^j}\big[\tau^{\tilde{B}^j}(1), A^j(u)/A^j(\tilde{L}_j)\big], \qquad \tilde{L}_j^- \le u \le \tilde{L}_j. \quad (3.21)$$

Moreover by scaling and because B^j is independent from W_κ, $\tilde{B}^j$ is, conditionally to W_κ, a standard Brownian motion starting from 0 and killed when it first hits 1.

Furthermore, even if W_κ appears in the expression of $\tilde{B}^j$, $\tilde{B}^j$ is independent of W_κ. Then, let

$$\mathbf{e}_j := \mathcal{L}_{\tilde{B}^j}\big[\tau^{\tilde{B}^j}(1), 0\big] = \mathcal{L}_{B^j}\big[\tau^{B^j}\big(A^j\big(\tilde{L}_j\big)\big), 0\big]/A^j\big(\tilde{L}_j\big). \tag{3.22}$$

Notice that by the first Ray-Knight theorem, $\mathbf{e}_j$ is exponentially distributed with mean 2. Since $\tilde{B}^j$ is independent of W_κ, $\mathbf{e}_j$ is also independent of W_κ. Also, the sequence $\mathbf{e}_j$, $1 \leq j \leq n_t$ is i.i.d. because the B^j are independent and the $\big(\tilde{R}_j, A^j(\tilde{L}_j)\big)$ are i.i.d., so $(\mathbb{L}_j, \tilde{\mathbb{H}}_j)$, $1 \leq j \leq n_t$, are also i.i.d. Moreover, (3.21) leads to

$$\mathcal{L}_j\big[H_j\big(\tilde{L}_j\big), \tilde{m}_j\big] = A^j\big(\tilde{L}_j\big)\mathcal{L}_{\tilde{B}^j}\big[\tau^{\tilde{B}^j}(1), 0\big] = A^j\big(\tilde{L}_j\big)\mathbf{e}_j = \mathbb{L}_j. \tag{3.23}$$

Now, for small $\varepsilon > 0$, thanks to Lemma 6.3, we have for large t,

$$\begin{aligned}\mathbb{P}\bigg(\bigcap_{j=1}^{n_t}\Big\{\Big|\tilde{h}_j - A^j(\tilde{L}_j)\mathbf{e}_j\tilde{R}_j\Big| \leq 2e^{-(1-3\varepsilon)h_t/6}A^j(\tilde{L}_j)\mathbf{e}_j\tilde{R}_j\Big\}\bigg) \quad &\geq \quad 1 - \frac{C_+ n_t}{e^{(c_-)\varepsilon h_t}} \\ &\geq \quad 1 - \frac{C_+}{e^{(c_-/2)\varepsilon h_t}},\end{aligned}$$

since $n_t = e^{o(1)h_t}$. Finally, this, together with (3.20), (3.23) and the equality of sequences at the start of this proof show (3.17) for some $D_- > 0$ and $d_- > 0$. So **(i)** is proved.

We now prove **(ii)**. The r.v. $\tilde{A}_j(\tilde{L}_j) = \int_{\tilde{m}_j}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(x)}\mathrm{d}x$ and $\tilde{R}_j$ are not independent, so we want to replace them by r.v. having better independence properties. Applying Lemma 6.4 with subscript 2 replaced by j for $1 \leq j \leq n_t$ gives the existence of R_j and S_j, independent and independent of $\mathbf{e_j}$, having the law claimed in **(ii)** and satisfying (6.5) with 2 replaced by j. This gives (3.18) since $n_t = e^{o(1)h_t}$. The fact that we can build these R_j and S_j with the claimed independence properties follows from the fact that $\big(\mathbf{e_j}, \tilde{R}_j, \tilde{A}^j(\tilde{L}_j)\big), 1 \leq j \leq n_t$ are i.i.d. □

Proof of Proposition 3.5: The existence and the law of the $\mathbf{e_j}$ come from Lemma 3.6 **(i)**. The existence and the law of the R_j and S_j, and the independence of R_j, S_j, $\mathbf{e_j}$, $1 \leq j \leq n_t$ come from Lemma 3.6 **(ii)**. Moreover, by Lemma 3.6 **(i)** and **(ii)**, there exist $d_1 > 0$ and $D_1 > 0$ such that for large t,

$$\begin{aligned}&\mathbb{P}\left(\cap_{j=1}^{n_t}\Big\{\big|U_j - \mathbf{e}_j S_j R_j\big| \leq \varepsilon_t \mathbf{e}_j S_j R_j,\ \big|\mathcal{L}\big(H\big(\tilde{L}_j\big), \tilde{m}_j\big) - \mathbf{e}_j S_j\big| \leq \varepsilon_t \mathbf{e}_j S_j\Big\}\right) \\ \geq\quad & 1 - e^{-D_1 h_t},\end{aligned}$$

which proves (3.16). So Proposition 3.5 is proved. □

4. Convergence toward the Lévy process $(\mathcal{Y}_1, \mathcal{Y}_2)$ and continuity

4.1. *Preliminaries.* We begin this section by the convergence of certain repartition functions. These key results are in the same spirit as the second part of Lemma 5.1 in Andreoletti and Devulder (2015).

Lemma 4.1. *Recall from Proposition 3.5 that $\ell_1 := \mathbf{e}_1 S_1$ and $\mathcal{H}_1 := \mathbf{e}_1 S_1 R_1$. Then for any $\varepsilon \in (0,1/3)$,*

$$\lim_{t\to+\infty} \sup_{x\in\left[e^{-(1-2\varepsilon)\phi(t)},+\infty\right[} \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P}\big(\ell_1/t > x\big) - \mathcal{C}_2 \right| = 0, \tag{4.1}$$

$$\lim_{t\to+\infty} \sup_{y\in\left[e^{-(1-3\varepsilon)\phi(t)},+\infty\right[} \left| y^\kappa e^{\kappa\phi(t)} \mathbb{P}\big(\mathcal{H}_1/t > y\big) - \mathcal{C}_2 \mathbb{E}\big[(\mathcal{R}_\kappa)^\kappa\big] \right| = 0, \tag{4.2}$$

with $\mathcal{C}_2$ a positive constant (see below (4.10)).
Moreover, for any $\alpha > 0$, $e^{\kappa\phi(t)}\mathbb{P}(\ell_1/t \geq x, \mathcal{H}_1/t \geq y)$ converges uniformly when t goes to infinity on $[\alpha,+\infty[\times[\alpha,+\infty[$ to $\nu\left([x,+\infty[\times[y,+\infty[\right)$, where ν is defined in (1.4).

Proof: Let $\varepsilon \in (0,1/3)$. □

Proof of (4.1)*:* We first prove that, as $t \to +\infty$, $x^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(S_1/t > x\right)$ converges uniformly in $x \in \left[e^{-(1-\varepsilon)\phi(t)}, +\infty\right[$ to a constant c, that is, we prove that

$$\lim_{t\to+\infty} \sup_{x\in\left[e^{-(1-\varepsilon)\phi(t)},+\infty\right[} \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P}\left(S_1/t > x\right) - c \right| = 0. \tag{4.3}$$

For that, with the change of variables $y = e^{(1-\varepsilon)\phi(t)}x$, we just have to prove that

$$\lim_{t\to+\infty} \sup_{y\in[1,+\infty[} \left| y^\kappa e^{\kappa\varepsilon\phi(t)} \mathbb{P}\left(S_1/e^{h_t+\varepsilon\phi(t)} > y\right) - c \right| = 0, \tag{4.4}$$

but this is equivalent to prove that for any function $f\ :\ ]0,+\infty[\to [1,+\infty[$,

$$\lim_{t\to+\infty} (f(t))^\kappa e^{\kappa\varepsilon\phi(t)} \mathbb{P}\left(S_1/e^{h_t+\varepsilon\phi(t)} > f(t)\right) = c. \tag{4.5}$$

First by definition (see Proposition 3.5), S_1 can be written as the sum of two independent random variables, that we denote by $F_1^+(h_t)$ and $G^+(h_t/2,h_t)$ for simplicity. That is,

$$S_1/t = \left(F_1^+(h_t) + G^+(h_t/2,h_t)\right)/t = e^{-\phi(t)}\left(e^{-h_t}F_1^+(h_t) + e^{-h_t}G^+(h_t/2,h_t)\right). \tag{4.6}$$

Since we know the asymptotic behavior of the Laplace transforms of $F^+(h_t)/e^{h_t}$ and $G^+(h_t/2,h_t)/e^{h_t}$, the proof of (4.5) is similar to the proof of a Tauberian theorem. First by (7.1) and (7.2) we have, using the independence of $F_1^+(h_t)$ and $G^+(h_t/2,h_t)$,

$$\begin{aligned} \forall \gamma > 0, \qquad \omega_{f,t}(\gamma) \quad &:= \quad \frac{1}{\gamma}\left(1 - \mathbb{E}\left[e^{-\gamma S_1/(f(t)e^{h_t+\varepsilon\phi(t)})}\right]\right) \\ &\underset{t\to+\infty}{\sim} \quad c'\gamma^{\kappa-1}(f(t))^{-\kappa}e^{-\kappa\varepsilon\phi(t)}, \end{aligned} \tag{4.7}$$

where $c' = \Gamma(1-\kappa)2^\kappa/\Gamma(1+\kappa)$. Note that by Fubini, $\omega_{f,t}$ is the Laplace transform of the measure $\mathrm{d}U_{f,t}(z) := \mathbb{1}_{\mathbb{R}_+}(z)\mathbb{P}\left(S_1/(f(t)e^{h_t+\varepsilon\phi(t)}) > z\right)\mathrm{d}z$, that is, $\omega_{f,t}(\gamma) = \int_0^\infty e^{-\gamma z}\mathrm{d}U_{f,t}(z)$. From (4.7), we have

$$\forall \gamma > 0, \qquad \frac{\omega_{f,t}(\gamma)}{\omega_{f,t}(1)} \underset{t\to+\infty}{\longrightarrow} \gamma^{\kappa-1}.$$

We can now follow the same line as in the proof of a classical Tauberian theorem, making the link between a Laplace transform and the repartition function, (see for example Feller, 1971 volume 2, section XIII.5, Theorem 1, page 442), we can deduce that

$$\forall z > 0, \qquad \frac{U_{f,t}([0,z])}{\omega_{f,t}(1)} \underset{t\to+\infty}{\longrightarrow} \frac{z^{1-\kappa}}{\Gamma(2-\kappa)}.$$

Then, e.g. as in the proof of Theorem 4 of the same reference page 446, or using inequalities similar to those at the end of the proof of Lemma 5.1 in Andreoletti and Devulder (2015), we deduce from the monotony of the densities of measures $U_{f,t}$ that

$$\forall z > 0, \qquad \frac{\mathbb{P}\left(S_1/(f(t)e^{h_t+\varepsilon\phi(t)}) > z\right)}{\omega_{f,t}(1)} \underset{t\to+\infty}{\longrightarrow} z^{-\kappa}\frac{1-\kappa}{\Gamma(2-\kappa)}.$$

Considering this convergence with $z=1$ we get exactly (4.5) for $c = c'(1-\kappa)/\Gamma(2-\kappa) = 2^\kappa/\Gamma(1+\kappa)$, so (4.3) follows.

Now, let $a_t := e^{\varepsilon\phi(t)}$. For any $x > 0$,

$$x^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x,\ \mathbf{e}_1 < a_t\right) = 2^{-1}\int_0^{a_t}(x/u)^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(S_1/t > x/u\right)u^\kappa e^{-u/2}\mathrm{d}u,$$

because $\mathbf{e}_1$ has law $\mathcal{E}(1/2)$ and is independent of S_1.
Taking x arbitrary in $[e^{-(1-2\varepsilon)\phi(t)},+\infty[$, we have $x/u \in [e^{-(1-\varepsilon)\phi(t)},+\infty[$ for every $u \in]0,a_t]$, so thanks to (4.3) we get

$$\lim_{t\to+\infty}\sup_{x\in\left[e^{-(1-2\varepsilon)\phi(t)},+\infty\right[}\left|x^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x,\ \mathbf{e}_1 < a_t\right) - \frac{c}{2}\int_0^{+\infty}\frac{u^\kappa}{e^{u/2}}\mathrm{d}u\right| = 0. \tag{4.8}$$

Now for t large enough such that $\forall y \geq 1$, $y^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(S_1/t > y\right) < 2c$ (see (4.3)), we have for any $x > 0$,

$$\begin{aligned}
&\left|x^\kappa e^{\kappa\phi(t)}\mathbb{P}\big(\mathbf{e}_1 S_1/t > x,\ \mathbf{e}_1 < a_t\big) - x^\kappa e^{\kappa\phi(t)}\mathbb{P}\big(\mathbf{e}_1 S_1/t > x\big)\right| \\
&= x^\kappa e^{\kappa\phi(t)}\mathbb{P}\big(\mathbf{e}_1 S_1/t > x,\ \mathbf{e}_1 \geq a_t\big) \\
&= 2^{-1}\int_{a_t}^{+\infty} x^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(S_1/t > x/u\right)e^{-u/2}\mathrm{d}u \\
&= 2^{-1}\int_{a_t}^{+\infty} u^\kappa (x/u)^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(S_1/t > x/u\right)\mathbb{1}_{x\leq u}e^{-u/2}\mathrm{d}u \\
&+ 2^{-1}\int_{a_t}^{+\infty} u^\kappa (x/u)^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(S_1/t > x/u\right)\mathbb{1}_{x> u}e^{-u/2}\mathrm{d}u \\
&\leq 2^{-1}e^{\kappa\phi(t)}\int_{a_t}^{+\infty} u^\kappa e^{-u/2}\mathrm{d}u + c\int_{a_t}^{+\infty} u^\kappa e^{-u/2}\mathrm{d}u.
\end{aligned} \tag{4.9}$$

For the second term in the inequality we used the fact that

$$(x/u)^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(S_1/t > x/u\right) < 2c$$

when $x \geq u$. Since $a_t = e^{\varepsilon\phi(t)}$, the right hand side of (4.9) converges to 0 when t goes to infinity. Combining this with (4.8), we get

$$\lim_{t\to+\infty}\sup_{x\in\left[e^{-(1-2\varepsilon)\phi(t)},+\infty\right[}\left|x^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x\right) - 2^{-1}c\int_0^{+\infty}u^\kappa e^{-u/2}\mathrm{d}u\right| = 0, \tag{4.10}$$

and this is exactly (4.1) with $\mathcal{C}_2 := 2^{-1}c\int_0^{+\infty} u^\kappa e^{-u/2}\mathrm{d}u = 2^\kappa\Gamma(\kappa+1)c = 4^\kappa$. $\square$

Proof of (4.2)*:* Let μ_{R_1} be the distribution of R_1. For any $y>0$, $a>0$ and $t>0$, we have by independence of $\mathbf{e}_1 S_1$ and R_1,

$$y^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1 R_1/t > y,\ R_1 < a\right) = \int_0^a (y/u)^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > y/u\right) u^\kappa \mu_{R_1}(\mathrm{d}u).$$

Taking $a = a_t = e^{\varepsilon\phi(t)}$ and y arbitrary in $[e^{-(1-3\varepsilon)\phi(t)}, +\infty[$, we have $y/u \in [e^{-(1-2\varepsilon)\phi(t)}, +\infty[$ for all $u \in]0, a_t]$, so thanks to (4.10), we get

$$\lim_{t\to+\infty} \sup_{y\in\left[e^{-(1-3\varepsilon)\phi(t)},+\infty\right[} \left| y^\kappa e^{\kappa\phi(t)}\mathbb{P}\big(\mathbf{e}_1 S_1 R_1/t > y,\ R_1 < a_t\big) - \mathcal{C}_2\int_0^{a_t} u^\kappa \mu_{R_1}(\mathrm{d}u)\right|$$
$$= 0,$$

where we used $\int_0^\infty u^\kappa \mu_{R_1}(\mathrm{d}u) = \mathbb{E}\left[(R_1)^\kappa\right] \le \mathbb{E}\left[(\mathcal{R}_\kappa)^\kappa\right] < \infty$, as explained in the following lines. By definition (see before Proposition 3.5 and (1.6)) and with $\widetilde{W}_\kappa^\uparrow$ an independent copy of $W_\kappa^\uparrow$, R_1 is equal in law to $\int_0^{\tau^{W_\kappa^\uparrow}(h_t/2)} e^{-W_\kappa^\uparrow(x)}\mathrm{d}x + \int_0^{\tau^{\widetilde{W}_\kappa^\uparrow}(h_t/2)} e^{-\widetilde{W}_\kappa^\uparrow(x)}\mathrm{d}x$, which itself converges almost surely to $\mathcal{R}_\kappa$ (defined in (1.2)) when t goes to infinity. This also shows that for each t, R_1 is stochastically inferior to $\mathcal{R}_\kappa$, which admits finite moments of any positive order by Lemma 6.6. In particular the family $(R_1)_{t>0}$ is bounded in all L^p spaces, and more precisely, $\mathbb{E}\left[(R_1)^p\right] \le \mathbb{E}\left[(\mathcal{R}_\kappa)^p\right] < \infty$ for every $p \in \mathbb{R}_+$. So by the dominated convergence theorem, $\int_0^{+\infty} u^\kappa \mu_{R_1}(\mathrm{d}u)$ converges to $\mathbb{E}\left[(\mathcal{R}_\kappa)^\kappa\right]$ when t goes to infinity. Hence,

$$\lim_{t\to+\infty} \sup_{y\in\left[e^{-(1-3\varepsilon)\phi(t)},+\infty\right[} \left| y^\kappa e^{\kappa\phi(t)}\mathbb{P}\big(\mathbf{e}_1 S_1 R_1/t > y,\ R_1 < a_t\big) - \mathcal{C}_2\mathbb{E}\big[(\mathcal{R}_\kappa)^\kappa\big]\right| = 0.$$

Finally, as the family $(R_1)_{t>0}$ is bounded in all L^p spaces, $e^{\kappa\phi(t)}\int_{a_t}^\infty u^\kappa \mu_{R_1}(\mathrm{d}u)$ converges to 0 as $t\to+\infty$. So we can proceed as before (as in (4.9), integrating with respect to R_1 instead of $\mathbf{e}_1$ and using (4.1) instead of (4.3)) to remove the event $R_1 < a_t$ and we thus get

$$\lim_{t\to+\infty} \sup_{y\in\left[e^{-(1-3\varepsilon)\phi(t)},+\infty\right[} \left| y^\kappa e^{\kappa\phi(t)}\mathbb{P}\big(\mathbf{e}_1 S_1 R_1/t > y\big) - \mathcal{C}_2\mathbb{E}\big[(\mathcal{R}_\kappa)^\kappa\big]\right| = 0, \quad (4.11)$$

which is (4.2). We now prove the last assertion. For any $x>0$, $y>0$, $a>0$ and $t>0$, we have

$$\begin{aligned}
& e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x,\ \mathbf{e}_1 S_1 R_1/t > y,\ R_1 < a\right) \\
= & \int_0^a e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x,\ \mathbf{e}_1 S_1/t > y/u\right)\mu_{R_1}(\mathrm{d}u) \\
= & \int_0^{a\wedge(y/x)} e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > y/u\right)\mu_{R_1}(\mathrm{d}u) \\
& \quad + \int_{a\wedge(y/x)}^a e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x\right)\mu_{R_1}(\mathrm{d}u), \\
= & \frac{1}{y^\kappa}\int_0^{a\wedge(y/x)} e^{\kappa\phi(t)}(y/u)^\kappa\mathbb{P}\left(\mathbf{e}_1 S_1/t > y/u\right)u^\kappa\mu_{R_1}(\mathrm{d}u) \\
& \quad + \frac{1}{x^\kappa}\int_{a\wedge(y/x)}^a e^{\kappa\phi(t)}x^\kappa\mathbb{P}\left(\mathbf{e}_1 S_1/t > x\right)\mu_{R_1}(\mathrm{d}u).
\end{aligned}$$

Taking $a = a_t = e^{\varepsilon\phi(t)}$ and x, y arbitrary in $[\alpha, +\infty[$ (for some $\alpha > 0$), we have $(y/u, x) \in [e^{-(1-2\varepsilon)\phi(t)}, +\infty[^2$, $\forall u \in]0, a_t]$ whenever t is large enough, so, thanks to (4.10) we get that $e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 S_1 R_1/t > y,\ R_1 < a_t\right)$ converges uniformly in $(x, y) \in [\alpha, +\infty[\times[\alpha, +\infty[$ toward

$$\mathcal{C}_2 x^{-\kappa}\mathbb{P}\big(\mathcal{R}_\kappa > y/x\big) + \mathcal{C}_2 y^{-\kappa}\mathbb{E}\big((\mathcal{R}_\kappa)^\kappa \mathbb{1}_{\mathcal{R}_\kappa \le y/x}\big) = \nu\big([x, +\infty[\times[y, +\infty[\big).$$

Then as before we can remove the event $\{R_1 < a_t\}$ since $e^{\kappa\phi(t)}\mathbb{P}(R_1 \ge a_t) \to 0$ as $t \to +\infty$ because the family $(R_1)_{t>0}$ is bounded in all L^p spaces, which gives the last assertion of Lemma 4.1. □

4.2. *Proof of Proposition 1.4.*
We start with the finite dimensional convergence. We recall that $(Y_1, Y_2)^t_s$ is defined just before Proposition 1.4, and $(\mathcal{Y}_1, \mathcal{Y}_2)$ before (1.4). We sometimes use the notation $(\mathcal{Y}_1, \mathcal{Y}_2)_s = (\mathcal{Y}_1(s), \mathcal{Y}_2(s))$ and $(Y_1, Y_2)^t_s = (Y^t_1(s), Y^t_2(s))$.

Lemma 4.2. *For any $k \in \mathbb{N}$ and $s_i > 0, i \le k$, $((Y_1, Y_2)^t_{s_i}, i \le k)$ converges in law as t goes to infinity to $((\mathcal{Y}_1, \mathcal{Y}_2)_{s_i}, i \le k)$.*

Proof: The proof is basic here, however we give some details as we deal with a two dimensional walk which increments depend on t itself. As $Y^t_1(s)$ and $Y^t_2(s)$ are sums of i.i.d sequences we only have to prove the convergence in law for the couple $(Y_1, Y_2)^t_s$ for any $s > 0$. For $b \ge 0$, we define $\big(Y_1^{>b}, Y_2^{>b}\big)$, obtained from $(Y_1, Y_2)^t$ by keeping only the increments larger than b, that is, $Y_1^{>b}(s) := \frac{1}{t}\sum_{j=1}^{\lfloor se^{\kappa\phi(t)}\rfloor} \ell_j \mathbb{1}_{\ell_j/t>b}$ and $Y_2^{>b}(s) := \frac{1}{t}\sum_{j=1}^{\lfloor se^{\kappa\phi(t)}\rfloor} \mathcal{H}_j \mathbb{1}_{\mathcal{H}_j/t>b}$ for every $s \ge 0$ and $t > 0$. Also let $Y_i^{\le b}(s) := Y^t_i(s) - Y_i^{>b}(s)$ for $i \in \{1, 2\}$. We first prove that for any $s > 0$,

$$\lim_{\varepsilon\to 0}\limsup_{t\to+\infty}\mathbb{P}\big(\big|\big|(Y_1^{\le\varepsilon}, Y_2^{\le\varepsilon})^t_s\big|\big| > \varepsilon^{1-\kappa(2-\kappa)}\big) = 0, \tag{4.12}$$

where for any $a = (a_1, a_2) \in \mathbb{R}^2$, $||a|| := \max(|a_1|, |a_2|)$, with $(Y_1^{\le\varepsilon}, Y_2^{\le\varepsilon})^t_s = \big(Y_1^{\le\varepsilon}(s), Y_2^{\le\varepsilon}(s)\big)$ and $1 - \kappa(2-\kappa) > 0$ since $\kappa < 1$.

Let $\varepsilon > 0$ and $s > 0$. We now give an upper bound for the first moments of $Y_1^{\le\varepsilon}(s)$ and $Y_2^{\le\varepsilon}(s)$. Let $\eta > 0$ be such that $\kappa - (1 - 3\eta) < 0$. Applying Fubini, we have for large t,

$$\begin{aligned}
& e^{\kappa\phi(t)}\mathbb{E}\left(\frac{\ell_1}{t}\mathbb{1}_{\ell_1/t\le\varepsilon}\right) \\
= \ & e^{\kappa\phi(t)}\mathbb{E}\left[\frac{\mathbf{e}_1 S_1}{t}\ \mathbb{1}_{\mathbf{e}_1 S_1/t\le\varepsilon}\right] \\
\le \ & \int_0^\varepsilon e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x\right)\mathrm{d}x \\
= \ & \int_0^{e^{-(1-2\eta)\phi(t)}} e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x\right)\mathrm{d}x + \int_{e^{-(1-2\eta)\phi(t)}}^\varepsilon e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x\right)\mathrm{d}x \\
\le \ & e^{(\kappa-(1-2\eta))\phi(t)} + \int_{e^{-(1-2\eta)\phi(t)}}^\varepsilon x^{-\kappa}x^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x\right)\mathrm{d}x.
\end{aligned} \tag{4.13}$$

The first term in (4.13) converges to 0 when t goes to infinity because $\kappa - (1-2\eta) < -\eta < 0$. Moreover, according to (4.1), for t large enough, we have

$$\forall x \ge e^{-(1-2\eta)\phi(t)}, \qquad x^\kappa e^{\kappa\phi(t)}\mathbb{P}\left(\mathbf{e}_1 S_1/t > x\right) \le 2\mathcal{C}_2.$$

For such t, the second term in (4.13) is less than

$$2\mathcal{C}_2 \int_0^{\varepsilon} x^{-\kappa} \mathrm{d}x = 2\mathcal{C}_2 \frac{\varepsilon^{1-\kappa}}{1-\kappa}.$$

So, we get for large t,

$$e^{\kappa\phi(t)} \mathbb{E}\left(\frac{\ell_1}{t} \mathbb{1}_{\ell_1/t \le \varepsilon}\right) \le e^{(\kappa-(1-2\eta))\phi(t)} + C_+ \varepsilon^{1-\kappa}. \tag{4.14}$$

Using the same method and applying this time (4.2), we get for large t,

$$e^{\kappa\phi(t)} \mathbb{E}\left(\frac{\mathcal{H}_1}{t} \mathbb{1}_{\mathcal{H}_1/t \le \varepsilon}\right) \le e^{(\kappa-(1-3\eta))\phi(t)} + C_+ \varepsilon^{1-\kappa}. \tag{4.15}$$

We thus obtain

$$\mathbb{E}\left(Y_1^{\le\varepsilon}(s)\right) \le s e^{(\kappa-(1-2\eta))\phi(t)} + C_+ s \varepsilon^{1-\kappa}, \tag{4.16}$$

$$\mathbb{E}\left(Y_2^{\le\varepsilon}(s)\right) \le s e^{(\kappa-(1-3\eta))\phi(t)} + C_+ s \varepsilon^{1-\kappa}, \tag{4.17}$$

then a Markov inequality leads to (4.12) since $\kappa - (1-3\eta) < 0$.

