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RENEWAL THEORY FOR FUNCTIONALS OF A MARKOV CHAIN WITH COMPACT STATE SPACE[1]

BY CLAUDIA KLÜPPELBERG AND SERGUEI PERGAMENCHTCHIKOV[2]

Munich University of Technology and Université de Rouen

Motivated by multivariate random recurrence equations we prove a new analogue of the Key Renewal Theorem for functionals of a Markov chain with compact state space in the spirit of Kesten [*Ann. Probab.* **2** (1974) 355–386]. Compactness of the state space and a certain continuity condition allows us to simplify Kesten's proof considerably.

1. Introduction.

We prove an analogue of Blackwell's Renewal Theorem or the Key Renewal Theorem in the following setup: $(x_n)_{n\geq 0}$ is a Markov chain with separable metric state space S and $(u_n)_{n\geq 0}$ is a sequence of random variables such that the conditional distribution of u_i, given all the x_j and u_j, $j \neq i$, depends on x_i and x_{i+1} only. Here the $v_n = \sum_{i=1}^{n} u_i$, $n \in \mathbb{N}$, take the role of the partial sums of i.i.d. random variables in ordinary renewal theory. The Key Renewal Theorem in this setup states that $\lim_{t\to\infty} E_x \sum_{n=0}^{\infty} g(x_n, t - v_n)$ exists for suitable functions g and is independent of x.

This is quoted from the abstract of Kesten's famous paper [14], which has attracted vast attention, particularly in the area of random recurrence equations; see, for example, [10, 11, 20]. Such equations play an important role in many applications as, for example, in queueing; see [4] and in financial time series; see [7]. The Key Renewal Theory is used in such models to derive the tail behavior and study extreme value theory of a stationary version of $(x_n)_{n\geq 0}$. Some special examples have been worked out as ARCH(1) and GARCH(1, 1); see [5, 10, 17].

In this paper we review and modify Kesten's paper [14] motivated by examples more general than the above. We consider multivariate random recurrence equations of the type

$$Y_n = A_n Y_{n-1} + \zeta_n, \qquad n \in \mathbb{N}, \tag{1.1}$$

where the Y_n and ζ_n are column vectors of size q and A_n are $(q \times q)$ matrices. Moreover, we assume that (A_n, ζ_n) are i.i.d.

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Under appropriate stability conditions [11], equation (1.1) has a stationary distribution defined by

$$Y = \zeta_1 + \sum_{k=2}^{\infty} A_1 \cdots A_{k-1}\zeta_k. \tag{1.2}$$

Questions of interest concern the tail behavior

$$\mathbf{P}(x'Y > t) \qquad \text{as } t \to \infty \tag{1.3}$$

for every $x \in S = \{z \in \mathbb{R}^q : |z| = 1\}$ and the extremal behavior of the corresponding stationary distribution and process, respectively. Vectors are always column vectors, x' denotes the transpose of x, and $|\cdot|$ denotes any norm in $\mathbb{R}^q$.

In the one-dimensional case ($q = 1$) Goldie [10] has solved the problem in a very elegant way and found the tail behavior (1.3). But for the multivariate model ($q > 1$) renewal theory is called for. One can show (see, e.g., [13] and [16]) that the function $t^\lambda\, \mathbf{P}(x'Y > t)$ (with a some $\lambda > 0$) is asymptotically equivalent to a renewal function, that is

$$t^\lambda\, \mathbf{P}(x'Y > t) \sim G(x,t) = \mathbf{E}_x \sum_{i=0}^{\infty} g(x_n, t - v_n), \qquad t \to \infty, \tag{1.4}$$

where $\sim$ means that the quotient of both sides tends to a positive constant. Here $g(\cdot,\cdot)$ is some continuous function satisfying condition (2.3). In the context of model (1.1) the processes $(x_n)_{n\ge 0}$ and $(v_n)_{n\ge 0}$ are defined as

$$x_0 = x \in S, \qquad x'_n = \frac{x'_{n-1}A_n}{|x'_{n-1}A_n|} = \frac{x'A_1\cdots A_n}{|x'A_1\cdots A_n|}, \qquad n \in \mathbb{N}, \tag{1.5}$$

and $v_0 = 0$ for $n \in \mathbb{N}$,

$$v_n = \sum_{i=1}^{n} u_i = \log|x'A_1\cdots A_n| \qquad \text{with } u_n = \log|x'_{n-1}A_n|. \tag{1.6}$$

To obtain the asymptotic behavior of $G(x,t)$ we apply the Key Renewal Theorem to (1.4). Unfortunately, to apply this theorem one has to check a "direct Riemann integrability" condition for the function $g(\cdot,\cdot)$; see [14], equation (1.11). This is a difficult task because it requires the explicit form of the infinite distributions of the processes (1.5) and (1.6). For matrices with non-negative elements Kesten [13] proved that his notion of "direct Riemann integrablity" is equivalent to our condition (2.3), which is in general weaker than Kesten's condition. Since models like ARCH(1) and GARCH(1, 1) play a prominent role as volatility models in finance, which are by nature positive, Kesten's results apply. When we consider more general models like autoregressive models with GARCH errors or random coefficient autoregressive models, elements of A_n are often normally distributed, and this means model (1.1) falls outside the scope of

Kesten's work. The tail behavior and extreme value theory of an AR(1) model with ARCH(1) errors was investigated in [3] by different (purely analytic) methods. It seems to be difficult, if not impossible, to extend these methods to higher order processes of this kind. For this reason we come back to Kesten's methods as an appropriate remedy. Our generalization in this respect goes in the same direction as LePage [16].