The next step is to prove that $(Y_1^{>\varepsilon}, Y_2^{>\varepsilon})_s^t$ can be written as the integral of a point process which converges to the desired limit. We have

$$\left(Y_1^{>\varepsilon}, Y_2^{>\varepsilon}\right)_s^t = \left(Y_1^{>\varepsilon}(s), Y_2^{>\varepsilon}(s)\right) = \left(\int_{x>\varepsilon}\int_0^s x\mathcal{P}_t^1(\mathrm{d}x, \mathrm{d}v), \int_{x>\varepsilon}\int_0^s x\mathcal{P}_t^2(\mathrm{d}x, \mathrm{d}v)\right)$$

where the measures $\mathcal{P}_t^1$ and $\mathcal{P}_t^2$ are defined by $\mathcal{P}_t^1 := \sum_{i=1}^{+\infty} \delta_{(t^{-1}\ell_i, e^{-\kappa\phi(t)} i)}$ and similarly $\mathcal{P}_t^2 := \sum_{i=1}^{+\infty} \delta_{(t^{-1}\mathcal{H}_i, e^{-\kappa\phi(t)} i)}$. Recall that $\mathcal{P}_t^1$ and $\mathcal{P}_t^2$ are not independent. We now prove that $(\mathcal{P}_t^1, \mathcal{P}_t^2)$ converges to a Poisson point measure. For that just use Lemma 4.1 together with Proposition 3.1 in Resnick (1986) after discretization, it implies that $(\mathcal{P}_t^1, \mathcal{P}_t^2)$ converges weakly to the Poisson random measure denoted by $(\mathcal{P}^1, \mathcal{P}^2)$ with intensity measure given by $\mathrm{d}s \times \nu$.

Then using that for any $\varepsilon > 0$, and $T < +\infty$, on $[0,T) \times (\varepsilon, +\infty) \times (\varepsilon, +\infty)$ $\mathrm{d}s \times \nu$ is finite, we have that $(Y_1^{>\varepsilon}, Y_2^{>\varepsilon})_s^t$ converges weakly to

$$(\mathcal{Y}_1^{>\varepsilon}, \mathcal{Y}_2^{>\varepsilon})_s := \left(\int_{x>\varepsilon}\int_0^s x\mathcal{P}^1(\mathrm{d}x, \mathrm{d}v), \int_{x>\varepsilon}\int_0^s x\mathcal{P}^2(\mathrm{d}x, \mathrm{d}v)\right).$$

We are left to prove that $(\mathcal{Y}_1^{>\varepsilon}, \mathcal{Y}_2^{>\varepsilon})$ converges to $(\mathcal{Y}_1, \mathcal{Y}_2)$ when $\varepsilon \downarrow 0$. This is a straightforward computation, that we detail for completeness. Let $\nu_1([x, +\infty[) := \nu\left([x, +\infty[\times \mathbb{R}_+\right) = \mathcal{C}_2/x^{\kappa}$, we have

$$\mathbb{E}\left(\int_{x\le\varepsilon}\int_0^s x\mathcal{P}^1(\mathrm{d}x, \mathrm{d}v)\right) = s\int_{x\le\varepsilon} x\nu_1(\mathrm{d}x) = C\varepsilon^{1-\kappa},$$

Then a Markov inequality proves that for any $s > 0$, the process $\int_{x\le\varepsilon}\int_0^s x\mathcal{P}^1(\mathrm{d}x, \mathrm{d}v)$ converges to zero (when ε goes to zero) in probability. The same is true for $\int_{x\le\varepsilon}\int_0^s x\mathcal{P}^2(\mathrm{d}x, \mathrm{d}v)$, so we obtain that $(\mathcal{Y}_1^{>\varepsilon}, \mathcal{Y}_2^{>\varepsilon})_s$ converges in probability to $(\mathcal{Y}_1, \mathcal{Y}_2)_s$ when $\varepsilon \to 0$. □

We now prove the tightness of $(\mathcal{D}(Y_1, Y_2)^t)_t$, the family of measures induced by processes $(Y_1, Y_2)^t$.

Lemma 4.3. *The family of laws $(\mathcal{D}(Y_1, Y_2)^t)_t$ is tight on $(D([0, +\infty), \mathbb{R}^2), J_1)$.*

Proof: We only have to prove that the family law of the restriction of the process to the interval $[0,T]$, $((Y_1,Y_2)^t|_{[0,T]})_t$ is tight. To prove this we use the following restatement of Theorem 1.8 in Billingsley (1999) using Aldous's tightness criterion (see Condition 1, and equation (16.22) page 176 in Billingsley, 1999) also used in Bovier (2010) page 100. We have to check the two following statements:
1) for any $\varepsilon > 0$, there exists a such that for any t large enough,

$$\mathbb{P}\Big(\sup_{s\in[0,T]} ||(Y_1,Y_2)^t_s|| \geq a\Big) \leq \varepsilon.$$

2) for any $\varepsilon > 0$, and $\eta > 0$ there exists δ, $0 < \delta < T$ and $t_0 > 0$ such that for $t > t_0$,

$$\mathbb{P}\big[\omega((Y_1,Y_2)^t,\delta,T) \geq \eta\big] \leq \varepsilon,$$

with $\omega((Y_1,Y_2)^t,\delta,T) := \sup_{0\leq r\leq T} \omega((Y_1,Y_2)^t,\delta,T,r)$, and

$$\begin{aligned}&\omega((Y_1,Y_2)^t,\delta,T,r)\\ := &\sup_{0\vee(r-\delta)\leq u_1<u<u_2\leq(r+\delta)\wedge T}\Big(\min\Big\{\big|\big|(Y_1,Y_2)^t_{u_2}-(Y_1,Y_2)^t_u\big|\big|,\\ &\qquad\qquad\big|\big|(Y_1,Y_2)^t_u-(Y_1,Y_2)^t_{u_1}\big|\big|\Big\}\Big).\end{aligned}$$

Also

$$\mathbb{P}(v((Y_1,Y_2)^t,0,\delta,T)\geq\eta)\leq\varepsilon, \text{ and } \mathbb{P}(v((Y_1,Y_2)^t,T,\delta,T)\geq\eta)\leq\varepsilon,$$

where $v((Y_1,Y_2)^t,u,\delta,T) := \sup_{(u-\delta)\vee 0\leq u_1\leq u_2\leq(u+\delta)\wedge T}\{||(Y_1,Y_2)^t_{u_1}-(Y_1,Y_2)^t_{u_2}||\}$.

We first check 1) since the process is monotone increasing,

$$\mathbb{P}(\sup_{s\in[0,T]}||(Y_1,Y_2)^t_s||\geq a) = \mathbb{P}(||(Y_1,Y_2)^t_T||\geq a) \leq \mathbb{P}(Y_1(T)\geq a)+\mathbb{P}(Y_2(T)\geq a). \tag{4.18}$$

Recall that $Y_1^{>b}$ is obtained from Y_1 where we remove the increments ℓ_j/t smaller than b and $Y_1^{\leq b} = Y_1 - Y_1^{>b}$. Define $N_u^{>b} := \sum_{i=1}^{\lfloor ue^{\kappa\phi(t)}\rfloor}\mathbb{1}_{\ell_j/t>b}$. Let $0<\delta_1<1$. A Markov inequality yields

$$\begin{aligned}\mathbb{P}\big(Y_1^t(T)\geq a\big) &\leq \mathbb{P}\left(Y_1^{\leq 1}(T)\geq\frac{a}{2}\right)+\mathbb{P}\left(Y_1^{>1}(T)\geq\frac{a}{2}\right)\\ &\leq\frac{2}{a}\mathbb{E}\left[Y_1^{\leq 1}(T)\right]+\frac{1}{a^{\delta_1}}\mathbb{E}\left(N_T^{>1}\right)+\mathbb{P}\left(Y_1^{>1}(T)\geq\frac{a}{2},N_T^{>1}\leq a^{\delta_1}\right).\end{aligned}\tag{4.19}$$

On $\{N_T^{>1}\leq a^{\delta_1}\}$ there is at most a^{δ_1} terms in the sum $Y_1^{>1}(T)$ so

$$\begin{aligned}\mathbb{P}\left(Y_1^{>1}(T)>a/2,N_T^{>1}\leq a^{\delta_1}\right) &\leq \sum_{1\leq i\leq a^{\delta_1}}\mathbb{P}\left(\ell_i/t\geq(a^{1-\delta_1}/2)|\ell_i/t\geq 1\right)\\ &\leq a^{\delta_1}\mathbb{P}\left(\ell_1/t\geq(a^{1-\delta_1}/2)|\ell_1/t\geq 1\right)\\ &\leq a^{\delta_1}2\frac{\mathcal{C}_2e^{-\kappa\phi(t)}a^{-\kappa(1-\delta_1)}2^\kappa}{\mathcal{C}_2e^{-\kappa\phi(t)}}\\ &= 2^{1+\kappa}\ a^{\delta_1-\kappa(1-\delta_1)},\end{aligned}\tag{4.20}$$

for all t large enough thanks to (4.1) and δ_1 such that $\delta_1-\kappa(1-\delta_1)<0$.

Also, notice that for any $b > 0$, $N_T^{>b}$ follows a binomial law with parameters $\big(\lfloor Te^{\kappa\Phi(t)}\rfloor, \mathbb{P}(\ell_1/t > b)\big)$. So, using (4.1) again and (4.16), we obtain for t large enough,

$$\mathbb{E}(N_T^{>b}) \leq 2\mathcal{C}_2 T b^{-\kappa}, \qquad \mathbb{E}\left[Y_1^{\leq b}(T)\right] \leq 2\mathcal{C}_2 T b^{1-\kappa}. \tag{4.21}$$

Collecting (4.20), (4.21) and (4.19), we get the existence of $t_1 > 0$ such that

$$\lim_{a\to+\infty} \sup_{t\geq t_1} \mathbb{P}(Y_1(T) \geq a) = 0. \tag{4.22}$$

The same arguments holds for Y_2 (using (4.2) instead of (4.1) and (4.17) instead of (4.16)) so (4.22) also holds for Y_2 instead of Y_1. We conclude the proof of 1) by putting (4.22) and its analogous for Y_2 in (4.18).

We now check 2) We first write, as usual,

$$\begin{aligned}
&\left\{\omega\big((Y_1, Y_2)^t, \delta, T\big) \geq \eta\right\}\\
\subset\ &\left\{\omega\big((Y_1^{\leq b}, Y_2^{\leq b})^t, \delta, T\big) \geq \eta/2\right\} \cup \left\{\omega\big(\big(Y_1^{>b}, Y_2^{>b}\big)^t, \delta, T\big) \geq \eta/2\right\}.
\end{aligned}$$

For $Y_\cdot^{\leq b}$, we have

$$\mathbb{P}\big[\omega\big(\big(Y_1^{\leq b}, Y_2^{\leq b}\big)^t, \delta, T\big) \geq \eta/2\big] \leq \mathbb{P}\big[\omega\big(Y_1^{\leq b}, \delta, T\big) \geq \eta/2\big] + \mathbb{P}\big[\omega\big(Y_2^{\leq b}, \delta, T\big) \geq \eta/2\big].$$

Moreover, by positivity of the increments,

$$\begin{aligned}
&\mathbb{P}\left(\omega\big(Y_1^{\leq b}, \delta, T\big) \geq \eta/2\right)\\
\leq\ &\mathbb{P}\left(\cup_{k\leq\lfloor T/2\delta\rfloor}\left\{Y_1^{\leq b}\big((k+1)2\delta\big) - Y_1^{\leq b}\big(k2\delta\big) \geq \eta/4\right\}\right)\\
\leq\ &\sum_{k\leq\lfloor T/\delta\rfloor} \mathbb{P}\left(Y_1^{\leq b}\big((k+1)2\delta\big) - Y_1^{\leq b}\big(k2\delta\big) \geq \eta/4\right).
\end{aligned} \tag{4.23}$$

For any k, $Y_1^{\leq b}((k+1)2\delta) - Y_1^{\leq b}(k2\delta)$ is the sum of at most $\lfloor 2\delta e^{\kappa\Phi(t)}\rfloor + 1$ i.i.d. random variables having the same law as ℓ_1/t. We get that for any integer k

$$\mathbb{P}\left(Y_1^{\leq b}((k+1)2\delta) - Y_1^{\leq b}(k2\delta) \geq \eta/4\right) \leq \mathbb{P}\left(Y_1^{\leq b}(3\delta) \geq \eta/4\right) \leq 8\mathcal{C}_2\delta b^{1-\kappa}/\eta,$$

where the first inequality holds for t large enough so that $2\delta e^{\kappa\Phi(t)} \geq 1$ and the second from the second expression in (4.21) (replacing T by 2δ). Combining with (4.23) we get for large t

$$\mathbb{P}\left(\omega(Y_1^{\leq b}, \delta, T) \geq \eta/2\right) \leq 24\mathcal{C}_2 T(1+2\delta)b^{1-\kappa}/\eta, \tag{4.24}$$

[note that δ will be chosen later (and will be less than 1)]. T and η are fixed so we choose b small enough so that the right hand side of (4.23) is less than $\varepsilon/4$. A similar estimate can be proved for $\mathbb{P}(\omega(Y_2^{\leq b}, \delta, T) \geq \eta/2)$.

For $Y_\cdot^{>b}$, we have again

$$\mathbb{P}(\omega((Y_1^{>b}, Y_2^{>b})^t, \delta, T) \geq \eta/2) \leq \mathbb{P}(\omega(Y_1^{>b}, \delta, T) \geq \eta/2) + \mathbb{P}(\omega(Y_2^{>b}, \delta, T) \geq \eta/2).$$

Since $Y_1^{>b}$ is piecewise constant with jumps larger than b, $\{\omega(Y_1^{>b}, \delta, T) > \eta/2\}$ implies that two jumps larger than b for Y_1^t occur in an interval smaller than

2δ. That is $\{\omega(Y_1^{>b},\delta,T)>\eta/2\}\subset\cup_{j=1}^{\lfloor Te^{\kappa\phi(t)}\rfloor}\cup_{i>j,(i-j)/e^{\kappa\phi(t)}\le 2\delta}^{\lfloor Te^{\kappa\phi(t)}\rfloor}\{\ell_j\wedge\ell_i/t>b\}$. Applying (4.1) for t large enough,

$$\mathbb{P}\left(\cup_{j=1}^{\lfloor Te^{\kappa\phi(t)}\rfloor}\cup_{i>j,(i-j)/e^{\kappa\phi(t)}\le 2\delta}^{\lfloor Te^{\kappa\phi(t)}\rfloor}\left\{\ell_j\wedge\ell_i/t>b\right\}\right)\le 8\mathcal{C}_2^2\delta T b^{-2\kappa},$$

which can be small choosing this time $\delta=\delta(b)$ properly. Again the same argument can be used for $\omega(Y_2^{>b},\delta,T)$. To finish the proof, we have to deal with $v()$, as again our processes are increasing,

$$\mathbb{P}(v((Y_1,Y_2)^t,0,\delta,T)\ge\eta)\le\mathbb{P}(||(Y_1,Y_2)_\delta^t||\ge\eta)$$

we can then proceed as for 1) decreasing the value of δ if needed, this also applies to $\mathbb{P}(v((Y_1,Y_2)^t,T,\delta,T)\ge\eta)$. □

Putting together the two preceding lemmata we obtain Proposition 1.4.

4.3. *Continuity of some functionals of* $(\mathcal{Y}_1,\mathcal{Y}_2)$ *in* J_1 *topology.* In this section, we study the continuity of some functionals which will be applied later to $(Y_1,Y_2)^t$ and to the Lévy processes $(\mathcal{Y}_1,\mathcal{Y}_2)$.

For our purpose, we are interested in the following mappings. We have already mentioned the first two in the introduction:

$$J:\ \begin{array}{ll} D(\mathbb{R}_+,\mathbb{R}) & \longrightarrow D(\mathbb{R}_+,\mathbb{R}) \\ f & \longmapsto f^\natural \end{array}\qquad I:\ \begin{array}{ll} (D(\mathbb{R}_+,\mathbb{R}),J_1) & \longrightarrow (D(\mathbb{R}_+,\mathbb{R}),U) \\ f & \longmapsto f^{-1} \end{array}$$

where U denotes uniform convergence on every compact subset of $\mathbb{R}_+$. Then we also need the compositions of these two: for any positive a, let

$$J_{I,a}:\ \begin{array}{ll} D(\mathbb{R}_+,\mathbb{R}^2) & \longrightarrow \mathbb{R} \\ f=(f_1,f_2) & \longmapsto f_1^\natural\left(f_2^{-1}(a)\right), \end{array}\qquad J_{I,a}^-:\ \begin{array}{ll} D(\mathbb{R}_+,\mathbb{R}^2) & \longrightarrow \mathbb{R} \\ f=(f_1,f_2) & \longmapsto f_1^\natural\left(f_2^{-1}(a)^-\right), \end{array}$$

$J_{I,a}$ (respectively $J_{I,a}^-$) produces the largest jump of f_1, between 0 and the time just after (respectively before) f_2 first reaches $(a,+\infty)$. We also define $K_{I,a}$, $K_{I,a}^-$, $\tilde{K}_{I,a}$ and $\tilde{K}_{I,a}^-$ as follows.

$$\begin{aligned} K_{I,a}:&\ \begin{array}{ll} D(\mathbb{R}_+,\mathbb{R}^2) & \longrightarrow \mathbb{R} \\ f=(f_1,f_2) & \longmapsto f_1\left(f_2^{-1}(a)\right), \end{array}\\ K_{I,a}^-:&\ \begin{array}{ll} D(\mathbb{R}_+,\mathbb{R}^2) & \longrightarrow \mathbb{R} \\ f=(f_1,f_2) & \longmapsto f_1\left(f_2^{-1}(a)^-\right), \end{array}\\ \tilde{K}_{I,a}:&\ \begin{array}{ll} D(\mathbb{R}_+,\mathbb{R}^2) & \longrightarrow \mathbb{R} \\ f=(f_1,f_2) & \longmapsto f_2\left(f_2^{-1}(a)\right), \end{array}\\ \tilde{K}_{I,a}^-:&\ \begin{array}{ll} D(\mathbb{R}_+,\mathbb{R}^2) & \longrightarrow \mathbb{R} \\ f=(f_1,f_2) & \longmapsto f_2\left(f_2^{-1}(a)^-\right). \end{array} \end{aligned}\tag{4.25}$$

Finally, with $\Delta f_1(s):=f_1(s)-f_1(s^-)$, define F^* by

$$F^*:\ \begin{array}{ll} D(\mathbb{R}_+,\mathbb{R}^2) & \longrightarrow \mathbb{R} \\ f=(f_1,f_2) & \longmapsto \inf\left\{s\in\left(0,f_2^{-1}(1)\right),\ \Delta f_1(s)=f_1^\natural\left(f_2^{-1}(1)^-\right)\right\}. \end{array}$$

We need this functional F^* for the characterization of the favorite site.

Lemma 4.4. *J is continuous in the J_1 topology.*

Proof: This fact is basic. However, we have not found a proof in the literature, so we give some details. To prove the continuity on $D(\mathbb{R}_+, \mathbb{R})$, we only have to prove it for every compact subset of $\mathbb{R}_+$, (see Whitt, 2002 Theorem 12.9.1). So let $f \in D(\mathbb{R}_+, \mathbb{R})$ and $T > 0$ at which f is continuous, let us prove that J_T defined by

$$\begin{aligned} J_T : \quad & D([0,T], \mathbb{R}) && \longrightarrow D([0,T], \mathbb{R}) \\ & g && \longmapsto g^{\natural} \end{aligned}$$

is continuous at the restriction $f_{|[0,T]}$. Let $\varepsilon > 0$ and $g \in D([0,T], \mathbb{R})$ such that $d_T(f_{|[0,T]}, g) \leq \frac{\varepsilon}{2}$. d_T is the usual metric d of the J_1-topology restricted to the interval $[0,T]$. By definition of d_T there exists a strictly increasing continuous mapping of $[0,T]$ onto itself, $e : [0,T] \longrightarrow [0,T]$ such that

$$\sup_{s \in [0,T]} |e(s) - s| \leq \frac{\varepsilon}{2} \text{ and } \sup_{s \in [0,T]} \left| g(e(s)) - f_{|[0,T]}(s) \right| \leq \frac{\varepsilon}{2}.$$

So for every $s \in [0,T]$ we have

$$\begin{aligned} \left| \Delta g(e(s)) - \Delta f_{|[0,T]}(s) \right| &= \left| (g(e(s)) - g(e(s)-)) - \left(f_{|[0,T]}(s) - f_{|[0,T]}(s-) \right) \right| \\ &\leq \left| g(e(s)) - f_{|[0,T]}(s) \right| + \left| g(e(s)-) - f_{|[0,T]}(s-) \right| \\ &\leq 2\frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

where $\Delta h(s) = h(s) - h(s^-)$. This implies $d_T\left(J_T\left(f_{|[0,T]}\right), J_T(g)\right) \leq \varepsilon$. $\square$

Lemma 4.5. *Fix $a > 0$. The mappings $J^-_{I,a}$, $J_{I,a}$, $K^-_{I,a}$, $K_{I,a}$, $\tilde{K}^-_{I,a}$ and $\tilde{K}_{I,a}$ are continuous for J_1-topology at every couple $(f^1, f^2) \in D(\mathbb{R}_+, \mathbb{R}^2)$ such that*

(1) *For any $\varepsilon > 0$, f^1 and f^2 have a finite number of jumps greater than ε on every compact subset of $\mathbb{R}^*_+$,*

(2) *f^2 is strictly increasing, with a limit equal to $+\infty$,*

(3) *$f^2(0) = 0$,*

(4) *f^2 has a jump at $I(f^2)(a)$ and $f^2(I(f^2)(a)-) < a < f^2(I(f^2)(a))$.*

Proof: This fact may also be known as we are looking at randomly stopped process, but once again we did not find what we need in the literature (Silvestrov, 2008, Whitt, 2002).

Let $(f^1_n, f^2_n)_n$ be a sequence of elements of $D(\mathbb{R}_+, \mathbb{R})$ which converges to (f^1, f^2) for the J_1 topology. To prove continuity, we prove that the sequence $(J^-_{I,a}(f^1_n, f^2_n))_n$ converges to $J^-_{I,a}(f^1, f^2)$, and the equivalent for $J_{I,a}$.

The first hypothesis guaranties that there exist neighborhoods of $I(f^2)(a)$ for which f^1 makes no jump greater than $1/4$ times its higher previous jump, that is to say there exists $\delta \in]0, I(f^2)(a)[$ (notice that $I(f^2)(a)$ exists tanks to (2) and is positive thanks to (3)) such that f^1 makes no jump greater than $J(f^1)(I(f^2)(a) - \delta)/4$ on $[I(f^2)(a) - \delta, I(f_2)(a)[$ and on $]I(f^2)(a), I(f^2)(a) + \delta]$. Note also that $J(f^1)$ is constant on $[I(f^2)(a) - \delta, I(f_2)(a)[$ and on $]I(f^2)(a), I(f^2)(a) + \delta]$.

Also δ can be made smaller (if needed) in such a way that $I(f^2)(a) + \delta$ is a point of continuity of (f^1, f^2) and $(f^1_n, f^2_n)_n$ for every $n \in \mathbb{N}$. By hypothesis

$d\left((f_n^1, f_n^2), (f^1, f^2)\right) \longrightarrow_{n\to+\infty} 0$ so

$$d_n := d_{[0,I(f^2)(a)+\delta]}\left((f_n^1, f_n^2)_{|[0,I(f^2)(a)+\delta]}, (f^1, f^2)_{|[0,I(f^2)(a)+\delta]}\right) \longrightarrow_{n\to+\infty} 0,$$

where $|[0, I(f^2)(a)+\delta]$ in index means restriction to $[0, I(f^2)(a)+\delta]$. Also by continuity of J (see Lemma 4.4) we also have $d\left(J(f_n^1), J(f^1)\right) \longrightarrow_{n\to+\infty} 0$ and therefore

$$d_n' := d_{[0,I(f^2)(a)+\delta]}\left(\left(J(f_n^1)\right)_{|[0,I(f^2)(a)+\delta]}, \left(J(f^1)\right)_{|[0,I(f^2)(a)+\delta]}\right) \longrightarrow_{n\to+\infty} 0.$$

Let h^- (respectively h^+) be the largest jump of f^1 just before (resp. just after) $I(f^2)(a)$. By definition of δ we have

$$h^- = J(f^1)\left(I(f^2)(a) - \delta\right), h^+ = J(f^1)\left(I(f^2)(a) + \delta\right).$$

We have two cases, either $J(f^1)$ is continuous at $I(f^2)(a)$ or it makes a jump.
Case $J(f^1)$ makes a jump, in this case the size of the jump is $h^+ - h^- > 0$.