On the other hand, all models we want to consider have compact state space; indeed, our models have state space $S = \{z \in \mathbb{R}^q : |z| = 1\}$. Kesten and LePage, however, work with Markov chains with general state space (which can be unbounded). Hence, in our context, Kesten's conditions and also proofs can be simplified considerably. We shall indicate this at the proper places throughout the paper.

The result of this paper is applied to various models in the accompanying paper [15].

Our paper is organized as follows. In Section 2 we state the conditions and the Key Renewal Theorem, which is our main result. In Section 3 we prove the for us necessary version of the Choquet–Deny lemma similar to the one used already in [8] for the proof of the classical Key Renewal Theorem. Section 4 ensures the existence of a limit for a time changed version of the Markov chain $(x_n)_{n\geq 0}$ under investigation and the overshoot of the corresponding time changed process $(\upsilon_n)_{n\geq 0}$. Some properties of the renewal function are investigated in Section 5 and, finally, the Key Renewal Theorem is proved in Section 6. Some technical results are summarized in the Appendix as not to disturb the flow of arguments in the paper.

2. Main result. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbf{P})$, that is, $(\mathcal{F}_n)_{n\geq 0}$ is a nondecreasing family of sub-σ-fields of $\mathcal{F}$. Let $(x_n)_{n\geq 0}$ be a homogeneous Markov chain with compact state space $S \subset \mathbb{R}^q$, on which a σ-field $\mathcal{G}$ is given. We suppose that $(x_n)_{n\geq 0}$ is an $\mathcal{F}_n$-adapted process, that is, x_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}_0$.

For this Markov process we denote the transition probabilities

$$\mathbf{P}_x(\Gamma) = \mathbf{P}(x_1 \in \Gamma | x_0 = x), \qquad \mathbf{P}_x^{(n)}(\Gamma) = \mathbf{P}(x_{n+k} \in \Gamma | x_k = x), \qquad k \in \mathbb{N},$$

for every $x \in S$ and every measurable set $\Gamma \subseteq S$.

We also consider the $(\mathcal{F}_n)_{n\geq 0}$-adapted stochastic process $(u_n)_{n\in\mathbb{N}}$. We request certain further conditions on the processes $(x_n)_{n\geq 0}$ and $(u_n)_{n\geq 0}$ as follows:

C1. For every bounded measurable function $f : \mathbb{R} \times \prod_{i=1}^{\infty}(S \times \mathbb{R}) \to \mathbb{R}$ and for every $\mathcal{F}_n$-measurable random variable η,

$$\mathbf{E}\big(f(\eta, x_{n+1}, u_{n+1}, \dots, x_{n+l}, u_{n+l}, \dots) | \mathcal{F}_n\big) = \Phi(\eta, x_n), \tag{2.1}$$

where $\Phi(a, x) = \mathbf{E}_x f(a, x_1, u_1, \dots, x_l, u_l, \dots)$ for every $a \in \mathbb{R}$ and $x \in S$. Moreover, if the function $f : \prod_{i=1}^{m}(S \times \mathbb{R}) \to \mathbb{R}$ is continuous then the function $\Phi(x) = \mathbf{E}_x\, f(x_1, u_1, \dots, x_m, u_m)$ is continuous for all $m \in \mathbb{N}$.

Now consider the sequence

$$v_0 = 0 \quad \text{and} \quad v_n = \sum_{i=1}^{n} u_i, \qquad n \geq 1. \tag{2.2}$$

Property (2.1) implies that the bivariate process $(x_n, v_n)_{n \geq 0}$ is a Markov process.

We study the asymptotic properties of the renewal function

$$\mathbf{E}_x \sum_{k=0}^{\infty} g(x_k, t - v_k),$$

where the function $g : S \times \mathbb{R} \to \mathbb{R}$ satisfies the following uniform direct Riemann integrability condition

$$\sum_{l=-\infty}^{\infty} \sup_{x \in S} \sup_{l \leq t \leq l+1} |g(x,t)| < \infty. \tag{2.3}$$

With this notation we can formulate the following conditions:

C2. There exists a probability measure $\pi(\cdot)$ on S, which is equivalent to Lebesgue measure such that

$$\|\mathbf{P}_x^{(n)}(\cdot) - \pi(\cdot)\| \to 0, \qquad n \to \infty, \tag{2.4}$$

for all $x \in S$, where $\| \cdot \|$ denotes total variation of measures on S. Note that this implies that $(x_n)_{n \geq 0}$ is recurrent. Moreover, there exists a constant $\beta > 0$ such that for all $x \in S$

$$\lim_{n \to \infty} \frac{v_n}{n} = \beta, \qquad \mathbf{P}_x\text{-a.s.}$$

C3. There exists a number $m \in \mathbb{N}$ such that for all $\nu \in \mathbb{R}$ and for all $\delta > 0$ there exist $y_{\nu,\delta} \in S$ and $\varepsilon_0 = \varepsilon_0(\nu, \delta) > 0$ such that $\forall\, 0 < \varepsilon < \varepsilon_0$

$$\inf_{x \in B_{\varepsilon,\delta,\nu}} \mathbf{P}_x(|x_m - y_{\nu,\delta}| < \varepsilon, |v_m - \nu| < \delta) = \rho_* = \rho_{\varepsilon,\delta,\nu} > 0, \tag{2.5}$$

where $B_{\varepsilon,\delta,\nu} = \{x \in S : |x - y_{\nu,\delta}| < \varepsilon\}$.