Let $\alpha = 8^{-1}\min\left(h^-, \delta, 1 - f^2\left(I(f^2)(a)^-\right), f^2\left(I(f^2)(a)\right) - 1\right)$, and $n_0 \in \mathbb{N}$ be such that for any $n \geq n_0$, $d_n < \alpha$ and $d_n' < \alpha$. $T = I(f^2)(a) + \delta$, there exist two homeomorphisms $e_n, e_n' : [0,T] \longrightarrow [0,T]$ such that:

- $\sup_{s\in[0,T]} |e_n(s) - s| \leq d_n$,
- $\sup_{s\in[0,T]} \left|\left|\left(f_n^1\left(e_n(s)\right), f_n^2\left(e_n(s)\right)\right)_{|[0,I(f^2)(a)+\delta]} - \left(f^1(s), f^2(s)\right)_{|[0,I(f^2)(a)+\delta]}\right|\right|_\infty \leq d_n.$
- $\sup_{s\in[0,T]} |e_n'(s) - s| \leq d_n'$,
- $\sup_{s\in[0,T]} \left|\left(J(f_n^1)\right)_{|[0,I(f^2)(a)+\delta]}\left(e_n'(s)\right) - \left(J(f^1)\right)_{|[0,I(f^2)(a)+\delta]}(s)\right| \leq d_n'.$

The second inequality implies that for any $n \geq n_0$,

$$f_n^2\left(e_n\left(I(f^2)(a)^-\right)\right) < a < f_n^2\left(e_n\left(I(f^2)(a)\right)\right),$$

so as we also have $f_n^2\left(I(f_n^2)(a)^-\right) \leq a \leq f_n^2\left(I(f_n^2)(a)\right)$ we get

$$I(f_n^2)(a) = e_n\left(I(f^2)(a)\right). \tag{4.26}$$

The fourth point implies that for any $n \geq n_0$,

$$J(f_n^1)\left(e_n'\left(I(f^2)(a) - \frac{1}{2}\delta\right)\right) \geq J(f^1)\left(I(f^2)(a) - \frac{1}{2}\delta\right) - \alpha = h^- - \alpha > \frac{1}{2}h^-. \tag{4.27}$$

The second point and the argument of the previous proof imply that for any $n \geq n_0$, each jump of f_n^1 on $[e_n\left(I(f^2)(a) - \delta\right), e_n\left(I(f^2)(a)\right)[$ is 2α-close to a jump of f^1 on $[I(f^2)(a) - \delta, I(f^2)(a)[$, but such jumps are less than $h^-/4$ because of the definition of δ. Thus, f_n^1 makes no jump larger than $h^-/2$ on the interval $[e_n\left(I(f^2)(a) - \delta\right), e_n\left(I(f^2)(a)\right))$. Moreover, the increases of e_n' and the first and third points imply that

$$e_n\left(I(f^2)(a) - \delta\right) \leq e_n'\left(I(f^2)(a) - \delta/2\right) \leq e_n\left(I(f^2)(a)\right).$$

So, combining this with (4.27), we get that $J(f_n^1)$ is constant on the interval $[e_n'\left(I(f^2)(a) - \delta/2\right), e_n\left(I(f^2)(a)\right))$.

Now by definition of $J^-_{I,a}$, with (4.26) and then collecting what have just done above yields

$$\forall n \geq n_0, \qquad J^-_{I,a}\left((f^1_n, f^2_n)\right) = J(f^1_n)\left(I(f^2_n)(a)^-\right) = J(f^1_n)\left(e_n\left(I(f^2)(a)\right)^-\right) \\ = J(f^1_n)\left(e'_n\left(I(f^2)(a) - \delta/2\right)\right). \tag{4.28}$$

From definition of $J^-_{I,a}$ and the constantness of $J(f^1)$ on $[I(f^2)(a) - \delta, I(f_2)(a)[$ we also have

$$J^-_{I,a}(f^1, f^2) := J(f^1)\left(I(f^2)(a)-\right) = J(f^1)\left(I(f^2)(a) - \delta/2\right). \tag{4.29}$$

Combining (4.28), (4.29) and the fourth point gives that, as n goes to infinity, $J^-_{I,a}\left((f^1_n, f^2_n)\right)$ converges to $J^-_{I,a}\left((f^1, f^2)\right)$.

For $J_{I,a}$, we prove in a similar way as above that $J(f^1_n)$ is constant on $[e_n\left(I(f^2)(a)\right), e'_n\left(I(f^2)(a) + \delta/2\right)]$ so, as in (4.28) we have for n large enough

$$J_{I,a}\left((f^1_n, f^2_n)\right) = J(f^1_n)\left(e'_n\left(I(f^2)(a) + \delta/2\right)\right),$$

which, combined with the analogous of (4.29)

$$J_{I,a}(f^1, f^2) = J(f^1)\left(I(f^2)(a) + \delta/2\right)$$

allows us to conclude, using the fourth point, that $J_{I,a}\left((f^1_n, f^2_n)\right)$ converges to $J_{I,a}\left((f^1, f^2)\right)$ as n goes to infinity. Therefore, both $J^-_{I,a}$ and $J_{I,a}$ are continue at (f^1, f^2). The continuity of the other functionals are proved similarly. □

Lemma 4.6. *For any (f^1, f^2) in $D(\mathbb{R}_+, \mathbb{R}^2)$ that satisfy the hypothesis of lemma 4.5 and such that the sizes of the jumps of f^1 are all distinct, F^* is continuous at (f^1, f^2) in the J_1 topology.*

Proof: The proof follows mainly the steps of Lemma 4.5, we keep the same notation. The jump which takes place at the instant $F^*(f^1, f^2)$ has value h^-. With the additional hypothesis that the values of the jumps for f^1 are all different we have unicity for the value h^-. Let us define h', the second highest jump f^1 before instant $I(f^2)(1)$. With the additional condition that $\alpha < \frac{1}{8}(h^- - h')$ we have with the same arguments as in the proof of the continuity of J that for any $n \geq n_0$, f^1_n effectuates at $e_n\left(F^*(f^1, f^2)\right)$ a jump larger than $h^- - 2\alpha$, and larger than all the other jumps of f^1_n before $e_n(I(f^2)(1)-) = I(f^2_n)(1)$ which are smaller than $h' + 2\alpha$. So for $n \geq n_0$, the largest jump of f^1 before $I(f^2_n)(1)$ is obtained for $e_n\left(F^*(f^1, f^2)\right)$, that is to say for any $n \geq n_0$,

$$F^*\left((f^1_n, f^2_n)\right) = e_n\left(F^*(f^1, f^2)\right),$$

this implies $F^*\left((f^1_n, f^2_n)\right) \longrightarrow_{n\to\infty} F^*(f^1, f^2)$. □

5. Supremum of the Local time - and other functionals

5.1. *Supremum of the local time (proof of Theorem 1.3).*

First, notice that since the diffusion X is almost surely transient to the right, the random variable $\sup_{x<0} \mathcal{L}(+\infty, x)$ is $\mathbb{P}$-almost surely finite. So almost surely,

$$\lim_{t\to+\infty} \sup_{x<0} \mathcal{L}(t, x)/t = 0.$$

As a consequence, we only have to study the asymptotic behavior of $\sup_{x\geq 0} \mathcal{L}(t, x)/t$ as $t \to +\infty$.

We start with the proof of the following proposition, which makes a link between the supremum of the local time and the process $(Y_1, Y_2)^t$.

Proposition 5.1. *Let $\alpha > 0$. For any $\varepsilon > 0$ and large t,*

$$\mathcal{P}_1^- - v(\varepsilon, t) \leq \mathbb{P}\left(\sup_{x\geq 0} \mathcal{L}(t,x)/t \leq \alpha\right) \leq \mathcal{P}_1^+ + v(\varepsilon, t),$$

where

$$\mathcal{P}_1^{\pm} := \mathbb{P}\left[\left(1 - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}-1}\right)\frac{\bar{\ell}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\ell}_{\mathcal{N}_t^{2\varepsilon}-1}}{(\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}-1})} \leq \alpha_t^{\pm},\ \max_{1\leq j\leq \mathcal{N}_t^{2\varepsilon}-1} \frac{\ell_j}{t} \leq \alpha_t^{\pm}\right],$$

and with $\bar{\mathcal{H}}_k := Y_2^t\big(ke^{-\kappa\phi(t)}\big) = \frac{1}{t}\sum_{i=1}^k \mathcal{H}_i$, $\bar{\ell}_k := Y_1^t\big(ke^{-\kappa\phi(t)}\big) = \frac{1}{t}\sum_{i=1}^k \ell_i$ for any $k \in \mathbb{N}$, $\mathcal{N}_t^{2\varepsilon} := \inf\big\{m \geq 1, \bar{\mathcal{H}}_m > 1 - 2\varepsilon\big\}$, $\alpha_t^{\pm} := \alpha\big(1 \pm (\log\log t)^{-1/2}\big)$, and v is a positive function such that $\lim_{t\to+\infty} v(\varepsilon,t) \leq const \times \varepsilon^{\kappa\wedge(1-\kappa)}$.

The proof of this proposition relies on the three following lemmata. The first one deals with the local time at the h_t-minima for which the diffusion X already escaped before time t. The second deals with the local time at the last h_t-minimum m_{N_t} in the remaining time before time t. Finally the last one is a technical point.

Lemma 5.2. *For any large $t > 0$, $2 \leq k \leq n_t$, any $x > 0$ and $\gamma > 0$ possibly depending on t, define the repartition function*

$$F_\gamma(x) := \mathbb{P}\left(\max_{1\leq j\leq k-1} \mathcal{L}\big(H\big(\tilde{L}_j\big), \tilde{m}_j\big) \leq \gamma t,\ \sum_{i=1}^{k-1} U_i \leq xt\right).$$

Then for large t, for all $2 \leq k \leq n_t$, $x > 0$ and $\gamma > 0$,

$$F_\gamma^-(x) - e^{-D_1 h_t} \leq F_\gamma(x) \leq F_\gamma^+(x) + e^{-D_1 h_t},$$

where $F_\gamma^{\pm}(x) := \mathbb{P}\left(\max_{1\leq j\leq k-1} \ell_j \leq \gamma_t^{\pm} t, \sum_{i=1}^{k-1} \mathcal{H}_i \leq x_t^{\pm} t\right)$ with $\gamma_t^{\pm} := \gamma(1 \pm 2\varepsilon_t)$, $x_t^{\pm} := x(1 \pm 2\varepsilon_t)$, ε_t and D_1 are given in Proposition 3.5.

Lemma 5.3. *For any $t > 0$, define for every $\gamma > 0$ and $0 < x < 1$ possibly depending on t,*

$$f_\gamma(x) := E\left[\mathbb{P}_{\tilde{m}_1}^{W_\kappa}\left(\mathcal{L}_{X'}(t(1-x), \tilde{m}_1) \leq \gamma t, H'(\tilde{L}_1) > t(1-x), H'(\tilde{L}_1) < H'(\tilde{L}_1^-)\right)\right].$$

For such t, γ and x, we also introduce

$$\tilde{f}_\gamma(x) := E\left(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}\left(\sup_{y\in\mathcal{D}_1} \mathcal{L}_{X'}[t(1-x), y] \leq \gamma t, H'(\tilde{L}_1) > t(1-x), H'(\tilde{L}_1) < H'(\tilde{L}_1^-)\right)\right),$$

with $\mathcal{D}_1$ defined in (3.13). Here X' is an independent copy of X starting at $\tilde{m}_1$, and the definition of H' for X' is the same as the definition of H for X. Let $\varepsilon \in (0, 1/2)$. There exists $c_2 > 0$ such that for large t, for every $x \in [\varepsilon, 1-\varepsilon]$,

$$f_\gamma^-(x) - o(n_t^{-1}) \leq \tilde{f}_\gamma(x) \leq f_\gamma(x) \leq f_\gamma^+(x) + o(n_t^{-1}), \tag{5.1}$$

with $f_\gamma^{\pm}(x) := \mathbb{P}\left(\frac{1}{R_1} \leq \frac{\gamma}{1-x}(1 \pm \varepsilon_t'), \mathcal{H}_1 > t(1-x)(1 \mp \varepsilon_t')\right)$ and $\varepsilon_t' = e^{-c_2 h_t}$.

Lemma 5.4. *For any $0 < a < 1/4$, we have for any $t > 0$,*

$$\sum_{1\leq k\leq n_t} \mathbb{P}\left[\bar{\mathcal{H}}_k > 1-a/2,\ 1-2a < \bar{\mathcal{H}}_{k-1} \leq 1-3a/4\right] \leq s(a,t), \tag{5.2}$$

with $s(a,t)$ such that $\lim_{t\to+\infty} s(a,t) = const \times a^{1-\kappa}$. For any $\varepsilon \in (0,1/2)$,

$$\forall t > 0, \qquad \mathbb{P}\left[\varepsilon t \leq H(m_{N_t}) \leq (1-\varepsilon)t\right] \geq 1 - \tilde{s}(\varepsilon,t), \tag{5.3}$$

with $\tilde{s}(\varepsilon,t)$ such that $\lim_{t\to+\infty} \tilde{s}(\varepsilon,t) = const \times \varepsilon^{(1-\kappa)\wedge\kappa}$.

We postpone the proof of these lemmata after the proof of Proposition 5.1.

Proof of Proposition 5.1: Recall from (1.3) that N_t is the largest index such that $\sup_{s\leq t} X(s) \geq m_{N_t}$. In particular, $H\big(\tilde{L}_j\big) \leq H(\tilde{m}_{j+1}) = H(m_{j+1}) \leq H(m_{N_t}) \leq t$ for every $1 \leq j \leq N_t - 1$ on $\mathcal{V}_t \cap \{N_t \leq n_t\}$. The main idea is to use the fact that the supremum of the local time at time t is achieved in the neighborhood of the h_t-minima m_i, $1 \leq i \leq n_t$.

We start with the upper bound. Let $\alpha > 0$ and $0 < \varepsilon < 1/2$. Notice that $\mathbb{P}(N_t = 0, \mathcal{V}_t) \leq \mathbb{P}[H(\tilde{m}_1) > t] \leq C_2 v_t$ by (3.1). Using (3.1), (3.2), (3.3), (5.3), Lemma 2.2 and Remark 2.3, we have for t large enough,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{x\in\mathbb{R}} \mathcal{L}(t,x) \leq \alpha t\right) \leq E\left[\mathbb{P}^{W_\kappa}\left(\max_{1\leq j\leq N_t} \mathcal{L}(t,m_j) \leq \alpha t\right)\right] \qquad (5.4)\\
\leq& E\left[\mathbb{P}^{W_\kappa}\left(\max_{1\leq j\leq N_t-1} \mathcal{L}\big[H\big(\tilde{L}_j\big),\tilde{m}_j\big] \leq \alpha t, \mathcal{L}(t,\tilde{m}_{N_t}) \leq \alpha t, \mathcal{Q}, \mathcal{B}_1(n_t), \mathcal{B}_2(n_t), \mathcal{V}_t\right)\right]\\
&+ \bar{s}(\varepsilon,t).
\end{aligned}$$

with $\mathcal{Q} := \{\varepsilon t \leq H(m_{N_t}) \leq (1-\varepsilon)t,\ 1 \leq N_t \leq n_t\}$ and $\bar{s}$ satisfying $\lim_{t\to+\infty} \bar{s}(\varepsilon,t) \leq C_+ \varepsilon^{(1-\kappa)\wedge\kappa}$. We will introduce in what follows different measures denoted by the letter ν; they depend on k but we do not write k as a subscript to simplify the notation. First, define two measures $\nu_1^{W_\kappa}$ and $\nu_2^{W_\kappa}$ on $(0,1)$ by, for every $0 < y < 1$,

$$\begin{aligned}
\nu_1^{W_\kappa}(y) :=& \nu_1^{W_\kappa}([0,y])\\
:=& \mathbb{P}^{W_\kappa}\Big(\max_{1\leq j\leq k-1} \mathcal{L}\big[H\big(\tilde{L}_j\big),\tilde{m}_j\big] \leq \alpha t, H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t, H(\tilde{m}_k) \leq yt\Big),\\
\nu_2^{W_\kappa}(y) :=& \nu_2^{W_\kappa}([0,y])\\
:=& \mathbb{P}^{W_\kappa}_{\tilde{m}_k}\Big(\mathcal{L}_{X'}[t(1-y),\tilde{m}_k] \leq \alpha t,\ H'(\tilde{m}_{k+1}) > t(1-y),\\
&\qquad H'\big(\tilde{m}_{k+1}\big) < H'\big(\tilde{L}_k^-\big),\ H'\big(\tilde{m}_{k+1}\big) - H'\big(\tilde{L}_k\big) \leq \tilde{v}_t\Big),
\end{aligned}$$

with X' a diffusion starting from $\tilde{m}_k$ independent of X (conditionally on W_κ), and H' has the same definition as H (see (1.1)) but for X'. Partitioning on the values of N_t, and $H(\tilde{m}_k)$, we obtain by the strong markov property (applied at time $H(\tilde{m}_k)$ under $\mathbb{P}^{W_\kappa}$), that the probability $E\big[\mathbb{P}^{W_\kappa}(.)\big]$ in the line below (5.4) is smaller than

$$\sum_{1\leq k\leq n_t} \int_\varepsilon^{1-\varepsilon} E\left(\nu_2^{W_\kappa}(x)\mathrm{d}\nu_1^{W_\kappa}(x)\right) = \sum_{1\leq k\leq n_t} E\left[\int_\varepsilon^{1-\varepsilon} \nu_2^{W_\kappa}(x)\mathrm{d}\nu_1^{W_\kappa}(x)\right]. \tag{5.5}$$

The next step is to prove that the previous expectation can be approximated by a product of expectations. First notice that both $y \to \nu_1^{W_\kappa}(y)$ and $y \to \nu_2^{W_\kappa}(y)$ are positive increasing. So integrating by parts

$$\begin{aligned}\int_\varepsilon^{1-\varepsilon} \nu_2^{W_\kappa}(x)\mathrm{d}\nu_1^{W_\kappa}(x) &= \Big[\nu_2^{W_\kappa}(x)\nu_1^{W_\kappa}(x)\Big]_\varepsilon^{1-\varepsilon} - \int_\varepsilon^{1-\varepsilon} \nu_1^{W_\kappa}(x)\mathrm{d}\nu_2^{W_\kappa}(x) \\ &\le \Big[\nu_2^{W_\kappa}(x)\nu_1^{W_\kappa}(x)\Big]_\varepsilon^{1-\varepsilon} - \int_\varepsilon^{1-\varepsilon} \widetilde{\nu}_1^{W_\kappa}(x)\mathrm{d}\nu_2^{W_\kappa}(x) \\ &= \Big[\nu_2^{W_\kappa}(x)\Big(\nu_1^{W_\kappa}(x) - \widetilde{\nu}_1^{W_\kappa}(x)\Big)\Big]_\varepsilon^{1-\varepsilon} + \mathcal{I}_1, \end{aligned} \tag{5.6}$$

with $\widetilde{\nu}_1^{W_\kappa}(x) := \mathbb{P}^{W_\kappa}\Big(\mathcal{G}_1, H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t, \sum_{i=1}^{k-1} U_i + \tilde{v}_t \le xt\Big) \le \nu_1^{W_\kappa}(x)$ and $\mathcal{G}_1 := \{\max_{1\le j\le k-1} \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) \le \alpha t\}$ and

$$\mathcal{I}_1 := \int_\varepsilon^{1-\varepsilon} \nu_2^{W_\kappa}(x)\mathrm{d}\tilde{\nu}_1(x) \le \int_\varepsilon^{1-\varepsilon} \nu_2^{W_\kappa}(x)\mathrm{d}\nu_3^{W_\kappa}(x) =: \mathcal{I}_1',$$

$$\nu_3^{W_\kappa}(x) := \mathbb{P}^{W_\kappa}\left(\mathcal{G}_1, \sum_{i=1}^{k-1} U_i + \tilde{v}_t \le xt\right).$$

First, we deal with what is going to be a negligible part, that is to say the first term in (5.6). As $\nu_1^{W_\kappa}(x) \le \mathbb{P}^{W_\kappa}\Big(\mathcal{G}_1, H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t, \sum_{i=1}^{k-1} U_i \le xt\Big)$ because by definition $\sum_{i=1}^{k-1} U_i < H(\tilde{m}_k)$, we have, for $\varepsilon < x < 1-\varepsilon$,

$$\Big|\nu_1^{W_\kappa}(x) - \widetilde{\nu}_1^{W_\kappa}(x)\Big| \le \mathbb{P}^{W_\kappa}\left(xt - \widetilde{v}_t < \sum_{i=1}^{k-1} U_i \le xt\right) =: h_k(x).$$

so $\big[\nu_2^{W_\kappa}(x)\big(\nu_1^{W_\kappa}(x) - \widetilde{\nu}_1^{W_\kappa}(x)\big)\big]_\varepsilon^{1-\varepsilon} \le \nu_2^{W_\kappa}(1-\varepsilon)h_k(1-\varepsilon) + \nu_2^{W_\kappa}(\varepsilon)h_k(\varepsilon)$. Notice that $\sum_{i=1}^{k-1} U_i$ is measurable with respect to $\sigma\big(X(s), 0 \le s \le H\big(\tilde{L}_{k-1}\big); W_\kappa(x), x \le \tilde{L}_{k-1}^+\big)$, since $\tilde{L}_{k-1} \le \tilde{L}_{k-1}^+$, whereas the event in the definition of $\nu_2^{W_\kappa}$ belongs to

$$\sigma\big(X'(s), 0 \le s \le \min\big(H'\big(\tilde{L}_k^-\big), H'\big(\tilde{m}_{k+1}\big)\big); W_\kappa(x) - W_\kappa(\tilde{m}_k), \tilde{L}_{k-1}^+ \le x \le \tilde{L}_{k+1}^+\big),$$

with X' an independent copy of X starting at $\tilde{m}_k$.
So independence of X and X', and independence of the two portions of the environment involved (see Lemma 2.2) imply independence between $\nu_2^{W_\kappa}$ and h_k. Hence,

$$\begin{aligned} & E\Big(\Big[\nu_2^{W_\kappa}(x)\Big(\nu_1^{W_\kappa}(x) - \widetilde{\nu}_1^{W_\kappa}(x)\Big)\Big]_\varepsilon^{1-\varepsilon}\Big) \\ \le\ & E\Big[\nu_2^{W_\kappa}(1-\varepsilon)\Big]E\big[h_k(1-\varepsilon)\big] + E\Big[\nu_2^{W_\kappa}(\varepsilon)\Big]E\big[h_k(\varepsilon)\big] \\ =\ & E\Big[\widetilde{\nu}_2^{W_\kappa}(1-\varepsilon)\Big]E\big[h_k(1-\varepsilon)\big] + E\Big[\widetilde{\nu}_2^{W_\kappa}(\varepsilon)\Big]E\big[h_k(\varepsilon)\big]. \end{aligned} \tag{5.7}$$

with for any x,

$$\begin{aligned}\widetilde{\nu}_2^{W_\kappa}(x) \ := \ & \mathbb{P}_{\tilde{m}_1}^{W_\kappa}\Big(\mathcal{L}_X[t(1-x), \tilde{m}_1] \le \alpha t,\ H(\tilde{m}_2) > t(1-x), \\ & \qquad H(\tilde{m}_2) < H\big(\tilde{L}_1^-\big),\ H\big(\tilde{m}_2\big) - H\big(\tilde{L}_1\big) \le \tilde{v}_t\Big).\end{aligned}$$

As $E\big(\tilde{\nu}_2^{W_\kappa}(x)\big) \le \mathbb{P}[U_1 > t(1-x) - \tilde{v}_t]$ and for every small $\varepsilon > 0$ and t large enough $h_k(x) \le \mathbb{P}^{W_\kappa}\Big((x-\varepsilon)t < \sum_{i=1}^{k-1} U_i \le xt\Big)$ we can apply Proposition 3.5, we get

$$\begin{aligned} & E\big[h_k(1-\varepsilon)\big]E\Big[\tilde{\nu}_2^{W_\kappa}(1-\varepsilon)\Big] \\ \le\ & \mathbb{P}\left(\frac{1-2\varepsilon}{1+\varepsilon_t} < \sum_{i=1}^{k-1}\frac{\mathcal{H}_i}{t} \le \frac{1-\varepsilon}{1-\varepsilon_t}\right)\mathbb{P}\left(\mathcal{H}_1 > \frac{t\varepsilon - \tilde{v}_t}{1+\varepsilon_t}\right) + 3e^{-D_1 h_t}. \end{aligned}$$

By (4.2) and the first part of Lemma 6.2, for any $0<a<1$ and $b>0$,

$$\begin{aligned} \lim_{t\to+\infty}\sum_{1\le k\le n_t}\mathbb{P}\left(1-a<\sum_{i=1}^{k-1}\frac{\mathcal{H}_i}{t}\le 1\right)\mathbb{P}(\mathcal{H}_1 > tb) &= \frac{\text{const}}{b^\kappa}\int_{1-a}^1 y^{\kappa-1}\mathrm{d}y \\ &\le \frac{\text{const}}{b^\kappa}(1-(1-a)^\kappa). \quad (5.8) \end{aligned}$$

Therefore, we obtain

$$\sum_{1\le k\le n_t} E\Big[\tilde{\nu}_2^{W_\kappa}(1-\varepsilon)\Big]E\big[h_k(1-\varepsilon)\big] \le C_+\cdot u(t,\varepsilon)$$

with u a positive function such that $\lim_{t\to+\infty} u(t,\varepsilon) = \max(\varepsilon^{1-\kappa},\varepsilon^\kappa)$. A similar argument also works for the second term in (5.7), which yields

$$\sum_{1\le k\le n_t} E\left[\Big[\nu_2^{W_\kappa}(x)\Big(\nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x)\Big)\Big]_\varepsilon^{1-\varepsilon}\right] \le 2C_+\cdot u(t,\varepsilon). \tag{5.9}$$

We now deal with $\mathcal{I}_1'$. By independence between X and X', and the independent parts of the potential W_κ involved in $\nu_2^{W_\kappa}(x)$ and $\nu_3^{W_\kappa}(x)$,

$$E(\mathcal{I}_1') = \int_\varepsilon^{1-\varepsilon}\nu_2(x)\mathrm{d}\nu_3(x), \tag{5.10}$$

with $\nu_2(x) := E\big(\nu_2^{W_\kappa}(x)\big) = E\big(\tilde{\nu}_2^{W_\kappa}(x)\big)$ and $\nu_3(x) := E\big(\nu_3^{W_\kappa}(x)\big)$.