C4. Let $\Phi : S \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Then there exists some $l \in \mathbb{N}$ such that the function $\Phi_1(x,t) = \mathbf{E}_x \Phi(x_l, v_l, t)$ satisfies the following property:

$$\sup_{|x-y|<\varepsilon} \sup_{t \in \mathbb{R}} |\Phi_1(x,t) - \Phi_1(y,t)| \to 0, \qquad \varepsilon \to 0.$$

REMARK 2.1. Condition (2.3) in combination with C4 replaces the direct Riemann integrability condition (1.11) in [14]. Compactness of the state space S simplifies the double sum of [14] to (2.3). Moreover, Kesten's construction of sets $(C_k)_{k \geq 0}$ can be avoided by using the geometric properties of the compact state space S in combination with the continuity condition C4. For details see Remark 5.4.

The following theorem is the main result of this paper.

THEOREM 2.2 (Key Renewal Theorem for Markov chains with compact state space). *Assume that conditions* C1–C4 *are satisfied. Then for every continuous bounded function g satisfying condition* (2.3),

$$\lim_{t\to\infty} \mathbf{E}_x \sum_{k=0}^{\infty} g(x_k, t-v_k) = \frac{1}{\beta}\int_S \pi(dx)\int_{-\infty}^{\infty} g(x,t)\,dt \qquad \forall x \in S. \tag{2.6}$$

REMARK 2.3. (a) For a non-negative sequence $(u_n)_{n\geq 0}$, this theorem follows directly from [14]. This case was also considered in [19] by analytic methods, and this result was proved for g satisfying a weaker condition than (2.3).

(b) This type of result can also be obtained by regeneration methods for Markov chains as developed in [1] and [2]. In these papers almost sure convergence of (2.6) with respect to the stationary distribution π was shown. In [9] finally the rate of this a.s. convergence was found. Unfortunately, we cannot use these results, since we want to apply Theorem 2.2 for a single value of $x \in S$ to obtain the tail behavior of certain models; see [15] for details.

3. The Choquet–Deny lemma. In this section we prove an analogue of the Choquet–Deny lemma for our situation under conditions C1–C4. Such a lemma is prominent in proofs of renewal theorems as in [14], Section 2, but also in the classical case; see [8], Lemma XI.2.1 and corollary to Lemma XI.9.1.

Define for $t \geq 0$,

$$N(t) = \inf\{n > 0 : v_n > t\}, \qquad N(0) = 0, \tag{3.1}$$

and

$$Z(t) = x_{N(t)}, \qquad W(t) = v_{N(t)} - t. \tag{3.2}$$

By (2.4) we have immediately for all $t \geq 0$ that $N(t) < \infty$, $\mathbf{P}_x$-a.s. for all $x \in S$ and that $\lim_{t\to\infty} N(t) = \infty$.

Let $f : S \times \mathbb{R}_+ \to \mathbb{R}$ be a uniformly continuous bounded function. Define H_0 and H by

$$H_0(x,t) = \mathbf{E}_x f\big(Z(t), W(t)\big)\chi_{\{t\geq 0\}}, \tag{3.3}$$

where χ_B denotes the indicator function of a set B, and

$$H(x,t) = \int_{-\infty}^{\infty} H_0(x, t+s)\theta(s)\,ds, \tag{3.4}$$

where $\theta(\cdot)$ is some continuous function with compact support.

The following lemma is a version of Lemma 1 of [14], which we can prove without Kesten's condition (2.2).

LEMMA 3.1. *Assume that conditions* C1–C4 *are satisfied. Then every sequence* $(t_n)_{n\in\mathbb{N}}$ *which tends to infinity as* $n \to \infty$ *contains a subsequence* $(t_{n_k})_{k\in\mathbb{N}}$ *such that for every* $s \in \mathbb{R}$,

$$\lim_{k\to\infty} H\bigl(x, t_{n_k} + s\bigr)$$

exists and is independent of x *and* s.

PROOF. Take $l \in \mathbb{N}$ fixed. We first investigate for $x \in S$ and $t \geq 0$ the function

$$H_l(x,t) = \int_{-\infty}^{\infty} H_{0,l}(x, t+s)\theta(s)\,ds$$

with

$$\begin{aligned} H_{0,l}(x,t) &= \mathbf{E}_x f\bigl(Z(t), W(t)\bigr)\chi_{\{N(t)>l\}}\chi_{\{t\geq 0\}} \\ &= \mathbf{E}_x \sum_{j=l+1}^{\infty} f(x_j, v_j - t)\chi_{\{N(t)=j\}} \\ &= \mathbf{E}_x \sum_{j=l+1}^{\infty} f(x_j, v_j - t)\chi_{\{v_1\leq t,\dots,v_{j-1}\leq t, v_j>t\}} \\ &= \mathbf{E}_x \chi_{\{v_1\leq t,\dots,v_{l-1}\leq t\}}\mathbf{E}\bigl(\tilde{f}(v_l, x_{l+1}, u_{l+1},\dots)|\mathcal{F}_l\bigr), \end{aligned}$$

where

$$\begin{aligned} &\tilde{f}(v_l, x_{l+1}, u_{l+1},\dots) \\ &\quad= \sum_{j=1}^{\infty} f\bigl(x_{j+l}, v_{j+l} - v_l - (t - v_l)\bigr)\chi_{\{v_l\leq t,\dots,v_{j+l-1}-v_l\leq t-v_l, v_{j+l}-v_l>t-v_l\}} \\ &\quad= \sum_{j=1}^{\infty} f\left(x_{j+l}, \sum_{i=l+1}^{j+l} u_i - (t-v_l)\right)\chi_{\{v_l\leq t,\dots,\sum_{i=l+1}^{j+l-1} u_i\leq t-v_l, \sum_{i=l+1}^{j+l} u_i>t-v_l\}}. \end{aligned}$$

By (2.1) we obtain that

$$H_{0,l}(x,t) = \mathbf{E}_x \chi_{\{v_1\leq t,\dots,v_{l-1}\leq t\}}\Phi(x_l, v_l, t),$$

where

$$
\begin{aligned}
\Phi(x, v, t) &= \mathbf{E}_x \tilde{f}(v, x_1, u_1, \dots) \\
&= \chi_{\{v \le t\}} \mathbf{E}_x \sum_{j=1}^{\infty} f\big(x_j, v_j - (t - v)\big) \chi_{\{v_1 \le t - v, \dots, v_{j-1} \le t - v, v_j > t - v\}} \\
&= \chi_{\{v \le t\}} \mathbf{E}_x \sum_{j=1}^{\infty} f\big(x_j, v_j - (t - v)\big) \chi_{\{N(t-v) = j\}} \qquad (3.5) \\
&= \chi_{\{v \le t\}} \mathbf{E}_x f\big(Z(t - v), W(t - v)\big) \\
&= H_0(x, t - v).
\end{aligned}
$$

Hence,

$$H_{0,l}(x, t) = \mathbf{E}_x H_0(x_l, t - v_l) - \delta_l(x, t)$$

with

$$\delta_l(x, t) = \mathbf{E}_x\big(1 - \chi_{\{v_1 \le t, \dots, v_{l-1} \le t\}}\big) H_0(x_l, t - v_l).$$

By the dominated convergence theorem $\delta_l(x, t) \to 0$ as $t \to \infty$ for all $x \in S$. Therefore

$$H_l(x, t) = \int_0^{\infty} H_{0,l}(x, u)\theta(u - t)\, du = H_{1,l}(x, t) - \Delta_l(x, t),$$

where

$$
\begin{aligned}
H_{1,l}(x, t) &= \int_0^{\infty} \mathbf{E}_x H_0(x_l, u - v_l)\theta(u - t)\, du = \mathbf{E}_x H(x_l, t - v_l), \\
\Delta_l(x, t) &= \int_0^{\infty} \delta_l(x, u)\theta(u - t)\, du.
\end{aligned}
$$

Notice that the function $H_{1,l}(x, t)$ (by condition C4) satisfies the property

$$\sup_{|x - y| < \varepsilon} \sup_{|t' - t''| < \delta} |H_{1,l}(x, t') - H_{1,l}(y, t'')| \to 0, \qquad \delta \to 0, \varepsilon \to 0$$

and $\Delta_l(x, t) \to 0$ as $t \to \infty$ for every $x \in S$. Therefore we can [by standard diagonal selection methods ([18], Theorem 7.23)] find a subsequence $(t_{n_k})_{k \in \mathbb{N}}$ for which

$$\lim_{k \to \infty} H_l\big(x, t_{n_k} + s\big)$$

exists for every $x \in S$ and $s \in \mathbb{R}$.

Since f is bounded we get, for every $l \in \mathbb{N}$ and $x \in S$,

$$H_0(x, t) - H_{0,l}(x, t) \to 0, \qquad t \to \infty. \qquad (3.6)$$

But this means that $H(x,t) - H_l(x,t) \to 0$ as $t \to \infty$. Therefore, for every $(x,s) \in S \times \mathbb{R}$, there exists the limit

$$G(x,s) = \lim_{k\to\infty} H\big(x, t_{n_k} + s\big).$$

Furthermore, taking into account that $\Delta_l(x,t) \to 0$ as $t \to \infty$ we conclude, for every $x \in S$ and $l \in \mathbb{N}$,

$$\lim_{t\to\infty} \big(H(x,t) - \mathbf{E}_x H(x_l, t - v_l)\big) = 0$$

and we have that

$$G(x,s) = \mathbf{E}_x G(x_1, s - u_1) = \cdots = \mathbf{E}_x G(x_l, s - v_l), \qquad l \geq 1. \tag{3.7}$$

By condition C4 this function satisfies the following continuity condition:

$$\sup_{|x-y|<\varepsilon} \sup_{|t'-t''|<\delta} |G(x,t') - G(y,t'')| \to 0, \qquad \delta \to 0, \varepsilon \to 0. \tag{3.8}$$

We show now that for each $x \in \mathbb{R}$,

$$G(x,s) = G(x, s+\nu), \qquad \nu \in \mathbb{R}. \tag{3.9}$$

Suppose that there exist $x_0 \in S$, $s_0 \in \mathbb{R}$, $\nu \in \mathbb{R}$ such that

$$G(x_0, s_0) < G(x_0, s_0 + \nu).$$

Set $\tilde{G}(x,s) = G(x, s+\nu)$. Notice that the sequences

$$\big(G(x_n, s_0 - v_n)\big)_{n\in\mathbb{N}} \quad \text{and} \quad \big(\tilde{G}(x_n, s_0 - v_n)\big)_{n\in\mathbb{N}}$$

are bounded martingales, which converge $\mathbf{P}_{x_0}$-a.s. to random variables G_∞ and $\tilde{G}_\infty$ such that, for all $n \in \mathbb{N}$,

$$G(x_0, s_0) = \mathbf{E}_{x_0} G(x_n, s_0 - v_n) = \mathbf{E}_{x_0} G_\infty,$$
$$\tilde{G}(x_0, s_0) = \mathbf{E}_{x_0} \tilde{G}(x_n, s_0 - v_n) = \mathbf{E}_{x_0} \tilde{G}_\infty.$$

Since $\mathbf{E}_{x_0} G_\infty < \mathbf{E}_{x_0} \tilde{G}_\infty$, there exists $a < b$ such that

$$\mathbf{P}_{x_0}(G_\infty < a < b < \tilde{G}_\infty) = r > 0. \tag{3.10}$$

Further define

$$A = \{(x,v) : G(x, s_0 - v) < a\} \subset S \times \mathbb{R},$$
$$B = \{(x,v) : \tilde{G}(x, s_0 - v) > b\} \subset S \times \mathbb{R}$$

and $C = A \cap B$. Denote by $z_n = (x_n, v_n)$ the bivariate Markov chain on $S \times \mathbb{R}$ with initial value $z_0 = (x_0, 0)$.