By the lower bound in Lemma 5.2, we have $\nu_3(x) = F_\alpha(x-\tilde{v}_t/t) \ge F_\alpha^-(x-\tilde{v}_t/t) - e^{-D_1h_t}$ for every $x>\varepsilon$ for large t. So, again since $y\to\nu_2(y)$ is positive increasing and ν_3 is a repartition function, integrating by parts twice as in (5.6) gives with the change of variables $u = x-\tilde{v}_t/t$,

$$\begin{aligned} \int_\varepsilon^{1-\varepsilon}\nu_2(x)\mathrm{d}\nu_3(x) \le\ & \int_{\varepsilon-\tilde{v}_t/t}^{1-\varepsilon-\tilde{v}_t/t}\nu_2(x+\tilde{v}_t/t)\mathrm{d}F_\alpha^-(x) + e^{-D_1h_t} \\ & + \Big[\Big(F_\alpha(x) - F_\alpha^-(x)\Big)\nu_2(x+\tilde{v}_t/t)\Big]_{\varepsilon-\tilde{v}_t/t}^{1-\varepsilon-\tilde{v}_t/t}. \end{aligned} \tag{5.11}$$

Recall (see before Lemma 3.1) that $\tilde{v}_t/t = 2/\log(h_t) = o(1)$ as $t\to+\infty$. Then we can prove in a similar way we have obtained (5.9) that:

$$\sum_{1\le k\le n_t}\Big[\Big(F_\alpha(x) - F_\alpha^-(x)\Big)\nu_2(x+\tilde{v}_t/t)\Big]_{\varepsilon-\tilde{v}_t/t}^{1-\varepsilon-\tilde{v}_t/t} \le C_+\cdot u(t,\varepsilon), \tag{5.12}$$

with as usual a possibly enlarged C_+. Indeed by Lemma 5.2, $-\big(F_\alpha(\varepsilon-\tilde{v}_t/t) - F_\alpha^-(\varepsilon-\tilde{v}_t/t)\big)\nu_2(\varepsilon) \le e^{-D_1t} = o(n_t^{-1})$ for every $1\le k\le n_t$ for large t, and $\big(F_\alpha - F_\alpha^-\big)(1-\varepsilon-\tilde{v}_t/t) \le \big(F_\alpha^+ - F_\alpha^-\big)(1-\varepsilon-\tilde{v}_t/t) + e^{-D_1t} \le \mathbb{P}\big(\max_{1\le j\le k-1}\ell_j \in [\gamma_t^-t,\gamma_t^+t]\big) + \mathbb{P}\big(\sum_{i=1}^{k-1}\mathcal{H}_i \in [x_t^-t, x_t^+t]\big) + e^{-D_1t}$ for every $k\le n_t$ for large t, with $\gamma =$

α and $x = 1-\varepsilon-\tilde{v}_t/t$. The first probability is less than $n_t\mathbb{P}(S_1\mathbf{e}_1 \in [\gamma_t^- t, \gamma_t^+ t]) = n_t\mathbb{E}\big(\int_{\gamma t(1-2\varepsilon_t)/S_1}^{\gamma t(1+2\varepsilon_t)/S_1} e^{-u/2}\mathrm{d}u/2\big) \le 8n_t\varepsilon_t \sup_{v\ge 0}(ve^{-v}) = o(1/n_t)$, whereas the second one is treated as (5.8), which leads to (5.12).

So the important term in the right hand side of inequality (5.11) comes from the integral. We now work on $\nu_2(x)$. We have,

$$\begin{aligned}
&\nu_2(x)\\
&\le E\big(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}\big[\mathcal{L}_X(t(1-x),\tilde{m}_1) \le \alpha t,\ H\big(\tilde{L}_1\big) > t(1-x)-\tilde{v}_t,\ H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\big]\big)\\
&\le E\big(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}\big[\mathcal{L}_X(t(1-x)-\tilde{v}_t,\ \tilde{m}_1) \le \alpha t,\\
&\qquad\qquad H\big(\tilde{L}_1\big) > t(1-x)-\tilde{v}_t,\ H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\big]\big) = f_\alpha(x+\tilde{v}_t/t),
\end{aligned}$$

as defined in Lemma 5.3. Then, as $F_\alpha^-(x)$ is positive and increasing in x, using Lemma 5.3 with $\gamma=\alpha$, we obtain

$$\int_{\varepsilon-\tilde{v}_t/t}^{1-\varepsilon-\tilde{v}_t/t} \nu_2(x+\tilde{v}_t/t)\mathrm{d}F_\alpha^-(x) \le \int_{\varepsilon-\tilde{v}_t/t}^{1-\varepsilon-\tilde{v}_t/t} f_\alpha^+(x+2\tilde{v}_t/t)\mathrm{d}F_\alpha^-(x) + o(n_t^{-1}). \tag{5.13}$$

Now, as $f_\alpha^+(x+2\tilde{v}_t/t)$ can be written (since $\mathcal{H}_k = \ell_k R_k$, see Proposition 3.5),

$$f_\alpha^+(x+2\tilde{v}_t/t) = \mathbb{P}\left((1-x-2\tilde{v}_t/t)\frac{\ell_k}{\mathcal{H}_k} \le \alpha(1+\varepsilon_t'), \mathcal{H}_k > t(1-x-2\tilde{v}_t/t)(1-\varepsilon_t')\right),$$

we get by independence of the random variables $((\ell_j,\mathcal{H}_j), j\le n_t)$,

$$\begin{aligned}
&\int_{\varepsilon-\tilde{v}_t/t}^{1-\varepsilon-\tilde{v}_t/t} f_\alpha^+(x+2\tilde{v}_t/t)\mathrm{d}F_\alpha^-(x)\\
&\le\ \mathbb{P}\bigg[\big(1-\bar{\mathcal{H}}_{k-1}\big)\frac{\bar{\ell}_k-\bar{\ell}_{k-1}}{\bar{\mathcal{H}}_k-\bar{\mathcal{H}}_{k-1}} \le \alpha+\tilde{\varepsilon}_t(k),\ \bar{\mathcal{H}}_k \ge 1-\delta_t',\\
&\qquad\qquad \max_{1\le j\le k-1}\frac{\ell_j}{t} \le \alpha,\ \bar{\mathcal{H}}_{k-1} \le 1-\varepsilon+\delta_t'\bigg],
\end{aligned} \tag{5.14}$$

with $\delta_t' := 3\tilde{v}_t/t$, $\tilde{\varepsilon}_t(k) := (\alpha+\ell_k/\mathcal{H}_k)\,\delta_t'$.

The idea now is to make appear the event $\{\mathcal{N}_t^{2\varepsilon}=k\}$ in the above probability (recall the definition of $\mathcal{N}_t^{2\varepsilon}$ given in Proposition 5.1) and then sum over k.

We first prove that the sum over $k\le n_t$, of the above probability is small if we intersect its event with the event $\{\mathcal{N}_t^{2\varepsilon}\ne k\}$. In other words, let us prove that

$$\textstyle\sum_1 := \displaystyle\sum_{1\le k\le n_t}\mathbb{P}\left[\bar{\mathcal{H}}_k \ge 1-\delta_t',\ \bar{\mathcal{H}}_{k-1}\le 1-\varepsilon+\delta_t',\ \mathcal{N}_t^{2\varepsilon}\ne k\right] \tag{5.15}$$

is small. As $\{\mathcal{N}_t^{2\varepsilon}\ne k\} = \{\bar{\mathcal{H}}_k \le 1-2\varepsilon\}\cup\{\bar{\mathcal{H}}_{k-1} > 1-2\varepsilon\}$, and since for t large enough, $\{\bar{\mathcal{H}}_k \ge 1-\delta_t'\}\cap\{\bar{\mathcal{H}}_k\le 1-2\varepsilon\} = \emptyset$, we have

$$\textstyle\sum_1 \le \displaystyle\sum_{1\le k\le n_t}\mathbb{P}\left[\bar{\mathcal{H}}_k \ge 1-\delta_t',\ 1-2\varepsilon < \bar{\mathcal{H}}_{k-1}\le 1-\varepsilon+\delta_t'\right].$$

Therefore, for t large enough, with $s(\varepsilon,t)$ defined in Lemma 5.4,

$$\textstyle\sum_1 \le \displaystyle\sum_{1\le k\le n_t}\mathbb{P}\left[\bar{\mathcal{H}}_k > 1-\varepsilon/2,\ 1-2\varepsilon < \bar{\mathcal{H}}_{k-1}\le 1-3\varepsilon/4\right] \le s(\varepsilon,t) \tag{5.16}$$

by (5.2). Finally, combining equations from (5.10) to (5.16) leads to

$$\begin{aligned}
&\sum_{1\le k\le n_t} E(\mathcal{I}_1') \\
\le\ & \mathbb{P}\left[(1-\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}-1})\frac{\bar{\ell}_{\mathcal{N}_t^{2\varepsilon}}-\bar{\ell}_{\mathcal{N}_t^{2\varepsilon}-1}}{\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}}-\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}-1}}\le \alpha+\tilde{\varepsilon}_t(\mathcal{N}_t^{2\varepsilon}),\ \max_{1\le j\le \mathcal{N}_t^{2\varepsilon}-1}\frac{\ell_j}{t}\le\alpha\right] \\
&+s(\varepsilon,t)+C_+u(t,\varepsilon)+o(1). \qquad (5.17)
\end{aligned}$$

To finish we have to deal with $\tilde{\varepsilon}_t(\mathcal{N}_t^{2\varepsilon})$, a basic computation partitioning on the values of $\mathcal{N}_t^{2\varepsilon}$, shows that $\mathbb{P}\big[\tilde{\varepsilon}_t(\mathcal{N}_t^{2\varepsilon}))\ge\alpha\sqrt{\delta_t'}/6\big]\le C_+\mathbb{P}\left(R_1\le\sqrt{\delta_t'}\right)=o(1)$ as R_1 converges in distribution to $\mathcal{R}_\kappa$ which is almost surely positive. Collecting this last fact, (5.4), (5.5), (5.6), (5.9) and (5.17) finish the proof of the upper bound.

Proof of the lower bound:

The proof here follows the same line as the upper bound. The main difference comes from the fact that we can no longer use the inequality $\sup_{x\in\mathbb{R}}\mathcal{L}(t,x)\ge\sup_{1\le j\le N_t}\mathcal{L}(t,m_j)$. So for this part of the proof we stress on what is different from the upper bound, and refer to the previous computations when very few changes occur.

Assume for the moment that

$$\mathbb{P}\Big(\Big\{\sup_{x\in\mathbb{R}}\mathcal{L}(t,x)\ge 2\tilde{w}_t\Big\}=:\mathcal{E}_2\Big)\ge 1-o(1), \qquad (5.18)$$

with $\tilde{w}_t:=te^{(\kappa(1+3\delta)-1)\phi(t)}$, and recall that δ is chosen small enough such that $\kappa(1+3\delta)<1$ (see Lemma 3.2). This fact (5.18) is a direct consequence of the upper-bound of $\mathbb{P}(\sup_{x\in\mathbb{R}}\mathcal{L}(t,x)\le\alpha t)$ (see at the beginning of the proof of Theorem 1.3 page 908 for a proof of (5.18)). Recall (3.13), and define for any $\ell\ge 1$,

$$\begin{aligned}
&\mathcal{E}_3(\ell):=\mathcal{E}_3^1(\ell)\cap\mathcal{E}_3^2(\ell),\quad \text{with} \\
&\mathcal{E}_3^1(\ell):=\bigcap_{j=1}^{\ell-1}\left\{\sup_{x\in\mathcal{D}_j}\Big[\mathcal{L}\big(H\big(\tilde{L}_j\big),x\big)-\mathcal{L}\big(H\big(\tilde{m}_j\big),x\big)\Big]\le t\tilde{\alpha}_t\right\}, \\
&\mathcal{E}_3^2(\ell):=\left\{\sup_{x\in\mathcal{D}_\ell}\Big[\mathcal{L}(t,x)-\mathcal{L}\big(H\big(\tilde{m}_\ell\big),x\big)\Big]\le t\tilde{\alpha}_t\right\},
\end{aligned}$$

with $\tilde{\alpha}_t:=(\alpha t-2\tilde{w}_t)/t$. Recall the definitions of the events $\mathcal{B}_i$, $1\le i\le 4$ in Sections 3.1 and 3.2. We have for large t,

$$\begin{aligned}
&\big\{\sup\nolimits_{x\in\mathbb{R}_+}\mathcal{L}(t,x)\le\alpha t\big\}\cap\mathcal{V}_t\cap\mathcal{E}_2\cap\{N_t\le n_t\}\cap\cap_{i=1}^4\mathcal{B}_i(n_t) \\
\supset\ & \mathcal{E}_3(N_t)\cap\mathcal{V}_t\cap\mathcal{E}_2\cap\{N_t\le n_t\}\cap\cap_{i=1}^4\mathcal{B}_i(n_t).
\end{aligned}$$

Indeed, $\mathcal{L}(t,x)\le\tilde{w}_t$ for every $x\in\big(\mathbb{R}_+-\cup_{j=1}^{n_t}\big[\tilde{L}_j^-,\tilde{L}_j\big]\big)$ on $\mathcal{B}_2(n_t)\cap\mathcal{B}_3(n_t)\cap\mathcal{V}_t\cap\{N_t\le n_t\}$, and on the same event intersected with $\mathcal{B}_4(n_t)$, $\mathcal{L}(t,x)\le\tilde{w}_t+te^{-2\phi(t)}<2\tilde{w}_t$ for every $x\in\cup_{j=1}^{n_t}\big(\big[\tilde{L}_j^-,\tilde{L}_j\big]\cap\overline{\mathcal{D}_j}\big)$, whereas for $x\in\mathcal{D}_j$, $\mathcal{L}(H(\tilde{m}_j),x)\le\tilde{w}_t$ if $j\le n_t$ and $\mathcal{L}(t,x)-\mathcal{L}\big(H\big(\tilde{L}_j\big),x\big)\le\tilde{w}_t$ if $j<N_t$. Notice that by Lemmata 2.2, 3.1, 3.2, 3.4 and the above assumption (5.18),

$$\mathbb{P}\big(\mathcal{V}_t\cap\mathcal{E}_2\cap\{N_t\le n_t\}\cap\cap_{i=1}^4\mathcal{B}_i(n_t)\big)\ge 1-o(1).$$

We now deal with $\mathbb{P}(\mathcal{E}_3(N_t)\cap\mathcal{B}_1(N_t)\cap\mathcal{B}_2(n_t)\cap\mathcal{V}_t\cap\{N_t\le n_t\})$. Using Lemma 2.2, the fact that $H(\tilde{L}_k)\le H(\tilde{m}_{k+1})$ and the strong Markov property with respect to

$\mathbb{P}^{W_\kappa}$, we obtain

$$\begin{aligned}&\mathbb{P}(\mathcal{E}_3(N_t)\cap\mathcal{B}_1(N_t)\cap\mathcal{B}_2(n_t)\cap\mathcal{V}_t\cap\mathcal{Q})\\&\geq\sum_{k=1}^{n_t}E\left(\int_\varepsilon^{1-\varepsilon}\nu_4^{W_\kappa}(y)\mathbb{P}^{W_\kappa}\left(\mathcal{E}_3^1(k),\mathcal{B}_1(k),\mathcal{B}_2(k-1),H(\tilde{m}_k)/t\in \mathrm{d}y\right)\right)-o(1)\end{aligned}$$

with

$$\begin{aligned}&\nu_4^{W_\kappa}(y)\\:=&\mathbb{P}^{W_\kappa}_{\tilde{m}_k}\left(\sup_{x\in\mathcal{D}_k}\mathcal{L}_X(t(1-y),x)\leq t\tilde{\alpha}_t,\ H(\tilde{L}_k)>t(1-y),\ H(\tilde{L}_k)<H(\tilde{L}_k^-)\right),\end{aligned}$$

Now, by computations similar to the ones giving the upper bounds in (5.9) and (5.10), we have

$$\begin{aligned}&\mathbb{P}(\mathcal{E}_3(N_t)\cap\mathcal{B}_1(N_t)\cap\mathcal{B}_2(n_t)\cap\mathcal{V}_t,\mathcal{Q})\\&\geq\sum_{k=1}^{n_t}\int_\varepsilon^{1-\varepsilon}E\left(\nu_4^{W_\kappa}(y)\mathrm{d}\nu_5^{W_\kappa}(y)\right)-o(1)=\sum_{k=1}^{n_t}\int_\varepsilon^{1-\varepsilon}\nu_4(y)\mathrm{d}\nu_5(y)-o(1).\end{aligned}$$

with $\nu_5^{W_\kappa}(y):=\mathbb{P}^{W_\kappa}\left(\mathcal{E}_3^1(k),\mathcal{B}_1(k),\mathcal{B}_2(k-1),\sum_{i=1}^{k-1}U_i/t\leq y\right)$, $\nu_4(y):=E\big(\nu_4^{W_\kappa}(y)\big)$ and $\nu_5(y):=E\big(\nu_5^{W_\kappa}(y)\big)$. The next step is to remove $\mathcal{B}_1(k)$ in the above expression. For that, we only have to prove that

$$\sum_{k=1}^{n_t}\int_\varepsilon^{1-\varepsilon}E\left(\nu_4^{W_\kappa}(y)\mathbb{P}^{W_\kappa}\left(\mathcal{E}_3^1(k),\bar{\mathcal{B}}_1(k),\mathcal{B}_2(k-1),\sum_{i=1}^{k-1}U_i/t\in\mathrm{d}y\right)\right)$$

is negligible, one can check that this quantity is smaller than

$$\begin{aligned}&\sum_{k=1}^{n_t}\int_\varepsilon^{1-\varepsilon}E\left[\mathbb{P}^{W_\kappa}_{\tilde{m}_k}(H(\tilde{L}_k)<H(\tilde{L}_k^-),H(\tilde{L}_k)>t(1-y))\right]\\&\qquad\qquad\mathbb{P}\left(\bar{\mathcal{B}}_1(k),\mathcal{B}_2(k-1),\sum_{i=1}^{k-1}U_i/t\in\mathrm{d}y\right)\\\leq\ &\sum_{k=1}^{n_t}\mathbb{P}\left(\sum_{i=1}^{k-1}U_i/t\leq 1,\sum_{i=1}^{k}U_i/t>1,\ \bar{\mathcal{B}}_1(k)\right)\\\leq\ &\mathbb{P}\left(\bar{\mathcal{B}}_1(n_t)\right)\leq C_2v_t=o(1),\end{aligned}$$

where the last inequality comes from (3.1). Therefore, collecting the above computations yields

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}}\mathcal{L}(t,x)\leq\alpha\right)\geq\sum_{k=1}^{n_t}\int_\varepsilon^{1-\varepsilon}\nu_4(y)\mathrm{d}\tilde{\nu}_5(y)-o(1),$$

with $\tilde{\nu}_5(y):=e^{-\kappa\phi(t)}\mathbb{P}\left(\mathcal{E}_3^1(k),\mathcal{B}_2(k-1),\sum_{i=1}^{k-1}U_i/t\leq y\right)$.

We start with an estimation of the repartition function $\tilde{\nu}_5(y)$. Recall that like in the proof of Lemma 3.6, by the strong Markov property, the occupation time formula (1.12) and (1.13) the sequence $(U_j,\{\mathcal{L}(H(\tilde{L}_j),x)-\mathcal{L}(H(\tilde{m}_j),x),x\in\mathcal{D}_j\},j\leq$

n_t) under $\mathcal{B}_2(n_t)$ is equal to a sequence $(H_j(\tilde{L}_j), \{\mathcal{L}_j(H_j(\tilde{L}_j), x), x \in \mathcal{D}_j\}, j \le n_t)$, with this time

$$\begin{aligned} H_j(\tilde{L}_j) \quad &:= \quad A^j(\tilde{L}_j) \int_{\tilde{L}_j^-}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B^j}[\tau^{B^j}(1), A^j(u)/A^j(\tilde{L}_j)] \mathrm{d}u, \\ \mathcal{L}_j(H_j(\tilde{L}_j), x) \quad &:= \quad A^j(\tilde{L}_j) e^{-\tilde{V}^{(j)}(x)} \mathcal{L}_{B^j}[\tau^{B^j}(1), A^j(x)/A^j(\tilde{L}_j)], \end{aligned}$$

where $A^j(u) = \int_{\tilde{m}_j}^u e^{\tilde{V}^{(j)}(x)} \mathrm{d}x$. Using Remark 2.3, Lemma 2.2, Fact 2.1 **(ii)**, and then (7.5) and (7.6), we have for large t for any $1 \le j \le n_t$ since $\phi(t) = o(\log t)$,

$$\begin{aligned} P\big[\tilde{\tau}_j(\kappa r_t/8) \le \tilde{m}_j + r_t \le \tilde{\tau}_j(r_t)\big] \ge 1 - C_+ e^{-(c_-) r_t}, \\ P\big[\tilde{\tau}_j^-(r_t) \le \tilde{m}_j - r_t \le \tilde{\tau}_j^-(\kappa r_t/8)\big] \ge 1 - C_+ e^{-(c_-) r_t}. \end{aligned} \tag{5.19}$$

with $c_- > 0$. Therefore for any j, $P\big(\mathcal{D}_j \subset \big[\tilde{\tau}_j^-(r_t), \tilde{\tau}_j(r_t)\big]\big) \ge 1 - 2C_+ e^{-(c_-) r_t}$. Then on $\{\mathcal{D}_j \subset [\tilde{\tau}_j^-(r_t), \tilde{\tau}_j(r_t)]\}$, for any $x \in \mathcal{D}_j$,

$$\mathcal{L}_j(H_j(\tilde{L}_j), x) \le A^j(\tilde{L}_j) \mathcal{L}_{B^j}[\tau^{B^j}(1), A^j(x)/A^j(\tilde{L}_j)].$$

Also with probability $\ge 1 - 2C_+ e^{-(c_-) r_t}$, $\mathcal{D}_j \subset [\tilde{\tau}_j^-(r_t), \tilde{\tau}_j(r_t)]$ so for any $x \in \mathcal{D}_j$,

$$A^j(\tilde{\tau}_j^-(r_t)) \le A^j(x) \le A^j(\tilde{\tau}_j(r_t)). \tag{5.20}$$

With Remark 2.3, Lemma 2.2, Fact 2.1 and (7.8), we obtain with a probability larger than $1 - e^{-(c_-) r_t}$,

$$\begin{aligned} -e^{-h_t/4} \quad &\le \quad -e^{2r_t} e^{-(1-1/2)h_t} \le \frac{A^j(\tilde{\tau}_j^-(r_t))}{A^j\big(\tilde{L}_j\big)} \le \frac{A^j(\tilde{\tau}_j(r_t))}{A^j\big(\tilde{L}_j\big)} \\ &\le \quad e^{2r_t} e^{-(1-1/2)h_t} \le e^{-h_t/4}. \end{aligned} \tag{5.21}$$

Therefore, applying (7.11) (with $\delta = e^{-h_t/4}$ and $\varepsilon = \delta^{1/3}$), we obtain with a probability larger than $1 - e^{-(c_-) r_t}$,

$$\sup_{x \in \mathcal{D}_j} A^j\big(\tilde{L}_j\big) \mathcal{L}_{B^j}\big(\tau^{B^j}(1), A^j(x)/A^j\big(\tilde{L}_j\big)\big) \le A^j\big(\tilde{L}_j\big) \mathcal{L}_{B^j}\big(\tau^{B^j}(1), 0\big)\big(1 + e^{-h_t/12}\big). \tag{5.22}$$

Collecting the different estimates we then obtain,

$$\tilde{\nu}_5(y) \ge \mathbb{P}\left(\max_{1 \le j \le k-1} \mathcal{L}_j\big(H_j\big(\tilde{L}_j\big), \tilde{m}_j\big) \le t\bar{\alpha}_t, \ \sum_{j=1}^{k-1} \frac{H_j\big(\tilde{L}_j\big)}{t} \le y \right) - C_+ e^{-(c_-) r_t},$$

with $\bar{\alpha}_t := \tilde{\alpha}_t\big(1 + e^{-h_t/12}\big)^{-1}$. We can then inverse the equality in law we have used above, and then obtain

$$\tilde{\nu}_5(y) \ge F_{\bar{\alpha}_t}(y) - C_+ e^{-(c_-) r_t},$$

with $F_{\bar{\alpha}_t}$ defined in Lemma 5.2. Then we can follow the same lines as for the upper bound (especially computations after (5.9)), and obtain via Lemma 5.2 and by choosing C_0 large enough in such a way that $(c_-) r_t/\phi(t) = (c_-) C_0 > \kappa(1 + \delta)$:

$$\int_\varepsilon^{1-\varepsilon} \nu_4(y) \mathrm{d}\tilde{\nu}_5(y) \ge \int_\varepsilon^{1-\varepsilon} \nu_4(y) \mathrm{d}F_{\bar{\alpha}_t}^{+}(y) - o(n_t^{-1}).$$

Remark also that (5.22) implies the concentration of the local time at the h_t-minima: with probability larger than $1 - C_+ e^{-(c_-)r_t}$,

$$\left|\sup_{y\in\mathcal{D}_j} \mathcal{L}_j\big(H_j\big(\tilde{L}_j\big), y\big) - \mathcal{L}_j\big(H_j\big(\tilde{L}_j\big), \tilde{m}_j\big)\right| \le e^{-h_t/12}\mathcal{L}_j\big(H_j\big(\tilde{L}_j\big), \tilde{m}_j\big). \tag{5.23}$$

We now work on $\nu_4(y)$. By the second part of Lemma 2.2 it is equal to

$$E\left(\mathbb{P}^{W_\kappa}_{\tilde{m}_1}\left(\sup_{z\in\mathcal{D}_1} \mathcal{L}_{X'}(t(1-y), z) \le t\tilde{\alpha}_t, H'\big(\tilde{L}_1\big) > t(1-y), H'\big(\tilde{L}_1\big) < H'\big(\tilde{L}_1^-\big)\right)\right)$$
$$=: \tilde{\nu}_4(y),$$

and by Lemma 5.3, $\tilde{\nu}_4(y) \ge f^-_{\tilde{\alpha}_t}(y) - o(n_t^{-1})$. Therefore

$$\int_\varepsilon^{1-\varepsilon} \nu_4(y)\mathrm{d}\tilde{\nu}_5(y) \ge \int_\varepsilon^{1-\varepsilon} f^-_{\tilde{\alpha}_t}(y)\mathrm{d}F^+_{\tilde{\alpha}_t}(y) - o(n_t^{-1}).$$

From now on, the computations are very close from that of the upper bound (see (5.13) and below) and we do not give more details. □

Proof of Lemmata 5.2, 5.3 and 5.4.