We shall use the following fact.

LEMMA 3.2 ([12], page 89). *Let $(\Gamma_n)_{n\ge 0}$ be a sequence of the measurable sets of the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\ge 0}, \mathbf{P})$. Then*

$$\lim_{n\to\infty} \mathbf{P}\left(\bigcap_{k\ge n} \Gamma_k \middle| \mathcal{F}_n\right) = \chi_{\Gamma_*}, \qquad a.s.,$$

where χ_{Γ_*} denotes the indicator function of the set $\Gamma_* = \liminf_{n\to\infty} \Gamma_n := \bigcup_{j\ge 1}\bigcap_{k\ge j} \Gamma_k$.

Setting $\mathcal{F}_n = \sigma\{z_0, \dots, z_n\}$, $\Gamma_n = \{z_n \in C\}$, $n \ge 0$, and taking into account that $(z_n)_{n\ge 0}$ is a Markov chain we obtain

$$\lim_{n\to\infty} \mathbf{P}_{z_n}\left(\bigcap_{k=0}^{\infty}\{z_k \in C\}\right) = \chi_{\liminf_{n\to\infty}\{z_n\in C\}}, \qquad \mathbf{P}_{z_0}\text{-a.s.}$$

From this and (3.10) we conclude

$$\mathbf{P}_{z_0}\left(\lim_{n\to\infty} \mathbf{P}_{z_n}\left(\bigcap_{k=0}^{\infty}\{z_k \in C\}\right) = 1\right) \ge r > 0. \tag{3.11}$$

Let $\nu \in \mathbb{R}$ be as before. Taking (3.8) into account, we can fix $\delta > 0$ and $\varepsilon > 0$ such that $y_{\nu,\delta} \in S$ is as in C3 and

$$\sup_{|x-y|<\varepsilon}\ \sup_{|t'-t''|<\delta} |G(x,t') - G(y,t'')| \le \frac{b-a}{4}. \tag{3.12}$$

By condition C2 we have $\pi(B_{\varepsilon,\nu,\delta}) > 0$ (π is equivalent to Lebesgue measure on S), therefore for every $x \in S$,

$$\mathbf{P}_x\left(\bigcap_{l\ge 1}\bigcup_{n\ge l}\{x_n \in B_{\varepsilon,\nu,\delta}\}\right) = 1. \tag{3.13}$$

By (3.11) there exist $z_n^* = (x_n^*, v_n^*)$ such that $\lim_{n\to\infty} \mathbf{P}_{z_n^*}(\bigcap_{k=0}^{\infty}\{z_k \in C\}) = 1$. Hence for all $\rho > 0$ there exists some $n_0 \in \mathbb{N}$ such that $\mathbf{P}_{z_n^*}(\bigcap_{k=0}^{\infty}\{z_k \in C\}) > 1 - \rho/2$ for all $n \ge n_0$. Combining this with (3.13) there exists $z_* = (x_*, v_*)$ with $x_* \in B_{\varepsilon,\nu,\delta}$ such that

$$\mathbf{P}_{z_*}\left(\bigcap_{k=0}^{\infty}\{z_k \in C\}\right) \ge 1 - \rho/2.$$

This means that $z_* \in C = A \cap B$, that is,

$$G(x_*, s_1) < a \quad \text{and} \quad \tilde{G}(x_*, s_1) > b,$$

and for every $k \in \mathbb{N}$, either

$$\mathbf{P}_{z_*}\big(G(x_k, s_0 - v_k) < a, \tilde{G}(x_k, s_0 - v_k) > b\big) > 1 - \rho/2$$

or

$$\mathbf{P}_{x_*}\big(G(x_k, s_1 - v_k) < a, \tilde{G}(x_k, s_1 - v_k) > b\big) > 1 - \rho/2,$$

where $s_1 = s_0 - v_*$. Setting $k = m$, $\rho = \rho_* > 0$ as in (2.5) and taking into account that $x_* \in B_{\varepsilon,\delta,\nu}$ we get

$$\mathbf{P}_{x_*}\big(G(x_m, s_1 - v_m) < a, \tilde{G}(x_m, s_1 - v_m) > b\big) > 1 - \rho_*/2,$$
$$\mathbf{P}_{x_*}(|x_m - y_{\nu,\delta}| < \varepsilon, |v_m - \nu| < \delta) > \rho_*.$$

Since the event in the following probability is the intersection of two events, one with probability $1 - \rho_*/2$ and the other with probability ρ_*, we conclude

$$\mathbf{P}_{x_*}\big(G(x_m, s_1 - v_m + \nu) > b, |x_m - y_{\nu,\delta}| < \varepsilon, |v_m - \nu| < \delta\big) > \rho_*/2 > 0.$$

On this event we have that

$$\begin{aligned}
b - a &< G(x_m, s_1 - v_m + \nu) - G(x_*, s_1) \\
&\le |G(x_m, s_1 - v_m + \nu) - G(y_{\nu,\delta}, s_1 - v_m + \nu)| \\
&\quad + |G(y_{\nu,\delta}, s_1 - v_m + \nu) - G(x_*, s_1)| \\
&\le 2 \sup_{|x-y|<\varepsilon} \sup_{|t'-t''|<\delta} |G(x, t') - G(y, t'')| \\
&\le \frac{b-a}{2}.
\end{aligned}$$

By means of this contraction we obtain for every $x \in S$ and $s \in \mathbb{R}$,

$$G(x, s) = G(x, 0)$$

and therefore, by condition C2, for each $x \in S$ and $s \in \mathbb{R}$,

$$\begin{aligned}
G(x, s) = G(x, 0) &= \mathbf{E}_x G(x_n, -v_n) = \mathbf{E}_x G(x_n, 0) \\
&= \lim_{n\to\infty} \mathbf{E}_x G(x_n, 0) = \int_S G(z, 0)\pi(dz).
\end{aligned}$$ □

The following corollary corresponds to the Corollary on page 366 of [14]. For sake of self-containedness of our paper we repeat the short proof.