Proof of Lemma 5.2: This is a direct consequence of Proposition 3.5. □

Proof of Lemma 5.3: To obtain the result we use a similar method than in Andreoletti and Diel (2011). That is to say, we study the inverse of the local time at $\tilde{m}_1$, and use our knowledge about $H(\tilde{L}_1)$. From the definitions of f_γ and $\tilde{f}_\gamma$ we have easily $\tilde{f}_\gamma(x) \le f_\gamma(x)$ for all x. So, to prove (5.1), we only need to prove the upper bound for f_γ and the lower bound for $\tilde{f}_\gamma$. We fix $\varepsilon \in (0, 1/2)$.
• *Upper bound for $f_\gamma(x)$.* Recall that $\sigma(u, \tilde{m}_1) = \inf\{s > 0, \mathcal{L}(s, \tilde{m}_1) \ge u\}$, $u \ge 0$. First, notice that for $0 < x < 1$, $f_\gamma(x)$ is equal to

$$\begin{aligned}
&E\left[\mathbb{P}^{W_\kappa}_{\tilde{m}_1}\left(\mathcal{L}(t(1-x), \tilde{m}_1) \le \gamma t, H\big(\tilde{L}_1\big) > t(1-x), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\right)\right] \\
&= E\left[\mathbb{P}^{W_\kappa}_{\tilde{m}_1}\left(\sigma(\gamma t, \tilde{m}_1) \ge t(1-x), H\big(\tilde{L}_1\big) > t(1-x), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\right)\right] && (5.24)\\
&= E\left[\mathbb{P}^{W_\kappa}_{\tilde{m}_1}\left(H\big(\tilde{L}_1\big) > \sigma(\gamma t, \tilde{m}_1) \ge t(1-x), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\right)\right] && (5.25)\\
&\qquad + E\left[\mathbb{P}^{W_\kappa}_{\tilde{m}_1}\left(\sigma(\gamma t, \tilde{m}_1) > H\big(\tilde{L}_1\big) > t(1-x), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\right)\right]. && (5.26)
\end{aligned}$$

Let us first study the expectation in (5.25). On $\big\{H\big(\tilde{L}_1\big) > \sigma(\gamma t, \tilde{m}_1), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\big\}$ under $\mathbb{P}^{W_\kappa}_{\tilde{m}_1}$, X remains between $\tilde{L}_1^-$ and $\tilde{L}_1$ until time $\sigma(\gamma t, \tilde{m}_1)$ which is finite. On this event and under $\mathbb{P}^{W_\kappa}_{\tilde{m}_1}$, considering (1.12) and (1.13) as in Shi (1998, p. 248), the inverse of the local time can be written for X starting at $\tilde{m}_1$ as

$$\sigma(\gamma t, \tilde{m}_1) = \int_{\tilde{L}_1^-}^{\tilde{L}_1} e^{-\tilde{V}^{(1)}(z)}\mathcal{L}_B\big(\sigma_B(\gamma t, 0), A^1(z)\big)\mathrm{d}z =: I, \tag{5.27}$$

where $A^1(z) = \int_{\tilde{m}_1}^z e^{\tilde{V}^{(1)}(y)}\mathrm{d}y$ and B is a standard Brownian motion independent of W_κ, such that B starts at $A^1(\tilde{m}_1) = 0$ and is killed when it first hits $A^1(\tilde{L}_1)$. In (5.27), we integrate only between $\tilde{L}_1^-$ and $\tilde{L}_1$ because under $\mathbb{P}^{W_\kappa}_{\tilde{m}_1}$,

$e^{-\tilde V^{(1)}(z)}\mathcal{L}_B\big(\sigma_B(\gamma t,0),A^1(z)\big)=\mathcal{L}(\sigma(\gamma t,\tilde m_1),z)=0$ for $z\notin\big[\tilde L_1^-,\tilde L_1\big]$ as explained after (5.26). We have

$$I=\gamma t\int_{\tilde L_1^-}^{\tilde L_1}e^{-\tilde V^{(1)}(z)}\mathcal{L}_{\tilde B}\big(\sigma_{\tilde B}(1,0),\tilde a(z)\big)\mathrm{d}z,$$

with $\tilde a(z):=(\gamma t)^{-1}A^1(z)=(\gamma t)^{-1}\int_{\tilde m_1}^z e^{\tilde V^{(1)}(y)}\mathrm{d}y$ and where $\tilde B:=B((\gamma t)^2.)/(\gamma t)$. By scale invariance $\tilde B$ is also a standard Brownian motion that we still denote by B in the sequel. Also, recall that $\sigma_U(r,y):=\inf\{s>0,\ \mathcal{L}_U(s,y)>r\}$ for $r>0$, $y\in\mathbb{R}$ is the inverse of the local time of the process U. Since we consider X starting at $\tilde m_1$, we have $H(\tilde L_1)=H(\tilde L_1)-H(\tilde m_1)=U_1$, for which Proposition 3.5 gives

$$\mathbb{E}\Big(\mathbb{P}_{\tilde m_1}^{W_\kappa}\Big\{\big|H(\tilde L_1)-\mathcal{H}_1\big|\le\varepsilon_t\mathcal{H}_1\Big\}\Big)=\mathbb{P}\left(\{|U_1\mathcal{H}_1|\le\varepsilon_t\mathcal{H}_1\}=:\mathcal{G}_1\right)\ge 1-e^{-D_1h_t},\tag{5.28}$$

with $\varepsilon_t:=e^{-d_1h_t}$, if $\delta>0$ is chosen small enough. This will explain the appearance of $\mathcal{H}_1$ in $f_\gamma^{\pm}(x)$. So, we now deal with I. Notice that $(\gamma t)^{-1}I$ can be split into two terms $(\gamma t)^{-1}I=I_1+I_2$, with

$$I_1:=\int_{\tilde\tau_1^-(h_t/2)}^{\tilde\tau_1(h_t/2)}e^{-\tilde V^{(1)}(z)}\mathcal{L}_B\big(\sigma_B(1,0),\tilde a(z)\big)\mathrm{d}z,$$

and $I_2:=(\gamma t)^{-1}I-I_1\ge 0$. We now prove that the main contribution in $(\gamma t)^{-1}I$ comes from I_1 and obtain its approximation in probability. Let $\varepsilon\in(0,1/100)$. First, using the second part of Lemma 2.2, followed by Remark 2.3, Fact 2.1 **(ii)** (for which we need $i\ge 2$), (7.8) and finally the first part of Lemma 2.2, we get

$$\begin{aligned}
&P\big[\big|A^1\big(\tilde\tau_1^-(h_t/2)\big)\big|\le e^{h_t(1+\varepsilon)/2},\big|A^1\big(\tilde\tau_1(h_t/2)\big)\big|\le e^{h_t(1+\varepsilon)/2}\big]\\
=\ &P\big[\big|A^2\big(\tilde\tau_2^-(h_t/2)\big)\big|\le e^{h_t(1+\varepsilon)/2},\big|A^2\big(\tilde\tau_2(h_t/2)\big)\big|\le e^{h_t(1+\varepsilon)/2}\big]\\
\ge\ &1-2P\big[F^+(h_t/2)>e^{h_t(1+\varepsilon)/2}\big]-P\big[\overline{\mathcal{V}_t}\big]\ge 1-C_+e^{-\kappa\varepsilon h_t/4}.
\end{aligned}\tag{5.29}$$

Therefore, since $\tilde a\big(\tilde\tau_1^-(h_t/2)\big)\le\tilde a(z)\le\tilde a\big(\tilde\tau_1(h_t/2)\big)$ for all $z\in\big[\tilde\tau_1^-(h_t/2),\tilde\tau_1(h_t/2)\big]$,

$$P\big(\forall z\in\big[\tilde\tau_1^-(h_t/2),\tilde\tau_1(h_t/2)\big],\ |\tilde a(z)|\le e^{-(\log t)(1-3\varepsilon)/2}\big)\ge 1-C_+e^{-\kappa\varepsilon h_t/4}.\tag{5.30}$$

Also, using (7.15) and the second Ray-Knight theorem (see before (7.15)), we have

$$\mathbb{P}\left(\sup_{|u|\le e^{-(\log t)(1-3\varepsilon)/2}}\Big|\mathcal{L}_B(\sigma_B(1,0),u)-1\Big|\ge\widehat\varepsilon_t\right)\le e^{-t^\varepsilon/16}.\tag{5.31}$$

with $\widehat\varepsilon_t:=t^{-(1-5\varepsilon)/4}$. So we obtain

$$\mathbb{E}\Big[\mathbb{P}_{\tilde m_1}^{W_\kappa}\big(\big|I_1-\tilde R_1\big|\le\widehat\varepsilon_t\tilde R_1\big)\Big]\ge 1-C_+e^{-\kappa\varepsilon h_t/4},\tag{5.32}$$

with $\tilde R_1:=\int_{\tilde\tau_1^-(h_t/2)}^{\tilde\tau_1(h_t/2)}e^{-\tilde V^{(1)}(z)}\mathrm{d}z$. We now prove that I_2 is negligible compared to the integral $\tilde R_1$ which appears in the previous equation, and then compared to I_1. First thanks to (7.16) and the second Ray-Knight theorem, we have

$$\mathbb{E}\left[\mathbb{P}_{\tilde m_1}^{W_\kappa}\left(\sup_{z\in[\tilde L_1^-,\tilde L_1]}\mathcal{L}_B\big[\sigma_B(1,0),\tilde a(z)\big]>e^{\varepsilon\log t}\right)\right]\le 2e^{-\varepsilon\log t}.$$

So with probability larger than $1-2e^{-\varepsilon\log t}$, we have

$$I_2 \leq e^{\varepsilon\log t}\left(\int_{\tilde{L}_1^-}^{\tilde{\tau}_1^-(h_t/2)} e^{-\tilde{V}^{(1)}(z)}\mathrm{d}z + \int_{\tilde{\tau}_1(h_t/2)}^{\tilde{L}_1} e^{-\tilde{V}^{(1)}(z)}\mathrm{d}z\right) =: e^{\varepsilon\log t} I_3.$$

By Lemma 6.8, with a probability larger than $1-2e^{-(c_-)\varepsilon h_t}$ for large t,

$$I_3 \leq C_+ h_t^2 e^{-(1-\varepsilon)h_t/2}.$$

Also, by Lemma 3.6, with probability larger $1-e^{-(D_-)h_t}$, $\tilde{R}_1 = R_1$ (which is the same R_1 as in (5.28)), which law is given by the sum of two independent copies of $F^-(h_t/2)$. So using (7.9), with a probability larger than $1-2e^{-(D_-)h_t}$,

$$\tilde{R}_1 = R_1 \geq e^{-\varepsilon h_t/2}.$$

We deduce from the last three inequalities that with a probability larger than $1-e^{-(c_-)\varepsilon h_t}$,

$$I_2 < R_1 e^{-(1-5\varepsilon)h_t/2} = \tilde{R}_1 e^{-(1-5\varepsilon)h_t/2}. \tag{5.33}$$

Finally, using $(\gamma t)^{-1} I = I_1 + I_2$ together with (5.32) and (5.33), we get

$$\begin{aligned} & E\Big[\mathbb{P}_{\tilde{m}_1}^{W_\kappa}\Big(\big|I-\gamma t R_1\big| \geq 2t^{-(1-5\varepsilon)/4}(\gamma t)R_1, \\ & \qquad\qquad H\big(\tilde{L}_1\big) > \sigma(\gamma t,\tilde{m}_1),\ H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\Big)\Big] \\ \leq\ & C_+ e^{-\varepsilon(c_-)h_t}. \end{aligned} \tag{5.34}$$

We recall that by (5.27), $\sigma(\gamma t,\tilde{m}_1) = I$ on $\{H(\tilde{L}_1) > \sigma(\gamma t,\tilde{m}_1), H(\tilde{L}_1) < H(\tilde{L}_1^-)\}$ under $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$. Hence, combining (5.34) with (5.28) gives for large t for every $x \in [\varepsilon, 1-\varepsilon]$,

$$\begin{aligned} & \Big\{H\big(\tilde{L}_1\big) > \sigma(\gamma t,\tilde{m}_1) \geq t(1-x), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\Big\} \\ & \subset \left\{\frac{1}{R_1} \leq \frac{\gamma}{1-x}(1+\varepsilon_t'), \mathcal{H}_1 > t(1-x)(1-\varepsilon_t'), H\big(\tilde{L}_1\big) > \sigma(\gamma t,\tilde{m}_1)\right\} \cup \mathcal{E}_\varepsilon^1, \end{aligned} \tag{5.35}$$

where $\mathcal{E}_\varepsilon^1$ is such that $E\big[\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(\mathcal{E}_\varepsilon^1)\big] \leq C_+ e^{-(\varepsilon c_-)h_t} + e^{-D_1 h_t}$ and where, as defined in the statement of the lemma, $\varepsilon_t' = e^{-c_2 h_t}$ with $c_2 > 0$ chosen small enough.

Now, let us study (5.26). On the event inside the probability in (5.26), $\sigma(\gamma t,\tilde{m}_1)$ might be infinite. We work under $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$. There exists a Brownian motion B such that, with T^1 playing under $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$ the same role as T does under $\mathbb{P}$ (see (1.12)), $H\big(\tilde{L}_1\big) = T^1\big(\tau^B\big(A^1\big(\tilde{L}_1\big)\big)\big)$ and $\sigma(\gamma t,\tilde{m}_1) = T^1(\sigma_B(\gamma t,0))$ (as in (5.27) and in Shi (1998) p. 248). Also by (1.12), notice for further use that under $\mathbb{P}_{\tilde{m}_1}^{W_\kappa}$,

$$\mathcal{L}\big(\sigma(yt,\tilde{m}_1),z\big) = e^{-\tilde{V}^{(1)}(z)}\mathcal{L}_B\big(\sigma_B(yt,0),A^1(z)\big), \qquad z\in\mathbb{R}, y\in(0,+\infty). \tag{5.36}$$

So, we have

$$\begin{aligned} \sigma(\gamma t,\tilde{m}_1) > H\big(\tilde{L}_1\big) \quad & \Leftrightarrow \quad \sigma_B(\gamma t,0) > \tau^B\big(A^1\big(\tilde{L}_1\big)\big) \\ & \Leftrightarrow \quad \mathcal{L}_B\big[\sigma_B(\gamma t,0),0\big] = \gamma t > \mathcal{L}_B\big[\tau^B\big(A^1\big(\tilde{L}_1\big)\big),0\big]. \end{aligned}$$

Now, note that, as in (3.22) in the proof of Lemma 3.6, $\mathcal{L}_B\big[\tau^B(A^1(\tilde{L}_1)),0\big] = A^1(\tilde{L}_1)\mathcal{L}_{\tilde{B}}\big(\tau^{\tilde{B}}(1),0\big)$, where $\tilde{B} := B((A^1(\tilde{L}_1))^2.)/A^1(\tilde{L}_1)$. Also, by definition of $\mathbf{e}_1$

given in (3.22), we have $\mathcal{L}_{\tilde{B}}\big(\tau^{\tilde{B}}(1),0\big) = \mathbf{e}_1$. As a consequence,

$$\sigma(\gamma t, \tilde{m}_1) > H\big(\tilde{L}_1\big) \Leftrightarrow \gamma t > A^1\big(\tilde{L}_1\big)\mathbf{e}_1 \Leftrightarrow \gamma t R_1 > A^1\big(\tilde{L}_1\big)\mathbf{e}_1 R_1.$$

Then, according to (3.18), we have $A^1\big(\tilde{L}_1\big) \geq \big(1 - e^{-(d_-)h_t}\big)S_1$ with probability greater than $1 - e^{-(D_-)h_t}$. Moreover, according to (5.28) and to the fact that under $\mathbb{P}^{W_\kappa}_{\tilde{m}_1}$ the diffusion X starts at $\tilde{m}_1$, we have $\mathcal{H}_1 = \mathbf{e}_1 S_1 R_1 \geq (1+\varepsilon_t)^{-1} H(\tilde{L}_1)$ with probability greater than $1 - e^{-(D_-)h_t}$. As a consequence,

$$\sigma(\gamma t, \tilde{m}_1) > H(\tilde{L}_1) \Rightarrow \gamma t R_1 > \big(1 - e^{-(d_-)h_t}\big)(1+\varepsilon_t)^{-1} H(\tilde{L}_1), \tag{5.37}$$

except on an event which probability $E\big[\mathbb{P}^{W_\kappa}_{\tilde{m}_1}(.)\big]$ is less than $2e^{-(D_-)h_t}$. Combining this with (5.28) we get for large t for every $x \in [\varepsilon, 1-\varepsilon]$,

$$\begin{aligned}
&\Big\{\sigma(\gamma t, \tilde{m}_1) > H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-))\Big\} \\
&\subset \left\{\frac{1}{R_1} \leq \frac{\gamma}{1-x}(1+\varepsilon_t'), \mathcal{H}_1 > t(1-x)(1-\varepsilon_t'), \sigma(\gamma t, \tilde{m}_1) > H(\tilde{L}_1)\right\} \cup \mathcal{E}^2_\varepsilon,
\end{aligned} \tag{5.38}$$

where $\mathcal{E}^2_\varepsilon$ is such that $E\big[\mathbb{P}^{W_\kappa}_{\tilde{m}_1}(\mathcal{E}^2_\varepsilon)\big] \leq 2e^{-(D_-)h_t} + e^{-D_1 h_t}$ and where, as before, $\varepsilon_t' = e^{-c_2 h_t}$ with $c_2 > 0$ possibly smaller than before.

Combining (5.25), (5.26) (5.35) and (5.38) with the strong Markov property, we get for large t for every $x \in [\varepsilon, 1-\varepsilon]$, since $\phi(t) = o(\log t)$,

$$\begin{aligned}
f_\gamma(x) &\leq \mathbb{P}\left(\frac{1}{R_1} \leq \frac{\gamma}{1-x}(1+\varepsilon_t'), \mathcal{H}_1 > t(1-x)(1-\varepsilon_t')\right) + o(n_t^{-1}). \\
&= f_\gamma^+(x) + o(n_t^{-1}).
\end{aligned}$$

• *Lower bound for $\tilde{f}_\gamma$.* Let $\tilde{\gamma} := \gamma\big(1+e^{-h_t/12}\big)^{-1}$ and $y := (1-x)/[R_1(1-4\widehat{\varepsilon}_t)]$. We have to distinguish the cases $H\big(\tilde{L}_1\big) > \sigma(\tilde{\gamma}t, \tilde{m}_1)$ and $\sigma(\tilde{\gamma}t, \tilde{m}_1) > H\big(\tilde{L}_1\big)$. We work under $\mathbb{P}^{W_\kappa}_{\tilde{m}_1}$. On $\big\{y \leq \tilde{\gamma}, H\big(\tilde{L}_1\big) > \sigma(\tilde{\gamma}t, \tilde{m}_1) \geq t(1-x), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\big\}$, we can express the local time of X at the inverse of its local time in $\tilde{m}_1$ at time yt in terms of the standard Brownian motion driving the diffusion. More precisely by (5.36) and by scale invariance, there exists a Brownian motion B such that for any $z \in \mathcal{D}_1$,

$$\mathcal{L}\big(\sigma(yt, \tilde{m}_1), z\big) = (yt)e^{-\tilde{V}^{(1)}(z)}\mathcal{L}_B\big(\sigma_B(1,0), \widehat{a}(z)\big) \tag{5.39}$$

with $\widehat{a}(z) := (yt)^{-1}\int_{\tilde{m}_1}^z e^{\tilde{V}^{(1)}(u)}\mathrm{d}u = A^1(z)/(yt)$. Notice that by (1.6), $F^-(h_t/2) \leq \tau^{W_\kappa^\uparrow}(h_t/2)$ in law, so $P\big[R_1 > 8h_t/\kappa\big] \leq 2P\big[F^-(h_t/2) > 4h_t/\kappa\big] \leq 2P\big[\tau^{W_\kappa^\uparrow}(h_t/2) > 4h_t/\kappa\big] \leq e^{-(c_-)h_t}$ for large t. Moreover, we prove with the same method used to prove (5.19) that $\tilde{\tau}^-(h_t/2) \leq \tilde{m}_1 - r_t \leq \tilde{m}_1 + r_t \leq \tilde{\tau}(h_t/2)$ with probability at least $1 - C_+e^{-(c_-)h_t}$. This and (5.29) give $-e^{h_t(1+\varepsilon)/2} \leq A^1[\tilde{\tau}^-(h_t/2)] \leq A^1(z) \leq A^1[\tilde{\tau}(h_t/2)] \leq e^{h_t(1+\varepsilon)/2}$ for any $z \in \mathcal{D}_1$ with probability $\geq 1 - e^{-(c_-)\varepsilon h_t}$. So, for large t for every $x \in [\varepsilon, 1-\varepsilon]$, $|\widehat{a}(z)| \leq e^{h_t(1+\varepsilon)/2}R_1/(t(1-x)) \leq e^{-(\log t)(1-3\varepsilon)/2}$ for these z with such probability. Hence with the same method we used to prove (5.32) from (5.30) and (5.31), we get for large t for every $x \in [\varepsilon, 1-\varepsilon]$,

$$E\left(\mathbb{P}^{W_\kappa}_{\tilde{m}_1}\left(\sup_{z\in\mathcal{D}_1}\Big|\mathcal{L}_B(\sigma_B(1,0), \widehat{a}(z)) - 1\Big| \leq \widehat{\varepsilon}_t\right)\right) \geq 1 - 2e^{-(c_-)\varepsilon h_t}.$$

The above inequality together with (5.39) imply that for large t for every $x \in [\varepsilon, 1-\varepsilon]$,

$$\begin{aligned} & E\left(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}\left(\left\{\exists z \in \mathcal{D}_1,\ \left|\mathcal{L}(\sigma(yt,\tilde{m}_1),z) - yte^{-\tilde{V}^{(1)}(z)}\right| \geq 2yte^{-\tilde{V}^{(1)}(z)}\widehat{\varepsilon}_t,\right.\right.\right. \\ & \qquad \left.\left.\left. y \leq \tilde{\gamma},\ H\big(\tilde{L}_1\big) > \sigma(\tilde{\gamma}t,\tilde{m}_1) \geq t(1-x),\ H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\right\}\right)\right) \\ \leq \;& 2e^{-(c_-)\varepsilon h_t}. \end{aligned} \tag{5.40}$$

On $\{y \leq \tilde{\gamma}, H\big(\tilde{L}_1\big) > \sigma(\tilde{\gamma}t,\tilde{m}_1) \geq t(1-x), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\}$, if $t(1-x) > \sigma(yt,\tilde{m}_1)$, then $\sigma(yt,\tilde{m}_1) - ytR_1 < -4tyR_1\widehat{\varepsilon}_t$, and by (5.34) (applied with γ replaced by y), this has on the previous event a probability $E\big(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(.)\big)$ less than $C_+e^{-(c_-)\varepsilon h_t}$. Thus on the previous event, we have $t(1-x) \leq \sigma(yt,\tilde{m}_1)$, except on a sub event of probability smaller than $C_+e^{-(c_-)\varepsilon h_t}$. This is true for every $x \in [\varepsilon, 1-\varepsilon]$ for large t.