COROLLARY 3.3. *Assume that conditions* C1–C4 *are satisfied. If there exists a measure* μ *on* $S \times \mathbb{R}$ *of finite total mass* m_0 *such that*

$$\lim_{t\to\infty} \int_{S\times\mathbb{R}} H(z, t - w)\mu(dz, dw) = \gamma,$$

then

$$\lim_{t\to\infty} H(x, t) = \gamma/m_0.$$

PROOF. Let $x_0 \in S$ and $(t_n)_{n\in\mathbb{N}}$ a sequence tending to infinity as $n \to \infty$ such that

$$\gamma^* = \lim_{n\to\infty} H(x_0, t_n).$$

By Lemma 3.1 there exists a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} H\big(z, t_{n_k} - w\big) = \gamma^*.$$

The result then follows by dominated convergence. □

4. Change of time theorem. In this section we obtain a limit theorem for the process $(Z(t), W(t))_{t\geq 0}$ as defined in (3.2). We use essentially the same method as Kesten applied to prove his Theorem 1 in [14] adapted to our conditions C1–C4. To be precise we replace Kesten's condition I.4 by C4. This allows us to prove this result directly without Lemma 5, which is necessary in [14].

THEOREM 4.1. *Assume that conditions* C1–C4 *are satisfied. Then there exists a random vector* (z_∞, w_∞) *such that for every initial value* $x \in S$

$$\big(Z(t), W(t)\big) \Rightarrow (z_\infty, w_\infty), \qquad t \to \infty, \tag{4.1}$$

where $\Rightarrow$ *denotes weak convergence with respect to the measure* $\mathbf{P}_x$.

PROOF. Let $f : S \times (0,\infty) \to \mathbb{R}$ be bounded and uniformly continuous. Define $H_0(x,t)$ and $H(x,t)$ as in (3.3) and (3.4) with continuous $\theta(\cdot)$ satisfying, for some $\eta > 0$,

$$\theta(s) \geq 0, \qquad \theta(s) = 0, \qquad |s| > \eta \quad \text{and} \quad \int_{-\infty}^{\infty} \theta(s)\, ds = 1. \tag{4.2}$$

As we show in the Appendix, there exists a probability measure μ on $S \times \mathbb{R}$ with support on $S \times \mathbb{R}_+$ [the invariant measure of $(Z(t), W(t))_{t\geq 0}$] such that

$$\begin{aligned} \lim_{t\to\infty} &\int_{S\times\mathbb{R}} H(z, t-w)\mu(dz, dw) \\ &= \int_{S\times\mathbb{R}} f(z,w)\mu(dz,dw) \\ &= \int_{S\times\mathbb{R}_+} f(z,w)\mu(dz,dw) \\ &= \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \int_0^{v_{\nu_1}} f(x_{\nu_1}, w)\, dw, \end{aligned} \tag{4.3}$$

where

$$\nu_1 = \inf\{n > 0 : v_n > 0\}$$

and $\sigma(\cdot)$ is some measure on $(S, \mathcal{G})$ with $0 < \sigma(S) < \infty$. Thus, if f is bounded and uniformly continuous, then Corollary 3.3 applies and

$$\lim_{t\to\infty} H(x,t) = \int_{S\times\mathbb{R}} f(z,w)\mu(dz,dw), \qquad x \in S. \tag{4.4}$$

We apply (4.4) first with $f(z,w) = k(w)$ for some uniformly continuous function $k(\cdot)$ satisfying $0 \le k(w) \le 1$ and for some $\eta > 0$,

$$k(w) = \begin{cases} 1, & \text{if } 0 \le w \le 4\eta, \\ 0, & \text{if } w \ge 5\eta. \end{cases}$$

For $t > \eta$ the corresponding H satisfies

$$\begin{aligned} H(x,t) &= \int_{-\infty}^{\infty} \theta(s)\mathbf{E}_x k\bigl(W(t+s)\bigr)\,ds \\ &\ge \int_{-\infty}^{\infty} \theta(s)\mathbf{P}_x\bigl(W(t+s) \le 4\eta\bigr)\,ds \ge \mathbf{P}_x\bigl(W(t+\eta) \le 2\eta\bigr). \end{aligned}$$

Thus, by (4.4),

$$\begin{aligned} &\limsup_{t\to\infty} \mathbf{P}_x\bigl(W(t-\eta) \le 2\eta\bigr) \\ &\quad \le \lim_{t\to\infty} H(x, t-2\eta) = \int_{S\times\mathbb{R}_+} k(w)\mu(dz,dw) \\ &\quad = \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \int_0^{v_{\nu_1}} k(w)\,dw \le \frac{5\eta\sigma(S)}{\beta} = \varepsilon_1(\eta) \to 0, \qquad \eta \to 0. \end{aligned}$$