Then since the local time is increasing in time, we have on the previous event for any $z \in \mathcal{D}_1$, $\mathcal{L}(t(1-x),z) \leq \mathcal{L}(\sigma(yt,\tilde{m}_1),z)$, which is according to (5.40) less than $yte^{-V^{(1)}(z)}(1+2\widehat{\varepsilon}_t) \leq yt(1+2\widehat{\varepsilon}_t)$ for every $z \in \mathcal{D}_1$ with probability $E\big(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(.)\big)$ at least $1-2e^{-(c_-)\varepsilon h_t}$. Combining this and the definition of our y gives for large t, for every $x \in [\varepsilon, 1-\varepsilon]$,

$$\begin{aligned} & E\left(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}\left(\left\{\frac{\sup_{z\in\mathcal{D}_1}\mathcal{L}(t(1-x),z)}{t} > \frac{(1-x)}{R_1}\frac{1+2\widehat{\varepsilon}_t}{1-4\widehat{\varepsilon}_t}\right\} =: \overline{\mathcal{G}_2}\right.\right. \\ & \qquad \left.\left. \cap \left\{y \leq \tilde{\gamma}, H\big(\tilde{L}_1\big) > \sigma(\tilde{\gamma}t,\tilde{m}_1) \geq t(1-x), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\right\}\right)\right) \\ \leq \;& (2+C_+)e^{-(c_-)\varepsilon h_t}. \end{aligned} \tag{5.41}$$

As a consequence, for t large enough so that $1+2\widehat{\varepsilon}_t \leq 1+e^{-h_t/12}$, we have for every $x \in [\varepsilon, 1-\varepsilon]$,

$$\begin{aligned} & \left\{y \leq \tilde{\gamma}, H\big(\tilde{L}_1\big) > \sigma(\tilde{\gamma}t,\tilde{m}_1) \geq t(1-x), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\right\} \\ \subset \;& \left\{\sup_{z\in\mathcal{D}_1}\mathcal{L}(t(1-x),z) \leq y(1+2\widehat{\varepsilon}_t)t \leq \gamma t,\ H\big(\tilde{L}_1\big) > \sigma(\tilde{\gamma}t,\tilde{m}_1)\right\} \cup \mathcal{E}_\varepsilon^3 \end{aligned} \tag{5.42}$$

by definition of $\tilde{\gamma}$, where $\mathcal{E}_\varepsilon^3$ is such that $E\big(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(\mathcal{E}_\varepsilon^3)\big) \leq (2+C_+)e^{-(c_-)\varepsilon h_t}$.

On the other hand, from the definition of $\sigma(.,\tilde{m}_1)$, (5.23) and the definition of $\tilde{\gamma}$, we have for large t for every $x \in [\varepsilon, 1-\varepsilon]$,

$$\begin{aligned} & \left\{y \leq \tilde{\gamma}, \sigma(\tilde{\gamma}t,\tilde{m}_1) > H\big(\tilde{L}_1\big) > t(1-x), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\right\} \\ & \subset \left\{\mathcal{L}\big(H\big(\tilde{L}_1\big),\tilde{m}_1\big) \leq \tilde{\gamma}t, \sigma(\tilde{\gamma}t,\tilde{m}_1) > H\big(\tilde{L}_1\big) > t(1-x), H\big(\tilde{L}_1\big) < H\big(\tilde{L}_1^-\big)\right\} \\ & \subset \left\{\sup_{z\in\mathcal{D}_1}\mathcal{L}\big(H\big(\tilde{L}_1\big),z\big) \leq \gamma t, \sigma(\tilde{\gamma}t,\tilde{m}_1) > H\big(\tilde{L}_1\big) > t(1-x)\right\} \cup \mathcal{E}_\varepsilon^4 \\ & \subset \left\{\sup_{z\in\mathcal{D}_1}\mathcal{L}(t(1-x),z) \leq \gamma t,\ \sigma(\tilde{\gamma}t,\tilde{m}_1) > H\big(\tilde{L}_1\big)\right\} \cup \mathcal{E}_\varepsilon^4, \end{aligned} \tag{5.43}$$

where $\mathcal{E}_\varepsilon^4$ is the event where (5.23) fails, it is such that $E\big(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(\mathcal{E}_\varepsilon^4)\big) \leq C_+e^{-(c_-)r_t}$.

Combining (5.42) and (5.43) we get for large t for every $x \in [\varepsilon, 1-\varepsilon]$, under $\mathbb{P}^{W_\kappa}_{\tilde{m}_1}$,

$$\left\{y \leq \tilde{\gamma}, H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-)\right\} \subset \left\{\sup_{z\in\mathcal{D}_1} \mathcal{L}(t(1-x), z) \leq \gamma t\right\} \cup \mathcal{E}_\varepsilon^5,$$

where $\mathcal{E}_\varepsilon^5$ is such that $E\big(\mathbb{P}^{W_\kappa}_{\tilde{m}_1}(\mathcal{E}_\varepsilon^5)\big) \leq C_+ e^{-(c_-)r_t} = C_+ e^{-(c_-)C_0\phi(t)} = o(n_t^{-1})$ as $t \to +\infty$ is we choose C_0 large enough. Combining this with (5.28), (3.2) and Proposition 3.5, we obtain for large t for every $x \in [\varepsilon, 1-\varepsilon]$,

$$\begin{aligned}
\tilde{f}_\gamma(x) &\geq \mathbb{P}\left(\frac{(1-x)}{R_1} \leq \gamma(1-\varepsilon_t'), \mathbf{e}_1 S_1 R_1 > t(1-x)(1+\varepsilon_t')\right) - o(n_t^{-1}) \\
&= f_\gamma^-(x) - o(n_t^{-1}),
\end{aligned}$$

where the constant c_2 in the definition of $\varepsilon_t' = e^{-c_2 h_t}$ has been decreased if necessary. This proves the lower bound for $\tilde{f}_\gamma(x)$ and then finishes the proof of the lemma. □

Proof of Lemma 5.4: Let $0 < a < 1/4$. We start with (5.2). By Proposition 3.5, the $\mathcal{H}_i$, $i \geq 1$ are i.i.d., so $\bar{\mathcal{H}}_{k-1}$ and $\bar{\mathcal{H}}_k - \bar{\mathcal{H}}_{k-1} = \mathcal{H}_k$ are independent for $k \geq 1$. Thus for $t > 0$,

$$\begin{aligned}
&\sum_{1\leq k\leq n_t} \mathbb{P}\left[\bar{\mathcal{H}}_k > 1-a/2,\ 1-2a < \bar{\mathcal{H}}_{k-1} \leq 1-3a/4\right] \\
&= \int_{1-2a}^{1-3a/4} d\mu_t(x) e^{\kappa\phi(t)} \mathbb{P}\left[\mathcal{H}_1 > 1-x-a/2\right],
\end{aligned} \tag{5.44}$$

where the measure μ_t is defined by $\int_0^x d\mu_t(y) := e^{-\kappa\phi(t)} \sum_{1\leq k\leq n_t} \mathbb{P}\left[\bar{\mathcal{H}}_{k-1} \leq x\right]$. We know that μ_t converges vaguely as $t \to +\infty$ to the measure μ which has a density with respect to the Lebesgue measure equal to $(\Gamma(\kappa)\mathcal{C}_\kappa)^{-1} x^{\kappa-1} \mathbb{1}_{x>0}$, with $\mathcal{C}_\kappa > 0$ (see Lemma 6.2). Also thanks to Lemma 4.1, $e^{\kappa\phi(t)} P\left[\mathcal{H}_1/t > x\right]$ converges uniformly on every compact subset of $(0, +\infty)$ to $\mathcal{C}_\kappa x^{-\kappa}/\Gamma(1-\kappa)$. Therefore,

$$\begin{aligned}
&\lim_{t\to+\infty} \sum_{1\leq k\leq n_t} \mathbb{P}\left[\bar{\mathcal{H}}_k > 1-a/2,\ 1-2a < \bar{\mathcal{H}}_{k-1} \leq 1-3a/4\right] \\
&= \frac{1}{\Gamma(\kappa)\Gamma(1-\kappa)} \int_{1-2a}^{1-3a/4} x^{\kappa-1}(1-x-a/2)^{-\kappa} \mathrm{d}x \\
&\leq \text{const} \times a^{1-\kappa}.
\end{aligned}$$

For (5.3), we apply (6.2) with $r = \varepsilon \in (0, 1/2)$ and $s = 1-\varepsilon$, which gives

$$\begin{aligned}
&\lim_{t\to+\infty} \mathbb{P}\left(\varepsilon t \leq H(m_{N_t}) \leq (1-\varepsilon)t\right) \\
&= 1 - \frac{\sin(\pi\kappa)}{\pi} \left(\int_0^\varepsilon x^{\kappa-1}(1-x)^{-\kappa} \mathrm{d}x + \int_{1-\varepsilon}^1 x^{\kappa-1}(1-x)^{-\kappa} \mathrm{d}x\right) \\
&\geq 1 - \frac{\sin(\pi\kappa)}{\pi} \left(\frac{(1-\varepsilon)^{-\kappa}}{\kappa} \varepsilon^\kappa + \frac{(1-\varepsilon)^{\kappa-1}}{1-\kappa} \varepsilon^{1-\kappa}\right),
\end{aligned}$$

which implies the result. □

Proof of Theorem 1.3: The proof of this theorem is a direct consequence of Propositions 5.1 and 1.4 and of Lemmata 4.4 and 4.5. Notice that the proof of the upper bound does not use the proof of the lower bound, but we use the upper bound for the proof of the lower bound. In particular from the upper bound of Theorem 1.3 (which makes use of the upper bound of Proposition 5.1 but not of its lower bound), we have $\limsup_{t\to+\infty}\mathbb{P}(\mathcal{L}^*(t) < 2\tilde{w}_t) \leq \mathbb{P}\big(\mathcal{Y}_1^\natural(\mathcal{Y}_2^{-1}(1)^-) \leq \varepsilon\big)$ for any $\varepsilon > 0$ as $\lim_{t\to+\infty}\tilde{w}_t/t = 0$. From this, as $\mathcal{Y}_1^\natural(\mathcal{Y}_2^{-1}(1)^-)$ is positive, we obtain $\lim_{t\to+\infty}\mathbb{P}(\mathcal{L}^*(t) < 2\tilde{w}_t) = 0$, which proves assertion (5.18) at the beginning of the proof of the lower bound of Proposition 5.1.

Thanks to Proposition 5.1 and to the remark before this proposition, we only need to study the convergence of $\mathcal{P}_1^\pm$ (the limit when t goes to infinity and then the limit when ε goes to 0). The latter can be written in term of functionals of $(Y_1,Y_2)^t$ as follows. Let $\mathbb{Y}_t := (Y_2^t)^{-1}(1-2\varepsilon)$; we have $\mathcal{N}_t^{2\varepsilon}e^{-\kappa\phi(t)} = \mathbb{Y}_t$, and

$$\begin{aligned}\mathcal{P}_1^\pm &= P\left[\big(1 - Y_2^t(\mathbb{Y}_t^-)\big)\frac{Y_1^t(\mathbb{Y}_t) - Y_1^t(\mathbb{Y}_t^-)}{Y_2^t(\mathbb{Y}_t) - Y_2^t(\mathbb{Y}_t^-)} \leq \alpha_t^\pm,\ (Y_1^t)^\natural(\mathbb{Y}_t^-) \leq \alpha_t^\pm\right]\\ &= P\Bigg[\big(1 - \tilde{K}_{I,1-2\varepsilon}^-((Y_1,Y_2)^t)\big)\frac{K_{I,1-2\varepsilon}((Y_1,Y_2)^t) - K_{I,1-2\varepsilon}^-((Y_1,Y_2)^t)}{\tilde{K}_{I,1-2\varepsilon}((Y_1,Y_2)^t) - \tilde{K}_{I,1-2\varepsilon}^-((Y_1,Y_2)^t)} \leq \alpha_t^\pm,\\ &\qquad\qquad J_{I,1-2\varepsilon}^-((Y_1,Y_2)^t) \leq \alpha_t^\pm\Bigg],\end{aligned}$$

with the notation $K_{I,a}$, $\tilde{K}_{I,a},\dots$ introduced in (4.25) and before. The hypotheses of Lemma 4.5 are: finite number of large jumps on compact intervals, strictly increasing, starting at 0, and jumping over 1 without reaching it. These properties are naturally almost surely satisfied by a κ-stable subordinator so, almost surely, the paths of $(\mathcal{Y}_1,\mathcal{Y}_2)$ satisfy the hypotheses of Lemma 4.5 (see e.g. Bertoin, 1996 III.2 p. 75). Therefore they are points of continuity for $J_{I,1-2\varepsilon}^-$, $K_{I,1-2\varepsilon}^-$, $K_{I,1-2\varepsilon}$, $\tilde{K}_{I,1-2\varepsilon}^-$ and $\tilde{K}_{I,1-2\varepsilon}$. Combining this continuity with Proposition 1.4, continuous mapping theorem, and replacing the functionals by their expressions, we obtain, when t goes to infinity, the convergence of $\mathcal{P}_1^\pm$ to

$$\begin{aligned}P\Bigg[\big(1-\mathcal{Y}_2\big(\mathcal{Y}_2^{-1}(1-2\varepsilon)^-\big)\big)\frac{\mathcal{Y}_1\big(\mathcal{Y}_2^{-1}(1-2\varepsilon)\big) - \mathcal{Y}_1\big(\mathcal{Y}_2^{-1}(1-2\varepsilon)^-\big)}{\mathcal{Y}_2\big(\mathcal{Y}_2^{-1}(1-2\varepsilon)\big) - \mathcal{Y}_2\big(\mathcal{Y}_2^{-1}(1-2\varepsilon)^-\big)} \leq \alpha,\\ \mathcal{Y}_1^\natural\big(\mathcal{Y}_2^{-1}(1-2\varepsilon)^-\big) \leq \alpha\Bigg].\end{aligned}$$

Then, note that almost surely $\mathcal{Y}_2\big(\mathcal{Y}_2^{-1}(1)^-\big) < 1$ so we have a.s. $\mathcal{Y}_2^{-1}(1-2\varepsilon) = \mathcal{Y}_2^{-1}(1)$ for all ε small enough. We deduce that the above expression converges to the repartition function of $\max(\mathcal{I}_1,\mathcal{I}_2)$ (see (1.5) for definitions of $\mathcal{I}_1$ and $\mathcal{I}_2$) when ε goes to 0, and this yields Theorem 1.3. □

5.2. *Favorite site (proof of Theorem 1.5).*

Thanks to Section 3, we know precisely the nature of the contribution of each h_t-valley to the local time. The difficulty in proving Theorem 1.3 was the need to consider only a part of the contribution of the last h_t-valley. The proofs of the first two points (1.8) and (1.9) of Theorem 1.5 are thus easier to obtain, since they do not require to "cut" the contribution of any valley. Let us prove the first point (1.8) (the second one, (1.9), is obtained similarly). We have, using (2.7),

$$\begin{aligned}
&\mathbb{P}\left[\mathcal{L}^*(H(m_{N_t+1})) \le \alpha t\right] \\
&\le \mathbb{P}\left(\mathcal{L}^*\big(H\big(\tilde{L}_{N_t}\big)\big) \le \alpha t,\ \mathcal{Q}, \mathcal{V}_t\right) + \mathbb{P}\left(\overline{\mathcal{Q}}\right) + P\left(\overline{\mathcal{V}}_t\right) + \mathbb{P}\left(\overline{\mathcal{B}_3(n_t)}\right) \\
&\le \mathbb{P}\Bigg(\sup_{1\le j\le N_t} \ell_j/t \le (1-\varepsilon_t)^{-1}\alpha,\ \mathcal{Q}, \mathcal{V}_t\Bigg) + \mathbb{P}\left(\overline{\mathcal{Q}}\right) + o(1),
\end{aligned}$$

where we fixed some $\varepsilon > 0$ and $\mathcal{Q} := \{\varepsilon t \le H(m_{N_t}) \le (1-\varepsilon)t,\ 1 \le N_t \le n_t\}$ as after (5.4) (from there we see that $\lim_{\varepsilon\to 0}\lim_{t\to+\infty}\mathbb{P}\big(\overline{\mathcal{Q}}\big) = 0$). In the last inequality we used Proposition 3.5, Lemma 2.2 and Lemma 3.2. To lighten notation, let $\tilde{\alpha}_t := (1-\varepsilon_t)^{-1}\alpha$. We have

$$\begin{aligned}
&\mathbb{P}\Bigg(\sup_{1\le j\le N_t} \ell_j/t \le \tilde{\alpha}_t,\ \mathcal{Q},\ \mathcal{V}_t\Bigg) \\
\le\ &\mathbb{P}\Bigg(\sup_{1\le j\le N_t} \ell_j/t \le \tilde{\alpha}_t,\ \mathcal{B}_1(n_t),\ \mathcal{Q},\ \mathcal{V}_t\Bigg) + \mathbb{P}\left(\overline{\mathcal{B}_1(n_t)}\right) \\
\le\ &\mathbb{P}\Bigg(\sup_{1\le j\le N_t} \ell_j/t \le \tilde{\alpha}_t,\ \bar{\mathcal{H}}_{N_t} \ge 1-\delta'_t,\ \bar{\mathcal{H}}_{N_t-1} \le 1-\varepsilon+\delta'_t,\ \mathcal{Q}\Bigg) + o(1),
\end{aligned}$$

with $\delta'_t = 3\tilde{v}_t/t$ and where we used (3.1) together with Proposition 3.5. Partitioning on the values of N_t we get that the above is less than

$$\sum_{1\le k\le n_t} \mathbb{P}\Bigg(\sup_{1\le j\le k} \ell_j/t \le \tilde{\alpha}_t,\ \bar{\mathcal{H}}_k \ge 1-\delta'_t,\ \bar{\mathcal{H}}_{k-1} \le 1-\varepsilon+\delta'_t,\ \mathcal{Q}\Bigg) + o(1).$$

Since the sum $\sum_1$ defined in the proof of the upper bound of Proposition 5.1 (see (5.15) and below) is smaller than $s(\varepsilon,t)$ satisfying $\lim_{\varepsilon\to 0}\lim_{t\to+\infty} s(\varepsilon,t) = 0$, we can intersect the event on the above probability with $\{k = \mathcal{N}_t^{2\varepsilon}\}$ and get

$$\mathbb{P}\left[\mathcal{L}^*(H(m_{N_t+1})) \le \alpha t\right] \le \mathbb{P}\Bigg(\sup_{1\le j\le \mathcal{N}_t^{2\varepsilon}} \ell_j/t \le \tilde{\alpha}_t\Bigg) + \mathbb{P}\left(\overline{\mathcal{Q}}\right) + s(\varepsilon,t) + o(1).$$

Then, as in the proof of Theorem 1.3 we have that $(\mathcal{Y}_1,\mathcal{Y}_2)$ almost surely satisfies the hypothesis of Lemma 4.5, and is therefore almost surely a point of continuity for $J_{I,12-\varepsilon}$ defined just above (4.25). From this continuity, Proposition 1.4 and the continuous mapping theorem we get

$$\sup_{1\le j\le \mathcal{N}_t^{2\varepsilon}} \ell_j/t = J_{I,1-2\varepsilon}\left((Y_1,Y_2)^t\right) \underset{t\to+\infty}{\overset{\mathcal{L}}{\to}} J_{I,1-2\varepsilon}(\mathcal{Y}_1,\mathcal{Y}_2) = \mathcal{Y}_1^\natural\big(\mathcal{Y}_2^{-1}(1-2\varepsilon)\big).$$

Then, as in the proof of Theorem 1.3 we have almost surely $\mathcal{Y}_2^{-1}(1-2\varepsilon) = \mathcal{Y}_2^{-1}(1)$ for all ε small enough so $\mathcal{Y}_1^\natural\big(\mathcal{Y}_2^{-1}(1-2\varepsilon)\big)$ converges almost surely to $\mathcal{Y}_1^\natural\big(\mathcal{Y}_2^{-1}(1)\big)$ when ε goes to 0. Thus, we get

$$\limsup_{t\to+\infty} \mathbb{P}\left[\mathcal{L}^*(H(m_{N_t+1})) \le \alpha t\right] \le \mathbb{P}\left(\mathcal{Y}_1^\natural\big(\mathcal{Y}_2^{-1}(1)\big) \le \alpha\right).$$

A lower bound is proved similarly, so we get the following, proving (1.8):

$$\lim_{t\to+\infty} \mathbb{P}\left[\mathcal{L}^*(H(m_{N_t+1})) \le \alpha t\right] = \mathbb{P}\left(\mathcal{Y}_1^\natural\big(\mathcal{Y}_2^{-1}(1)\big) \le \alpha\right).$$

To obtain the result (1.10) for the favorite site, we first argue that we essentially need to obtain the asymptotic behavior of N_t^*/N_t, where $N_t^* := \min\{j \geq 1, \mathcal{L}(m_j,t) = \max_{1\leq k\leq N_t}\mathcal{L}(m_k,t)\}$. Indeed, define for any $\varepsilon\in(0,1/2)$,

$$\begin{aligned}\mathcal{K}_1 &:= \left\{(1-\varepsilon)m_{N_t} \leq X(t) \leq (1+\varepsilon)m_{N_t}\right\},\\ \mathcal{K}_2 &:= \left\{(1-\varepsilon)m_{N_t^*} \leq F_t^* \leq (1+\varepsilon)m_{N_t^*}\right\}.\end{aligned}$$

Then, we have, $\lim_{t\to+\infty}\mathbb{P}(\mathcal{K}_1)=1$ by the localization result Theorem 1.2 combined with the fact that $X(t)/t^\kappa$ converges in law under $\mathbb{P}$ to a positive limit as $t\to+\infty$ by Kawazu and Tanaka (1997).

Let us now justify that $\lim_{t\to+\infty}\mathbb{P}(\mathcal{K}_2)=1$. According to (5.18) proved at the start of the proof of Theorem 1.3, to Lemma 3.4 and (3.3), we have

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}}\mathcal{L}(t,x)\geq 2\tilde{w}_t, \mathcal{B}_4(n_t), N_t\leq n_t\right) \underset{t\to+\infty}{\longrightarrow} 1.$$

Notice that on the event inside the above probability, for t large enough so that $2\tilde{w}_t \geq te^{-2\phi(t)}$, we have $F_t^*\in\mathcal{D}_{N_t^*}$ (recall the definition of $\mathcal{D}_j$ in (3.13)). Since $\mathcal{D}_{N_t^*}$ is centered at $m_{N_t^*}$ and its half-length is deterministic and equal to $r_t = C_0\phi(t)$ we only need to justify that

$$\mathbb{P}\left(\varepsilon m_{N_t^*}\geq C_0\phi(t)\right) \underset{t\to+\infty}{\longrightarrow} 1.$$

We have $m_{N_t^*}\geq m_1$ and $\mathbb{P}(m_1\geq C_0\phi(t)/\varepsilon)\geq\mathbb{P}(\tilde{m}_1\geq C_0\phi(t)/\varepsilon)-o(1)$ by Lemma 2.2. So using (6.13), we thus deduce that $\lim_{t\to+\infty}\mathbb{P}(\mathcal{K}_2)=1$.