Observe now that there is no ladder height of $(v_n)_{n\ge 0}$ in $(t-\eta, t+\eta)$ and hence we have for $|s| < \eta$ on the set

$$\{W(t-\eta) > 2\eta\} = \bigl\{v_{N(t-\eta)} > t+\eta\bigr\}$$

the following identities:

$$\begin{aligned} N(t+s) &= N(t) = N(t-\eta), \\ Z(t+s) &= Z(t) = Z(t-\eta), \\ W(t+s) &= W(t) - s. \end{aligned}$$

Thus

$$\begin{aligned} H(x,t) &= \mathbf{E}_x \int_{-\infty}^{\infty} f\bigl(Z(t+s), W(t+s)\bigr)\theta(s)\,ds \\ &= \mathbf{E}_x \int_{-\infty}^{\infty} \chi_{\{W(t-\eta)>2\eta\}} f\bigl(Z(t), W(t)-s\bigr)\theta(s)\,ds \\ &\quad + \mathbf{E}_x \int_{-\infty}^{\infty} \chi_{\{W(t-\eta)\le 2\eta\}} f\bigl(Z(t+s), W(t+s)\bigr)\theta(s)\,ds. \end{aligned}$$

From this we obtain

$$\begin{aligned}&\big|\mathbf{E}_x f\big(Z(t), W(t)\big) - H(x,t)\big| \\ &\qquad \le \sup_{z\in S}\sup_{|t-s|\le\eta} |f(z,t) - f(z,s)| + 2\sup_{z\in S, t\in\mathbb{R}} |f(z,t)|\varepsilon_1(\eta).\end{aligned}$$

Taking (4.4) into account and letting $\eta \to 0$, the right-hand side converges to 0 and we obtain (4.1) for every random vector with distribution μ. □

5. Properties of the renewal function. In this section we study the properties of the renewal function

$$G(x,t) = \mathbf{E}_x \sum_{n=0}^{\infty} g(x_n, t - v_n). \tag{5.1}$$

The proof of this result is much simpler than in [14], since we can take advantage of the topological properties of the compact state space in combination with the continuity condition C4.

We begin with the intersection inequality for the embedded Markov chains in $z_n = (x_n, v_n)_{n\ge 0}$, which is a generalization of (3.58) in [14].

Let $(\nu_n)_{n\ge 0}$ be an increasing sequence of stopping times such that $v_{\nu_n} \ge v_n$ a.s. and the process $\widehat{z}_n = (\widehat{x}_n, \widehat{v}_n) = (x_{\nu_n}, v_{\nu_n})$ constitutes a homogeneous Markov chain, that is, for every bounded measurable function $f : \prod_{i=1}^{\infty}(S\times\mathbb{R}) \to \mathbb{R}$,

$$\mathbf{E}\big(f(\widehat{z}_n, \widehat{z}_{n+1}, \dots, \widehat{z}_{n+p}, \dots)\big|\widehat{\mathcal{F}}_n\big) = \Phi(\widehat{z}_n), \tag{5.2}$$

where $\widehat{\mathcal{F}}_n = \sigma\{\widehat{z}_1, \dots, \widehat{z}_n\}$ and for $z = (x, v)$,

$$\begin{aligned}\Phi(z) &= \mathbf{E}_z f(z, \widehat{z}_1, \dots, \widehat{z}_p, \dots) \\ &= \mathbf{E}_x\, f(z, \widehat{x}_1, \widehat{v}_1 + v, \dots, \widehat{x}_p, \widehat{v}_p + v, \dots).\end{aligned}$$

PROPOSITION 5.1. *Assume that the sequence* $(\nu_n)_{n\ge 0}$ *satisfies condition* (5.2). *If conditions* C1, C2 *and* C4 *hold*, *then for each* $b > 0$ *there exists a positive constant* $M_b > 0$ *such that*

$$\sup_{t\in\mathbb{R}}\sup_{x\in S} \mathbf{E}_x \sum_{n=0}^{\infty} \chi_{\{t\le \widehat{v}_n \le t+b\}} \le M_b. \tag{5.3}$$

REMARK 5.2. Notice that condition C1 ensures that the sequence $(\nu_n)_{n\ge 0}$ with $\nu_n = n$ satisfies (5.2) and that for each $b > 0$ (5.3) also holds for $(v_n)_{n\ge 0}$:

$$\sup_{t\in\mathbb{R}}\sup_{x\in S} \mathbf{E}_x \sum_{n=0}^{\infty} \chi_{\{t\le v_n \le t+b\}} \le M_b. \tag{5.4}$$

We shall use this inequality in the Appendix to construct the invariant measure for a special sequence of stopping times; see (A.5).

PROOF. For $l \in \mathbb{N}$ define the sets

$$C_j = \left\{x \in S : \mathbf{P}_x\left(\bigcap_{m \geq j} \{v_{m+l} \geq m/j\}\right) > 1/4\right\}, \qquad j \in \mathbb{N},$$

which constitute an increasing sequence, and by condition $\mathbf{C}_1$,

$$\mathbf{P}_x\left(\bigcap_{m \geq j} \{v_{m+l} \geq m/j\}\right) = \mathbf{E}_x \mathbf{P}\left(\bigcap_{m \geq j} \{v_{m+l} \geq m/j\} \middle| \mathcal{F}_l\right)$$
$$= \mathbf{E}_x \Phi_j(x_l, v_l),$$

where $\Phi_j(x, v) = \mathbf{P}_x(\bigcap_{m \geq j}\{v_m + v \geq m/j\})$. By C4 the integer $l \in \mathbb{N}$ can be chosen such that the function $\mathbf{P}_x(\bigcap_{m \geq j}\{v_{m+l} - v_l > m/j\})$ is continuous in x and therefore the C_j are open sets (in the topology on S). By condition C2 we have that

$$S = \bigcup_{j \geq 1} C_j.$$

Since S is compact, there exists some $k \in \mathbb{N}$ such that

$$S = \bigcup_{j=1}^{k} C_j = C_k.$$

Define the sets

$$\widehat{C}_j = \left\{x \in S : \mathbf{P}_x\left(\bigcap_{m \geq j} \{\widehat{v}_{m+l} \geq m/j\}\right) > 1/4\right\}, \qquad j \in \mathbb{N}. \tag{5.5}$$