We can now write for $x>0$,

$$\mathbb{P}\left[\frac{F_t^*}{X(t)}\leq x\right] = \mathbb{P}\left[\frac{F_t^*}{X(t)}\leq x,\mathcal{K}_1,\mathcal{K}_2\right]+v(\varepsilon,t)\leq \mathbb{P}\left[\frac{m_{N_t^*}}{m_{N_t}}\leq x\frac{1+\varepsilon}{1-\varepsilon}\right]+v(\varepsilon,t).$$

where $v(\varepsilon,t)\geq 0$, satisfies $\lim_{\varepsilon\to 0}\lim_{t\to+\infty}v(\varepsilon,t)=0$. Similarly, we have

$$\mathbb{P}\left[\frac{F_t^*}{X(t)}\leq x\right]\geq \mathbb{P}\left[\frac{m_{N_t^*}}{m_{N_t}}\leq x\frac{1-\varepsilon}{1+\varepsilon}\right]-v(\varepsilon,t).$$

Hence, we obtain

$$\mathbb{P}\left[\frac{m_{N_t^*}}{m_{N_t}}\leq x\frac{1-\varepsilon}{1+\varepsilon}\right]-v(\varepsilon,t)\leq \mathbb{P}\left[\frac{F_t^*}{X(t)}\leq x\right]\leq \mathbb{P}\left[\frac{m_{N_t^*}}{m_{N_t}}\leq x\frac{1+\varepsilon}{1-\varepsilon}\right]+v(\varepsilon,t). \tag{5.45}$$

So, we observe that we only have to study the random variable $\frac{m_{N_t^*}}{m_{N_t}}$. For that we first remark that N_t^* and N_t diverge when t goes to infinity. Indeed by Lemma 6.1, the correct normalisation for the convergence in law of N_t is $e^{\kappa\phi(t)}$, so $\mathbb{P}(N_t\geq e^{(1-\varepsilon)\kappa\phi(t)})=1-o(1)$. For N_t^*, we first notice that the previous result for N_t also gives for t large, $\mathbb{P}(N_t\geq e^{(1-\varepsilon/2)\kappa\phi(t)})=1-o(1)$. Therefore

$$\mathbb{P}\Big(N_t^*\leq e^{(1-\varepsilon)\kappa\phi(t)}\Big)\leq\mathbb{P}\Big(\max_{k\leq e^{(1-\varepsilon)\kappa\phi(t)}}\mathcal{L}(m_k,t)\geq\max_{k<e^{(1-\varepsilon/2)\kappa\phi(t)}}\mathcal{L}(m_k,t)\Big)+o(1).$$

Now, since $\mathcal{L}(m_k,t)=\mathcal{L}\big(\tilde{m}_k,H\big(\tilde{L}_k\big)\wedge\big(H\big(\tilde{m}_k\big)+H_{\tilde{m}_k\to\tilde{L}_k^-}\big)\big)=:\widehat{\ell}_k$ for $k<N_t$ on $\mathcal{V}_t\cap\{N_t\leq n_t\}\cap\mathcal{B}_2(n_t)$ which has probability $1-o(1)$ by Lemmas 2.2 and 3.1,

$$\begin{aligned}&\mathbb{P}\Big(\max_{k\leq e^{(1-\varepsilon)\kappa\phi(t)}}\mathcal{L}(m_k,t)\geq\max_{k<e^{(1-\varepsilon/2)\kappa\phi(t)}}\mathcal{L}(m_k,t)\Big)\\ \leq\ &\mathbb{P}\Big(\max_{k\leq e^{(1-\varepsilon)\kappa\phi(t)}}\widehat{\ell}_k\geq\max_{k<e^{(1-\varepsilon/2)\kappa\phi(t)}}\widehat{\ell}_k\Big)+o(1),\end{aligned}$$

with $\left(\widehat{\ell}_k, k \le e^{(1-\varepsilon/2)\kappa\phi(t)}\right)$ i.i.d. random variables under $\mathbb{P}$ by strong Markov property and the second part of Lemma 2.2, and with queue distributions given by (4.1) and Proposition 3.5.
It is then clear that for large t, $\mathbb{P}(\max_{k\le e^{(1-\varepsilon)\kappa\phi(t)}} \widehat{\ell}_k \ge \max_{k<e^{(1-\varepsilon/2)\kappa\phi(t)}} \widehat{\ell}_k) = o(1)$, and we therefore obtain that $\mathbb{P}(N_t^* \ge e^{(1-\varepsilon)\kappa\phi(t)}) = 1 - o(1)$.
Then, following the work of Faggionato (2009), we know that $(m_i - m_{i-1}, i \ge 2)$ are i.i.d. random variables with a known Laplace transform (given by (2.19) in Faggionato, 2009), this allows to compute the first and fourth moments of $\Delta m_1 := m_2 - m_1$ and therefore obtain after an elementary but tedious computation that for large t, $\mathbb{E}(\Delta m_1) \sim C_7 e^{\kappa h_t}$ ($C_7 > 0$, see also (2.17) in Faggionato, 2009) and $\mathbb{E}((\Delta m_1 - \mathbb{E}(\Delta m_1))^4) \sim C_8 e^{4\kappa h_t}$ ($C_8 > 0$), which yields as $t \to +\infty$ and $k \to +\infty$,

$$\mathbb{E}\left[(m_k/k - \mathbb{E}(\Delta m_1))^4\right] \sim C_8 e^{4\kappa h_t}/k^2.$$

These facts allow us to write by a Markov inequality that

$$\begin{aligned}
& \mathbb{P}\left[|m_{N_t} - \mathbb{E}(\Delta m_1)N_t| > \varepsilon\mathbb{E}(\Delta m_1)N_t\right] \\
\le\ & \sum_{j\ge e^{(1-\varepsilon)\kappa\phi(t)}} \mathbb{P}\Big[\big|m_j - \mathbb{E}(\Delta m_1)j\big| > \varepsilon\mathbb{E}(\Delta m_1)j\Big] + o(1) \\
\le\ & \sum_{j\ge e^{(1-\varepsilon)\kappa\phi(t)}} \frac{2C_8(C_7)^{-4}}{\varepsilon^4 j^2} + o(1) \\
\le\ & C_+\varepsilon^{-4} e^{-(1-\varepsilon)\kappa\phi(t)} + o(1).
\end{aligned}$$

This yields that $\{|m_{N_t} - \mathbb{E}(\Delta m_1)N_t| \le \varepsilon\mathbb{E}(\Delta m_1)N_t\}$ as well as (with a similar computation) $\{\big|m_{N_t^*} - \mathbb{E}(\Delta m_1)N_t^*\big| \le \varepsilon\mathbb{E}(\Delta m_1)N_t^*\}$ are realized with a probability close to one.

Now including these events in the probability in (5.45), eventually enlarging $v(\varepsilon,t)$ we get

$$\mathbb{P}\left[\frac{N_t^*}{N_t} \le x\frac{(1-\varepsilon)^2}{(1+\varepsilon)^2}\right] - v(\varepsilon,t) \le \mathbb{P}\left[\frac{F_t^*}{X(t)} \le x\right] \le \mathbb{P}\left[\frac{N_t^*}{N_t} \le x\frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}\right] + v(\varepsilon,t).$$

Notice that the random variables involved now (N_t^* and N_t) only depend of what happens in the bottom of the h_t-valleys, and we have to deal with

$$\mathbb{P}\left[\frac{N_t^*}{N_t} \le y\right] = \mathbb{P}\big[N_t^* = N_t\big]\mathbb{1}_{\{y=1\}} + \mathbb{P}\left[\frac{N_t^*}{N_t} \le y,\ N_t^* < N_t\right]\mathbb{1}_{\{y\le 1\}} + \mathbb{1}_{\{y>1\}},$$

for any $y > 0$. We are now interested in the limit when t goes to infinity of the above two probabilities. We first use the same lines as for the proof of Section 5.1, that is to say we give a lower and an upper bound of this probability involving the i.i.d. sequences (ℓ_j, j) and $(\mathcal{H}_j, j)$. In the same way we have obtained Proposition 5.1, we then have for any $\varepsilon > 0$ and large t,

$$\tilde{\mathcal{P}} - v(\varepsilon,t) \le \mathbb{P}\left(N_t^* = N_t\right) \le \tilde{\mathcal{P}} + v(\varepsilon,t)$$

with

$$\tilde{\mathcal{P}} := \mathbb{P}\left[(1 - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}-1})\frac{\bar{\ell}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\ell}_{\mathcal{N}_t^{2\varepsilon}-1}}{(\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}} - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}-1})} > \max_{1\le j\le \mathcal{N}_t^{2\varepsilon}-1} \frac{\ell_j}{t}\right],$$

recall that $\bar{\mathcal{H}}_k = Y_2(ke^{-\kappa\phi(t)}) = \frac{1}{t}\sum_{i=1}^k \mathcal{H}_i$, $\bar{\ell}_k = Y_1(ke^{-\kappa\phi(t)}) = \frac{1}{t}\sum_{i=1}^k \ell_i$, $\mathcal{N}_t^{2\varepsilon} := \inf\{m \ge 1, \bar{\mathcal{H}}_m > 1 - 2\varepsilon\}$, and v is a positive function such that $\lim_{t\to+\infty} v(\varepsilon,t) \le$

const $\times\ \varepsilon^{\kappa\wedge(1-\kappa)}$ with an eventually larger const than in Proposition 5.1. In the same way, for any $y>0$, $\varepsilon>0$ and t large enough,

$$\bar{\mathcal{P}}_1^- - v(\varepsilon,t) \le \mathbb{P}\left[\frac{N_t^*}{N_t}\le y,\ N_t^*<N_t\right]\mathbb{1}_{y\le 1}\le \bar{\mathcal{P}}_1^+ + v(\varepsilon,t),$$

where

$$\begin{aligned}&\tilde{\mathcal{P}}_1^\pm\\ &:=\mathbb{P}\left[\mathcal{N}_t^*/\mathcal{N}_t^{2\varepsilon}\le y\pm\varepsilon,(1-\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}-1})\frac{\bar{\ell}_{\mathcal{N}_t^{2\varepsilon}}-\bar{\ell}_{\mathcal{N}_t^{2\varepsilon}-1}}{(\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}}-\bar{\mathcal{H}}_{\mathcal{N}_t^{2\varepsilon}-1})}\le \max_{1\le j\le \mathcal{N}_t^{2\varepsilon}-1}\frac{\ell_j}{t}\right]\mathbb{1}_{y\le 1},\end{aligned}$$

with $\mathcal{N}_t^*:=\min\{j\ge 1,\ell_j=\max_{k\le\mathcal{N}_t^{2\varepsilon}}\ell_k\}$. This together with Lemma 4.6 yields for large t,

$$\left|\mathbb{P}\left[N_t^*=N_t\right]-\mathbb{P}\left[\mathcal{I}_1<\mathcal{I}_2\right]\right|\le \lim_{t\to+\infty} v(\varepsilon,t)+o(1)$$

and

$$\begin{aligned}&\left|\mathbb{P}\left[\frac{N_t^* e^{-\kappa\phi(t)}}{N_t e^{-\kappa\phi(t)}}\le y, N_t^*<N_t\right]-\mathbb{P}\left[\frac{F^*(\mathcal{Y}_1,\mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)}\le y,\ \mathcal{I}_1\ge\mathcal{I}_2\right]\right|\\ \le\ &\lim_{t\to+\infty} v(\varepsilon,t),+o(1),\end{aligned}$$

where F^* is defined at the beginning of Section 4.3. Replacing y by $x\frac{(1-\varepsilon)^2}{(1+\varepsilon)^2}$ for the lower bound and by $x\frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}$ for the upper bound and taking the limit when t goes to infinity and then $\varepsilon\to 0$ we obtain for $0<x<1$,

$$\lim_{t\to+\infty}\mathbb{P}\left[\frac{N_t^*}{N_t}\le x\right]=\mathbb{P}\left[\frac{F^*(\mathcal{Y}_1,\mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)}\le x,\ \mathcal{I}_1\ge\mathcal{I}_2\right].$$

To finish the proof of the last result of Theorem 1.5 we finally have to prove Lemma 5.5 below.

Lemma 5.5. *The random variable* $\frac{F^*(\mathcal{Y}_1,\mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)}$ *follows a uniform law* $U_{[0,1]}$ *and is independent of the couple* $(\mathcal{I}_1,\mathcal{I}_2)$.

Proof: For any $s>0$, let $\mathcal{G}_1(s):=\inf\{u\le s,\ \mathcal{Y}_1(u)-\mathcal{Y}_1(u-)=\mathcal{Y}_1^\sharp(s)\}$. The fact that for every $s>0$, $\mathcal{G}_1(s)/s$ follows a uniform distribution is basic. Since the independence that we seek is specific we give some details.

The process of the jumps of $(\mathcal{Y}_1,\mathcal{Y}_2)$ in $[0,s]$ is a Poisson point process in $[0,s]\times(\mathbb{R}_+)^2$ (the coordinate in $[0,s]$ for the instant when the jump occurs and the other coordinate for the jump) with intensity measure $\lambda\times\nu$ where λ is the Lebesgue measure on $[0,s]$ and ν, as defined in the introduction, is the Lévy measure of $(\mathcal{Y}_1,\mathcal{Y}_2)$. Let us give a particular construction of the process $(\mathcal{Y}_1,\mathcal{Y}_2)$ on $[0,s]$:

Let $(P_n)_{n\ge1}$ be a countable partition of $(\mathbb{R}_+)^2$ by Borelian sets such that $\forall n\ge 1,\ 0<\nu(P_n)<+\infty$. For each n we define an *i.i.d.* sequence $(S_k^n)_{k\ge1}$ of random variables in $(\mathbb{R}_+)^2$, an *i.i.d.* sequence $(U_k^n)_{k\ge1}$ of random variables in $[0,s]$ and a random variable T_n such that

- $\forall n\ge1,\ S_1^n\sim\nu(.\cap P_n)/\nu(P_n),\ U_1^n\sim U_{[0,s]},\ T_n\sim\mathcal{P}(s\,\nu(P_n))$,
- For any $n\ge1$, the variables $(S_k^n)_{k\ge1}$, $(U_k^n)_{k\ge1}$ and T_n are independent,
- The triplets $\left((S_k^n)_{k\ge1},(U_k^n)_{k\ge1},T_n\right)_{n\ge1}$ are independent,

where U stands for uniform and $\mathcal{P}(.)$ for Poisson distribution. We know that the random set

$$\mathcal{S}_n := \{(U_k^n, S_k^n),\ n \geq 1,\ 1 \leq k \leq T_n\}$$

is a Poisson point process in $[0, s] \times (\mathbb{R}_+)^2$ with intensity measure $\lambda \times \nu$. Since $(\mathcal{Y}_1, \mathcal{Y}_2)$ is pure jump, its restriction to $[0, s]$ is equal in law to the process $(\mathcal{Z}_1, \mathcal{Z}_2)$ defined by

$$\forall r \in [0, s], \qquad (\mathcal{Z}_1, \mathcal{Z}_2)(r) = \sum_{n \geq 1, 1 \leq k \leq T_n} S_k^n\ \mathbb{1}_{U_k^n \leq r}.$$

In particular, with $\pi_i(x_1, x_2) := x_i$ for $(x_1, x_2) \in \mathbb{R}^2$, $i \in \{1, 2\}$ and $\mathcal{G}_1^{\mathcal{Z}}(s) := \inf\left\{u \leq s,\ \mathcal{Z}_1(u) - \mathcal{Z}_1(u^-) = \mathcal{Z}_1^\sharp(s)\right\} \overset{\mathfrak{L}}{=} \mathcal{G}_1(s)$, we have

$$\begin{aligned}
&\mathcal{Z}_1^\sharp(s) = \max\{\pi_1(S_k^n),\ n \geq 1,\ 1 \leq k \leq T_n\}, \\
&\mathcal{G}_1^{\mathcal{Z}}(s) = \inf\left\{U_k^n,\ n \geq 1,\ 1 \leq k \leq T_n,\ \pi_1(S_k^n) = \mathcal{Z}_1^\sharp(s)\right\}, \\
&\mathcal{Z}_1(s) = \sum_{n \geq 1, 1 \leq k \leq T_n} \pi_1(S_k^n), \qquad \mathcal{Z}_2(s) = \sum_{n \geq 1, 1 \leq k \leq T_n} \pi_2(S_k^n).
\end{aligned}$$

We thus have that $\mathcal{G}_1(s)/s \overset{\mathfrak{L}}{=} U_{[0,1]}$ and it is independent from $(\mathcal{Y}_1^\sharp(s), \mathcal{Y}_1(s), \mathcal{Y}_2(s))$ and from the sigma-field $\sigma((\mathcal{Y}_1, \mathcal{Y}_2)(t + s) - (\mathcal{Y}_1, \mathcal{Y}_2)(s),\ t \geq 0)$.

We now have to replace s by $\mathcal{Y}_2^{-1}(1)$. For that we can consider for example the dyadic approximations of $\mathcal{Y}_2^{-1}(1)$, that is, $(t_n := \max\left\{k \in \mathbb{N}, \frac{k}{2^n} < \mathcal{Y}_2^{-1}(1)\right\}, n)$. Then, partitioning on the values of t_n, using the independence we just proved and the fact that $\mathcal{G}_1(s)/s$ follows a uniform distribution on $[0, 1]$ we get that $\mathcal{G}_1(t_n)/t_n$ follows a uniform distribution on $[0, 1]$ and is independent from

$$\big((\mathcal{Y}_1^\sharp(t_n),\ \mathcal{Y}_2(t_n),\ \mathcal{Y}_1(t_n + 2^{-n}) - \mathcal{Y}_1(t_n),\ \mathcal{Y}_2(t_n + 2^{-n}) - \mathcal{Y}_2(t_n)\big). \tag{5.46}$$

We let n goes to infinity, t_n converges almost surely to $\mathcal{Y}_2^{-1}(1)$ from below. As a consequence, $\mathcal{G}_1(t_n)/t_n$ converges almost surely to $\frac{F^*(\mathcal{Y}_1, \mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)}$ while the quadruple in (5.46) converges almost surely to

$$\begin{aligned}
&\big(\mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1)-),\ \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-), \\
&\qquad\qquad \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)-),\ \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)-)\big).
\end{aligned}$$

As a consequence, $\frac{F^*(\mathcal{Y}_1, \mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)}$ follows a uniform distribution on $[0, 1]$ and is independent from the above quadruple for which $(\mathcal{I}_1, \mathcal{I}_2)$ is a measurable function, this yields the lemma. □

6. Results and additional arguments from the paper Andreoletti and Devulder (2015)

6.1. *Some estimates on the diffusion X.* The first lemma below gives the right normalisation in law of the number of h_t-valleys visited by X before time t.

Lemma 6.1 (number of visited h_t-valleys)**.** *Assume that $0 < \kappa < 1$. Then, under the annealed law $\mathbb{P}$, $N_t e^{-\kappa\phi(t)} \to_{t \to +\infty} \mathcal{N}$ in law. The law of $\mathcal{N}$ is determined by its Laplace transform:*

$$\forall u > 0, \qquad \mathbb{E}\left(e^{-u\mathcal{N}}\right) = \sum_{j=0}^{+\infty} \frac{1}{\Gamma(\kappa j + 1)} \left(\frac{-u}{\mathcal{C}_\kappa}\right)^j, \tag{6.1}$$

where $\mathcal{C}_\kappa$ is a positive constant. Moreover $\mathbb{P}(N_t > n_t) \leq e^{-\phi(t)}$.

Proof: The convergence in distribution is exactly Proposition 1.6 of Andreoletti and Devulder (2015). For the second fact we have $\mathbb{P}(N_t \geq n_t) \leq \mathbb{P}(\tilde{N}_t \geq n_t) + \mathbb{P}(\overline{\mathcal{V}}_t) \leq \mathbb{P}(\tilde{N}_t \geq n_t) + e^{[-\kappa/2+o(1)]h_t}$ by Lemma 2.2, with $\tilde{N}_t := \max\{j \geq 1,\ \tilde{m}_j \leq \sup_{s\leq t} X(s)\}$. Then equation (5.3) in Andreoletti and Devulder (2015) gives $\mathbb{P}(\tilde{N}_t \geq n_t) \leq \exp(-2\phi(t))$, which yields the result. □

The lemma below deals with the renewal structure we speak about on the introduction, and the consequence on the hitting time $H(m_{N_t})$ of the ultimate h_t-valley visited by X before time t.

Lemma 6.2. *Assume $0 < \kappa < 1$ and $0 < \delta < \inf\{2/27, \kappa^2/2\}$. For $t > 0$, let μ_t be the positive measure on $\mathbb{R}_+$ such that*

$$\forall x \geq 0, \qquad \mu_t([0,x]) := e^{-\kappa\phi(t)} \sum_{j=1}^{n_t} \mathbb{P}\left(\bar{\mathcal{H}}_j \leq x\right).$$

Recall that for any k, $\bar{\mathcal{H}}_k := \sum_{j=1}^k \mathcal{H}_j/t$, and $\mathcal{H}_1 = R_1 S_1 \mathbf{e_1}$ is defined in Proposition 3.5. Then, $(\mu_t)_t$ converges vaguely as $t \to +\infty$ to μ defined by

$$\mathrm{d}\mu(x) := (\mathcal{C}_\kappa \Gamma(\kappa))^{-1} x^{\kappa-1} \mathbb{1}_{(0,+\infty)}(x)\mathrm{d}x,$$

with $\mathcal{C}_\kappa$ is the same constant as in Lemma 6.1. For $0 \leq r < s \leq 1$,

$$\lim_{t\to+\infty} \mathbb{P}\left(1 - s \leq \frac{H(m_{N_t})}{t} \leq 1 - r\right) = \frac{\sin(\pi\kappa)}{\pi} \int_{1-s}^{1-r} x^{\kappa-1}(1-x)^{-\kappa}\mathrm{d}x. \tag{6.2}$$

Proof: The first part of the above lemma is very close to Lemma 5.1 of Andreoletti and Devulder (2015), indeed Proposition 3.5 gives the proximity between the random variables $(U_i, i \leq n_t)$ and the random variables $(\mathcal{H}_i, i \leq n_t)$, moreover an important preliminary result in Andreoletti and Devulder (2015) (Proposition 4.1) states that $e^{\kappa\phi(t)}(1 - \mathbb{E}(e^{-\lambda U_1/t})) = \mathcal{C}_\kappa \lambda^\kappa + o(1)$ for large t. So we also know that

$$e^{\kappa\phi(t)}(1 - \mathbb{E}(e^{-\lambda\mathcal{H}_1/t})) = \mathcal{C}_\kappa\lambda^\kappa + o(1), \tag{6.3}$$

notice that this result could also be deduced from (4.2) with the help of a Tauberian theorem. Then by independence of the random variables $\mathcal{H}_j$ and the fact that they are i.d., for any $\lambda > 0$

$$\int_0^{+\infty} e^{-\lambda x}\mathrm{d}\mu_t(x) \quad = \quad \frac{1}{e^{\kappa\phi(t)}} \sum_{j=1}^{n_t} \left(\mathbb{E}\left(e^{-\lambda\ \frac{\mathcal{H}_1}{t}}\right)\right)^j$$

By (6.3) as $n_t e^{-\kappa\phi(t)} \to_{t\to+\infty} +\infty$, $[\mathbb{E}\left(e^{-\lambda\mathcal{H}_1/t}\right)]^{n_t+1} = o(1)$. Hence, we get as $t \to +\infty$, again by 6.3

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda x}\mathrm{d}\mu_t(x) &= \frac{e^{-\kappa\phi(t)}(1+o(1))}{1 - \mathbb{E}\left(e^{-\lambda\mathcal{H}_1/t}\right)} + o(1) = \frac{1}{\mathcal{C}_\kappa\lambda^\kappa} + o(1) \\ &= \int_0^{+\infty} \frac{e^{-\lambda x}x^{\kappa-1}}{\mathcal{C}_\kappa\Gamma(\kappa)}\mathrm{d}x + o(1),\end{aligned}$$

which gives the vague convergence of measure $(\mu_t)_t$. Also (6.2) is equation (1.2) of Corollary 1.5 in Andreoletti and Devulder (2015). □

In Lemma 6.3 below, we approximate $\tilde{h}_j$, the exit time of h_t-valley number j (if X leaves it on the right), by a product of 3 simpler random variables. To this aim, we recall that with the notation of Lemma 3.6 and of its proof, for each $1 \leq j \leq n_t$, $\tilde{R}_j = \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j(h_t/2)} e^{-\tilde{V}^{(j)}(x)}\mathrm{d}x$, and $A^j(u) = \int_{\tilde{m}_j}^u e^{\tilde{V}^{(j)}(x)}\mathrm{d}x$, $u \in \mathbb{R}$. Moreover, for some independent Brownian motions B^j, $1 \leq j \leq n_t$, independent of W_κ,

$$\begin{aligned}\tilde{h}_j &= \int_{\tilde{L}_j^-}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)}\mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)]\mathrm{d}u,\\ \mathbf{e}_j &= \mathcal{L}_{B^j}\left[\tau^{B^j}(A^j(\tilde{L}_j)), 0\right]/A^j(\tilde{L}_j).\end{aligned}$$

Lemma 6.3. *Let $0 < \varepsilon < \inf\{2/27, \kappa^2/2\}$. For large t, we have for every $1 \leq j \leq n_t$,*

$$\mathbb{P}\left(\left|\tilde{h}_j - A^j(\tilde{L}_j)\mathbf{e}_j\tilde{R}_j\right| > 2e^{-(1-3\varepsilon)h_t/6}A^j(\tilde{L}_j)\mathbf{e}_j\tilde{R}_j\right) \leq C_+e^{-(c_-)\varepsilon h_t}. \tag{6.4}$$

Proof: We first notice that $\left(\tilde{h}_j, A^j(\tilde{L}_j), \mathbf{e}_j, \tilde{R}_j\right)$ is measurable with respect to the σ-field generated by $\left(\tilde{V}^{(j)}(x + \tilde{L}_{j-1}^+),\ 0 \leq x \leq \tilde{L}_j^+ - \tilde{L}_{j-1}^+\right)$ and B^j, so, thanks to the second fact of Lemma 2.2, its law under $\mathbb{P}$ does not depend on j. Thus, the left hand side of (6.4) does not depend on j. Hence we just have to prove (6.4) for $j = 2$.

This is actually already proved in Andreoletti and Devulder (2015), for which it is an important step. Indeed in this paper Andreoletti and Devulder (2015), our A^j, $\tilde{B}^2$ and $\tilde{h}_2$ are denoted respectively by $\tilde{A}_j$, B and $\mathbf{U}$, as defined in Andreoletti and Devulder (2015, eq. (3.17) and (3.18)), and our $\tilde{R}_2$ and $\mathbf{e}_2$ by $\mathcal{I}^-$ and $\mathbf{e}_1$, as defined in Andreoletti and Devulder (2015, after eq. (4.17)). Hence our (6.4) for $j = 2$ is exactly Andreoletti and Devulder (2015, Lemma 4.7), which proves our lemma.

The proof of Andreoletti and Devulder (2015, Lemma 4.7) is quite technical, however we can give a simple heuristic in order for the present paper to be more self-contained. The idea of the proof of Andreoletti and Devulder (2015, Lemma 4.7) is that, loosely speaking, for u close to $\tilde{m}_j$, that is for $u \in \left[\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j(h_t/2)\right]$, $\mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)]$ is nearly $\mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), 0] = A^j(\tilde{L}_j)\mathbf{e_j}$, whereas for u far from $\tilde{m}_j$, that is for $u \in [\tilde{L}_j^-, \tilde{L}_j]$ but $u \notin [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j(h_t/2)]$, $e^{-\tilde{V}^{(j)}(x)}$ is "nearly" 0, with large probability. Finally, combining these heuristics gives $\tilde{h}_j \approx A^j(\tilde{L}_j)\mathbf{e_j}\tilde{R}_j$.

□

The following lemma is used to prove Lemma 3.6 and uses the notation of this lemma, and where the independent r.v. $G^+(h_t/2, h_t)$, $F_1^+(h_t)$, $F_2^-(h_t/2)$ and $F_3^-(h_t/2)$ defined before Proposition 3.5.

Lemma 6.4. *Assume $0 < \delta < \inf\{2/27, \kappa^2/2\}$. For large t, possibly on an enlarged probability space, there exists $R_2 \overset{\mathfrak{L}}{=} F_2^-(h_t/2) + F_3^-(h_t/2)$ and $S_2 \overset{\mathfrak{L}}{=} F_1^+(h_t) + G^+(h_t/2, h_t)$, such that R_2, S_2 and $\mathbf{e}_2$ are independent and*

$$P\left(\left\{\left|\int_{\tilde{m}_2}^{\tilde{L}_2} e^{\tilde{V}^{(2)}(x)}\mathrm{d}x - S_2\right| \leq e^{-(d_-)h_t}S_2,\ \tilde{R}_2 = R_2\right\}\right) \geq 1 - e^{-(D_-)h_t}, \tag{6.5}$$

where $D_- > 0$.

Proof: Due to Andreoletti and Devulder (2015, Lemma 4.5) with its notation, we have $\mathcal{I}_0^+ := \int_{m_2}^{\tau_2(h_t)} e^{V^{(2)}(x)}\mathrm{d}x \overset{\mathfrak{L}}{=} F^+(h_t)$, $\mathcal{I}_2^+ := \int_{\tau_2(h_t)}^{L_2} e^{V^{(2)}(x)}\mathrm{d}x \overset{\mathfrak{L}}{=} G^+(h_t/2, h_t)$, $\mathcal{I}_1^- := \int_{m_2}^{\tau_2(h_t/2)} e^{-V^{(2)}(x)}\mathrm{d}x \overset{\mathfrak{L}}{=} F^-(h_t/2)$ and finally $\mathcal{I}_2^- := \int_{\tau_2^-(h_t/2)}^{m_2} e^{-V^{(2)}(x)}\mathrm{d}x \overset{\mathfrak{L}}{=} F^-(h_t/2)$ with $L_2 := \inf\{x > \tau_2(h_t),\, V^{(2)}(x) = h_t/2\}$. The problem is that $\mathcal{I}_0^+$ is not independent of $\mathcal{I}_1^-$, so we would like to replace it by some $\mathcal{I}_1^+ \overset{\mathfrak{L}}{=} \mathcal{I}_0^+$ of it with better independence properties. It is proved in Andreoletti and Devulder (2015, at the top of page 32) that for large t, possibly in an enlarged probability space, there exists $\mathcal{I}_1^+$ such that $|\mathcal{I}_0^+ - \mathcal{I}_1^+| \leq e^{-(1-3\delta)h_t/2}\mathcal{I}_1^+$ with probability greater than $1 - 4e^{-\kappa\delta h_t/2}$ and where $\mathcal{I}_1^+ \overset{\mathfrak{L}}{=} F^+(h_t)$ by Andreoletti and Devulder (2015, eq. (4.35)).

Let $S_2 := \mathcal{I}_1^+ + \mathcal{I}_2^+ \geq \mathcal{I}_1^+$. Notice that on $\mathcal{V}_t$, by Remark 2.3, $\tilde{R}_2 = \mathcal{I}_1^- + \mathcal{I}_2^- =: R_2$ and $\int_{\tilde{m}_2}^{\tilde{L}_2} e^{\tilde{V}^{(2)}(x)}\mathrm{d}x = \int_{m_2}^{L_2} e^{V^{(2)}(x)}\mathrm{d}x = \mathcal{I}_0^+ + \mathcal{I}_2^+$. The two previous inequalities give $\left| \int_{\tilde{m}_2}^{\tilde{L}_2} e^{\tilde{V}^{(2)}(x)}\mathrm{d}x - S_2 \right| = \left|\mathcal{I}_0^+ - \mathcal{I}_1^+\right| \leq e^{-(1-3\delta)h_t/2} S_2$ and $\tilde{R}_2 = R_2$ with probability at least $1 - 5e^{-\kappa\delta h_t/2}$ thanks to Lemma 2.2. This proves (6.5).

Moreover, by Andreoletti and Devulder (2015, Prop. 4.4 (i)), $\mathcal{I}_1^+$, $\mathcal{I}_2^+$, $\mathcal{I}_1^-$, $\mathcal{I}_2^-$ and $\mathbf{e}_2$ (which is denoted by $\mathbf{e}_1$ in Andreoletti and Devulder, 2015) are independent. So, $\mathbf{e}_2$, $S_2 = \mathcal{I}_1^+ + \mathcal{I}_2^+$ and $R_2 = \mathcal{I}_1^- + \mathcal{I}_2^-$ are independent, and $R_2 \overset{\mathfrak{L}}{=} F_2^-(h_t/2) + F_3^-(h_t/2)$ and $S_2 \overset{\mathfrak{L}}{=} F_1^+(h_t) + G^+(h_t/2, h_t)$. □

The last lemma of this section tells that with large probability, the diffusion X leaves every h_t-valley $[\tilde{L}_j^-, \tilde{L}_j]$, $1 \leq j \leq n_t$ from its right. Recall that B^j is defined after (3.19).

Lemma 6.5. *For large t, there exists $c_- > 0$ such that*

$$\mathbb{P}\left[\cap_{j=1}^{n_t} \left\{ \max_{u < \tilde{L}_j^-} \mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] = 0 \right\} \right] \geq 1 - e^{-(c_-)h_t}. \tag{6.6}$$

Proof: (6.6) is essentially Lemma 3.2 in Andreoletti and Devulder (2015):

Indeed, recall the definition of $\mathcal{A}_j := \{\max_{u<\tilde{L}_j^-} \mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] = 0\}$, we have $\cap_{j=1}^{n_t}\mathcal{A}_j = \cap_{j=1}^{n_t}\{H_j(\tilde{L}_j) < \{H_j(\tilde{L}_j^-)\}$, with, for any $\tilde{L}_j^- \leq x \leq \tilde{L}_j$, $H_j(x) = \inf\{s > 0, B_j(s) = x\}$, with B_j a Brownian motion. Therefore $\mathbb{P}^{W_\kappa}(\mathcal{A}_j)$ is equal to the probability $\mathbb{P}^{W_\kappa}(\overline{\mathcal{E}}_j)$ of Lemma 3.2 in Andreoletti and Devulder (2015). It is proved in this lemma see (3.10) that for large t, $P(\mathcal{B} := \{\mathbb{P}^{W_\kappa}(\overline{\mathcal{E}}_j) \leq e^{-(\kappa/2)h_t}\}) \geq 1 - 3e^{-\kappa\delta h_t}$, so we obtain (6.6) as $\mathbb{P}(\overline{\mathcal{E}}_j) \leq E(\mathbb{P}^{W_\kappa}(\overline{\mathcal{E}}_j)\mathbb{1}_{\mathcal{B}}) + P(\overline{\mathcal{B}}) \leq e^{-c_- h_t}/n_t$, for $c_- > 0$ small enough. □

6.2. *Some estimates on the potential W_κ and its functionals.*
We start this section with the Laplace transform of the important functional $\mathcal{R}_\kappa$:

Lemma 6.6. *Recall that $0 < \kappa < 1$. For any $\gamma > 0$,*

$$E\big(e^{-\gamma\mathcal{R}_\kappa}\big) = \left(\frac{(2\gamma)^{\kappa/2}}{\kappa\Gamma(\kappa)I_\kappa(2\sqrt{2\gamma})} \right)^2. \tag{6.7}$$

Moreover, $\mathcal{R}_\kappa$ admits moments of any positive order.

Proof: $\int_0^{+\infty} e^{-W_\kappa^\uparrow(u)}\mathrm{d}u$ is the limit in law under P of $\int_0^{\tau^{W_\kappa^\uparrow}(x)} e^{-W_\kappa^\uparrow(u)}\mathrm{d}u$ as $x \to +\infty$. This limit is given by Andreoletti and Devulder (2015, Lemma 4.2), which proves (6.7). Note that in Andreoletti and Devulder (2015, Lemma 4.2), $W_\kappa^\uparrow$ is denoted by R, and $\int_0^{\tau^{W_\kappa^\uparrow}(x)} e^{-W_\kappa^\uparrow(u)}\mathrm{d}u$ is denoted respectively by $F^-(x)$. Moreover the Laplace transform of $\mathcal{R}_\kappa$ is of class C^∞ on a neighborhood of 0 since $x \mapsto x^\kappa/I_\kappa(x)$ is C^∞ on such a neighborhood (see e.g. Borodin and Salminen, 2002 p. 638). Therefore $\mathcal{R}_\kappa$ admits moments of any positive order. $\square$

The following Lemma is a series of estimates concerning the different coordinates of valleys.

Lemma 6.7. *For t large enough, for every $1 \le i \le n_t$,*

$$P(0 < M_0 < m_1) \le C_+ h_t e^{-\kappa h_t}, \tag{6.8}$$

$$P(\tilde{\tau}^*_{i+1}(h_t) \neq \tilde{\tau}_{i+1}(h_t)) \le C_+ h_t e^{-\kappa h_t}, \tag{6.9}$$

$$P\Big(\inf_{[\tilde{\tau}_i^-(h_t^+),\tilde{\tau}_i^-(h_t)]} \tilde{V}^{(i)} < h_t/2\Big) \le e^{-\kappa h_t/8}, \tag{6.10}$$

$$P(\tilde{L}_i^+ - \tilde{L}_i^- \ge 40 h_t^+/\kappa) \le e^{-\kappa h_t/8}, \tag{6.11}$$

$$P(\tilde{\tau}_i(h) - \tilde{m}_i \ge 8h/\kappa) \le C_+ e^{-\kappa h/(2\sqrt{2})}, \qquad 0 \le h \le h_t, \tag{6.12}$$

$$P(\tilde{m}_1 \le r) \le e^r \exp\Big(\big(\kappa/2 - \sqrt{2+\kappa^2/4}\big)h_t^+\Big) = o(1), \qquad \forall r = o(h_t^+). \tag{6.13}$$

Proof: (6.8) follows from eq. (2.8) of Andreoletti and Devulder (2015); (6.9) is eq. (3.41) of Andreoletti and Devulder (2015). (6.10) and (6.11) are respectively eq. (2.34) and (2.32) of Lemma 2.7 of Andreoletti and Devulder (2015). Moreover, (6.12) is eq. (2.22) of the same reference. For (6.13), we know from definitions in (2.3) that $\tilde{m}_1 \ge \tilde{L}_1^\sharp = \tau^{W_\kappa}(-h_t^+)$, where $\tau^{W_\kappa}(-h_t^+)$ is the first positive time the drifted Brownian motion W_κ reaches $-h_t$. Using a Markov inequality together with (2.0.1) page 295 of Borodin and Salminen (2002) we obtain $P(\tau^{W_\kappa}(-h_t^+) \le r) = P(e^{-\tau^{W_\kappa}(-h_t^+)} \ge e^{-r}) \le e^r e^{(\kappa/2-\sqrt{2+\kappa^2/4})h_t^+}$, which is exactly (6.13). $\square$

The lemma below deals with two functionals involving coordinates far from the bottom $\tilde{m}_1$ of the first visited h_t-valley $[\tilde{L}_1^-, \tilde{L}_1]$.

Lemma 6.8. *There exists $c_- > 0$ such that for any $\varepsilon > 0$ and t large enough,*

$$P\left(\int_{\tilde{\tau}_1(h_t/2)}^{\tilde{L}_1} e^{-\tilde{V}^{(1)}(x)}\mathrm{d}x \le C_+ h_t^2 e^{-(1-\varepsilon)h_t/2}\right) \ge 1 - e^{-(c_-)\varepsilon h_t},$$

$$P\left(\int_{\tilde{L}_1^-}^{\tilde{\tau}_1^-(h_t/2)} e^{-\tilde{V}^{(1)}(x)}\mathrm{d}x \le C_+ h_t^2 e^{-(1-\varepsilon)h_t/2}\right) \ge 1 - e^{-(c_-)\varepsilon h_t}.$$

Proof: The proof is inspired from steps 1 and 2 of Lemma 4.7 of Andreoletti and Devulder (2015). For the first integral, let

$$\mathcal{A}_1 := \big\{\inf_{[\tilde{\tau}_1(h_t/2)),\tilde{\tau}_1(h_t)]} \tilde{V}^{(1)} > (1-\varepsilon)h_t/2\big\}, \qquad \mathcal{A}_2 := \big\{\tilde{L}_1^+ - \tilde{L}_1^- \le 40h_t^+/\kappa\big\}.$$

We have on $\mathcal{A}_1 \cap \mathcal{A}_2$,

$$\int_{\tilde{\tau}_1(h_t/2))}^{\tilde{L}_1} e^{-\tilde{V}^{(1)}(u)}\mathrm{d}u \le e^{-(1-\varepsilon)h_t/2}\big[\tilde{L}_1 - \tilde{\tau}_1(h_t/2))\big] \le \frac{40h_t^+ h_t}{\kappa} e^{-(1-\varepsilon)h_t/2}. \tag{6.14}$$

Now, Fact 2.1, equation (7.3) with $\alpha = 1/2$, $\gamma = (1-\varepsilon)/2$ and $\omega = 1$, and Lemma 2.2 give

$$P\big(\overline{\mathcal{A}_1}\big) \leq P\big[\inf_{[\tau_1(h_t/2),\tau_1(h_t)]} V^{(1)} \leq (1-\varepsilon)h_t/2, \mathcal{V}_t\big] + P(\overline{\mathcal{V}}_t) \leq 3e^{-\kappa\varepsilon h_t/2}.$$

Moreover, $P\big(\overline{\mathcal{A}_2}\big) \leq e^{-\kappa h_t/8} \leq e^{-\kappa\varepsilon h_t/2}$ by (6.11) since we can take $\varepsilon < 1/4$. The second inequality, can be proved similarly. □

Lemma 6.9. *Recall that for $h > 0$, $\beta_0(h) := E\left(\int_0^{\tau_1^*(h)} e^{W_\kappa(u)}\mathrm{d}u\right)$, with $\tau_1^*(h) := \inf\{u \geq 0,\ W_\kappa(u) - \inf_{[0,u]} W_\kappa \geq h\}$. For large h,*

$$\beta_0(h) \leq C_+ e^{(1-\kappa)h}. \tag{6.15}$$

Proof: (6.15) is Andreoletti and Devulder (2015, eq. (3.38)), since in Andreoletti and Devulder (2015), $\beta_0(h)$ is defined at the top of page 23 and $\tau_1^*(h)$ in its Lemma 3.6. □

7. Appendix

7.1. *Some estimates for Brownian motion, Bessel processes, $W_\kappa^\uparrow$ and their functionals.* We provide in this section some known formulas for some processes that appear in our study. The first lemma is about Laplace transforms of the exponential functionals defined in (1.6) and (1.7). Its proof can be found in Andreoletti and Devulder (2015, Lemma 4.2). Recall that C_+ (respectively c_-) is a positive constant that is as large (resp. small) as needed.

Lemma 7.1. *There exist $C_9 > 0$, $M > 0$ and $\eta_1 \in (0,1)$ such that $\forall y > M, \forall \gamma \in (0, \eta_1]$,*

$$\left|E\left(e^{-\gamma F^+(y)/e^y}\right) - [1 - 2\gamma/(\kappa+1)]\right| \leq C_9 \max(e^{-\kappa y}, \gamma^{3/2}), \tag{7.1}$$

$$\left|E\left(e^{-\gamma G^+(y/2,y)/e^y}\right) - [1 - \Gamma(1-\kappa)(2\gamma)^\kappa/\Gamma(1+\kappa)]\right| \leq C_9 \max(\gamma^\kappa e^{-\kappa y/2}, \gamma). \tag{7.2}$$

Moreover, there exists $C_{10} > 0$ such that for all $y > 0$, $E\left(F^+(y)/e^y\right) \leq C_{10}$.

Recall that $W_\kappa^\uparrow$ is a $(-\kappa/2)$-drifted Brownian motion W_κ Doob-conditioned to stay positive (see above (1.6)). We have,

Lemma 7.2. *Let $0 < \gamma < \alpha < \omega$. For all h large enough, we have*

$$P^{\alpha h}\left(\tau^{W_\kappa^\uparrow}(\gamma h) < \tau^{W_\kappa^\uparrow}(\omega h)\right) \leq 2e^{-\kappa(\alpha-\gamma)h}, \tag{7.3}$$

$$P\left(\tau^{W_\kappa^\uparrow}(\omega h) - \tau^{W_\kappa^\uparrow}(\alpha h) \leq 1\right) \leq 4e^{-[(\omega-\alpha)h]^2/3}, \tag{7.4}$$

$$P\big(\tau^{W_\kappa^\uparrow}(h) > 8h/\kappa\big) \leq C_+ e^{-\kappa h/(2\sqrt{2})}, \tag{7.5}$$

$$P\big(\tau^{W_\kappa^\uparrow}(h) \leq h\big) \leq C_+ e^{-(c_-)h}, \tag{7.6}$$

$$P\big(\tau^{W_\kappa^\uparrow}(\gamma h) \leq 1\big) \leq C_+ e^{-(c_-)[\gamma h]^2}, \tag{7.7}$$

where $P^{\alpha h}$ denotes the law of $W_\kappa^\uparrow$ starting from αh. Moreover the first inequality is still true if ω is a function of h such that $\lim_{h\to+\infty}\omega(h) = +\infty$.

Proof: The first 3 inequalities come from Andreoletti and Devulder (2015, Lemma 2.6). The fact that, in (7.3), ω can actually be taken as a function of h comes directly from eq. (2.31) of Andreoletti and Devulder (2015), which shows that the right hand side of (7.3) is equivalent to $e^{-\kappa(\alpha-\gamma)h}$ as $h \to +\infty$ if $w = w(h) \to_{h\to+\infty} +\infty$. (7.7) is a consequence of (7.4) with $\omega = \gamma$ and $\alpha = \gamma/2$. We turn to (7.6). By Andreoletti and Devulder (2015, eq. (2.7)) and Fact 2.1, coming from Faggionato (2009), $E\big(e^{-\alpha\tau^{W_\kappa^\uparrow}(h)}\big) \sim_{h\to+\infty} \text{const}.e^{h(\kappa/2-\sqrt{2\alpha+\kappa^2/4})}$, in particular for $\alpha = 1-\kappa$. Then a Markov inequality for $P\big(e^{-\alpha\tau^{W_\kappa^\uparrow}(h)} > e^{-\alpha h}\big)$ proves (7.6) since $1-\kappa/2-\sqrt{2(1-\kappa)+\kappa^2/4} < 0$. □

We also need the following lemma, focusing only on some exponential functionals.

Lemma 7.3. *Recall that $F^\pm$ and G^+ are defined in (1.6) and (1.7). For all $0 < \zeta \leq 1$ and $0 < \varepsilon < 1$, for h large enough,*

$$P\left[e^{(1-\varepsilon)\zeta h} \leq F^+(\zeta h) \leq e^{(1+\varepsilon)\zeta h}\right] \geq 1 - 4e^{-\kappa\varepsilon\zeta h/2}, \tag{7.8}$$

$$P\big[F^-(h) \geq e^{-\varepsilon h}\big] \geq 1 - e^{-(c_-)\varepsilon^2 h^2}, \tag{7.9}$$

$$P\big[G^+(\alpha h, h) \leq b(h)e^h\big] \geq 1 - C_+[b(h)]^{-\kappa}, \qquad 0<\alpha<1,\, b(h)>0. \tag{7.10}$$

Proof: By Markov inequality and the last line of Lemma 7.1,

$$P\big[F^+(\zeta h) > e^{(1+\varepsilon)\zeta h}\big] \leq C_{10}e^{-\varepsilon\zeta h} \leq e^{-\kappa\varepsilon\zeta h/2}$$

for large h. For the lower bound, we have by Andreoletti and Devulder (2015, eq. (2.29)) for large h,

$$P\big[F^+(\zeta h) \geq e^{(1-\varepsilon)\zeta h}\big] \geq 1 - 3e^{-\kappa\varepsilon\zeta h/2}.$$

These two inequalities prove (7.8). For (7.9), first $F^-(h) \geq e^{-\varepsilon h}\tau^{W_\kappa^\uparrow}(\varepsilon h)$, and using (7.7), $\tau^{W_\kappa^\uparrow}(\varepsilon h) \geq 1$ with a probability larger than $1-e^{-(c_-)\varepsilon^2 h^2}$, which proves (7.9). Finally, notice that in law $G^+(\alpha h, h) \leq e^h \int_0^{+\infty} e^{W_\kappa(x)}\mathrm{d}x = e^h A_\infty$. By Dufresne (2000), $2/A_\infty$ is a gamma variable of parameter $(\kappa, 1)$, and so has a density equal to $e^{-x}x^{\kappa-1}\mathbb{1}_{\mathbb{R}_+}(x)/\Gamma(\kappa)$, which leads to (7.10). □

The following lemma is exactly Lemma 4.3 in Andreoletti and Devulder (2015) which proof can be found in that paper.

Lemma 7.4. *Let $(B(s),\, s \in \mathbb{R})$ be a standard two-sided Brownian motion. For every $0 < \varepsilon < 1$, $0 < \delta < 1$ and $x > 0$,*

$$\mathbb{P}\left(\sup_{u\in[-\delta,\delta]} \left|\mathcal{L}_B\big(\tau^B(1), u\big) - \mathcal{L}_B\big(\tau^B(1), 0\big)\right| > \varepsilon\mathcal{L}_B\big(\tau^B(1), 0\big)\right) \leq C_+\frac{\delta^{1/6}}{\varepsilon^{2/5}}, \tag{7.11}$$

$$\mathbb{P}\left(\sup_{u\in[0,1]} \mathcal{L}_B\big(\tau^B(1), u\big) \geq x\right) \leq 4e^{-x/2}, \tag{7.12}$$

$$\mathbb{P}\left(\sup_{u\leq 0} \mathcal{L}_B\big(\tau^B(1), u\big) \geq x\right) \leq 4/x. \tag{7.13}$$

The next lemma says that with large probability, a 2-dimensional squared Bessel Process is bounded by some deterministic function. This lemma may be of independent interest.

Lemma 7.5. *Let $(Q_2(u),\ u \geq 0)$ be a Bessel process of dimension 2, starting from 0, and two functions $a(.)$ and $k(.)$ from $(0, +\infty)$ to $(0, +\infty)$, having limit $+\infty$ on $+\infty$. We have for large t,*

$$P\Big(\forall u \in (0, k(t)],\ Q_2^2(u) \leq 2e\big[a(t) + 4\log\log[ek(t)/u]\big]u\Big) \geq 1 - C_+ \exp[-a(t)/2].$$

Proof: We consider for $t > 0$ and $i \in \mathbb{N}$,

$$\mathcal{A}_{1,i} := \Big\{\sup_{[k(t)/e^{i+1}, k(t)/e^i]} Q_2^2 \leq 2\frac{k(t)}{e^i}[a(t) + 4\log(i+1)]\Big\}, \qquad \mathcal{A}_2 := \bigcap_{i=0}^{\infty} \mathcal{A}_{1,i}.$$

We recall that there exist two standard independent Brownian motions $(B_1(u),\ u \geq 0)$ and $(B_2(u),\ u \geq 0)$ such that $(Q_2^2(u),\ u \geq 0)$ is equal in law to $(B_1^2(u) + B_2^2(u),\ u \geq 0)$. So for $i \in \mathbb{N}$,

$$\begin{aligned} P\big(\overline{\mathcal{A}}_{1,i}\big) &\leq 2P\big(\sup\nolimits_{[k(t)/e^{i+1}, k(t)/e^i]} B_1^2 > k(t)e^{-i}[a(t) + 4\log(i+1)]\big) \\ &\leq 4P\Big(\sup\nolimits_{[0, k(t)/e^i]} B_1 > \sqrt{k(t)e^{-i}[a(t) + 4\log(i+1)]}\Big) \\ &= 4P\left(|B_1(1)| > \sqrt{a(t) + 4\log(i+1)}\right) \\ &\leq 8\exp[-a(t)/2 - 2\log(i+1)] \end{aligned}$$

for large t so that $a(t) \geq 1$, by scaling, and since $B_1 \overset{\mathcal{L}}{=} -B_1$, $\sup_{[0,1]} B_1 \overset{\mathcal{L}}{=} |B_1(1)|$ and $P(B_1(1) \geq x) \leq e^{-x^2/2}$ for $x \geq 1$. Consequently for large t,

$$P\big(\overline{\mathcal{A}}_2\big) \leq \sum_{i=0}^{\infty} P\big(\overline{\mathcal{A}}_{1,i}\big) \leq 8\exp[-a(t)/2] \sum_{i=0}^{\infty} \frac{1}{(i+1)^2} = C_+ \exp[-a(t)/2]. \tag{7.14}$$

Now, let $0 < u \leq k(t)$. There exists $i \in \mathbb{N}$ such that $k(t)/e^{i+1} < u \leq k(t)/e^i$. We have, $e^i \leq k(t)/u$, so $e^{i+1} \leq ek(t)/u$ and then $\log(i+1) \leq \log\log[ek(t)/u]$. Consequently on $\mathcal{A}_2$,

$$Q_2^2(u) \leq 2\big(k(t)/e^i\big)[a(t) + 4\log(i+1)] \leq 2eu\big[a(t) + 4\log\log[ek(t)/u]\big].$$

This, combined with (7.14), proves the lemma. □

We also need some estimates on the local time of B at a given coordinate $y \in \mathbb{R}$ at the inverse of the local time of B at 0. Recall that $\sigma_B(r, y) = \inf\{s > 0,\ \mathcal{L}_B(s, y) > r\}$ for $r > 0$, $y \in \mathbb{R}$. By the second Ray-Knight Theorem, the processes $(\mathcal{L}_B(\sigma_B(r, 0), y), y \in \mathbb{R}_+)$ and $(\mathcal{L}_B(\sigma_B(r, 0), -y), y \in \mathbb{R}_+)$ are two independent squared Bessel processes of dimension 0 starting at r. The following lemma is proved in Talet (2007, Lemma 3.1); the results are stated for a Bessel process but are actually true for a *squared* Bessel process; see also Diel (2011, Lemma 2.3).

Lemma 7.6. *We denote by $(Q_0(y),\ y \geq 0)$ the square of a 0-dimensional Bessel process starting at 1. Let $M > 0$, $u > 0$ and $v > 0$. Then,*

$$P\left(\sup_{0 \leq y \leq v} \big|Q_0(y) - 1\big| \geq u\right) \leq 4\frac{\sqrt{(1+u)v}}{u} \exp\left[-u^2/(8(1+u)v)\right], \tag{7.15}$$

$$P\left(\sup_{y \geq 0} Q_0(y) \geq M\right) = 1/M. \tag{7.16}$$

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