The inequalities $\widehat{v}_n \geq v_n$, $n \geq 0$, imply that $C_j \subseteq \widehat{C}_j$, therefore $S = \widehat{C}_k$. It means that every $x \in S$ belongs to $\widehat{C}_k$, that is, we have for every $n \in \mathbb{N}$ by (5.5), invoking (5.2),

$$\mathbf{P}\left(\bigcap_{m \geq k} \{\widehat{v}_{m+l+n} - \widehat{v}_n \geq m/k\} \middle| \widehat{\mathcal{F}}_n\right) = \mathbf{P}_{\widehat{x}_n}\left(\bigcap_{m \geq k} \{\widehat{v}_{m+l} \geq m/k\}\right) > 1/4.$$

Define the stopping times

$$\tau_0 = \min\{n \geq 0 : \widehat{v}_n \in [t, t+b]\}$$

and for $j \geq 0$ (recall that $k, l \in \mathbb{N}$ are fixed by the construction),

$$\tau_{j+1} = \min\left\{n \geq \tau_j + k + l : \widehat{v}_n \in [t, t+b], \widehat{v}_n - \widehat{v}_{\tau_j} < \frac{n - \tau_j - l}{k}\right\}.$$

Further,

$$
\begin{aligned}
&\mathbf{P}(\tau_{j+1} < \infty | \widehat{\mathcal{F}}_{\tau_j}) \\
&\quad = \chi_{\{\tau_j < \infty\}} \mathbf{P}(\tau_{j+1} < \infty | \widehat{\mathcal{F}}_{\tau_j}) \\
&\quad \le \sum_{n=0}^{\infty} \chi_{\{\tau_j = n\}} \mathbf{P}\left(\widehat{v}_{\tau_{j+1}} - \widehat{v}_n < \frac{\tau_{j+1} - n - l}{k} \Big| \widehat{\mathcal{F}}_n\right) \\
&\quad \le \sum_{n=0}^{\infty} \chi_{\{\tau_j = n\}} \left(1 - \mathbf{P}\left(\bigcap_{m \ge k} \{\widehat{v}_{m+n+l} - \widehat{v}_n \ge m/k\} \Big| \widehat{\mathcal{F}}_n\right)\right) \\
&\quad \le \frac{3}{4} \chi_{\{\tau_j < \infty\}},
\end{aligned}
$$

and therefore, for $x \in S$,

$$
\mathbf{P}_x(\tau_{j+1} < \infty) = \mathbf{E}_x \chi_{\{\tau_j < \infty\}} \mathbf{P}(\tau_{j+1} < \infty | \widehat{\mathcal{F}}_{\tau_j}) \le \tfrac{3}{4} \mathbf{P}_x(\tau_j < \infty).
$$

We obtained that for all $j \in \mathbb{N}$ and $x \in S$,

$$
\mathbf{P}_x(\tau_j < \infty) \le \left(\tfrac{3}{4}\right)^j. \tag{5.6}
$$

Further, we have

$$
\sum_{n=0}^{\infty} \chi_{\{t \le \widehat{v}_n \le t+b\}} \le \sum_{j=0}^{\infty} \chi_{\{\tau_j < \infty\}} \sum_{\tau_j \le n < \tau_{j+1}} \chi_{\{t \le \widehat{v}_n \le t+b\}}.
$$

If $t \le \widehat{v}_n \le t + b$ and $\tau_j + k + l \le n < \tau_{j+1}$ then

$$
t + \frac{n - \tau_j - l}{k} \le \widehat{v}_{\tau_j} + \frac{n - \tau_j - l}{k} \le \widehat{v}_n \le t + b,
$$

that is, $n - \tau_j \le l + bk$. Thus, denoting $m_j = \max\{\tau_j + k + l \le n < \tau_{j+1}\}$ if $v_n \in [t, t+b]$ and $m_j = \tau_j + k + l - 1$ if $v_n \notin [t, t+b]$ for all $\tau_j + k + l \le n < \tau_{j+1}$ we obtain

$$
\begin{aligned}
\sum_{\tau_j \le n < \tau_{j+1}} \chi_{\{t \le v_n \le t+b\}} &\le k + l + \sum_{\tau_j + k + l \le n \le m_j} \chi_{\{t \le v_n \le t+b\}} \\
&\le k + l + (m_j - \tau_j - k - l + 1) \chi_{\{m_j \ge \tau_j + k + l\}} \\
&\le 1 + l + bk
\end{aligned}
$$

and

$$
\mathbf{E}_x \sum_{n=0}^{\infty} \chi_{\{t \le v_n \le t+b\}} \le (1 + l + bk) \sum_{j=0}^{\infty} \mathbf{P}_x(\tau_j < \infty).
$$

By making use of inequality (5.6) we obtain (5.3) with $M = 4(1 + l + bk)$. □

PROPOSITION 5.3. *Let $g : S \times \mathbb{R} \to \mathbb{R}$ be a bounded (by $g^* \in \mathbb{R}$) jointly continuous function such that $g(x,t) = 0$ for $|t| > L$ for some $L > 0$. If conditions* C1, C2 *and* C4 *hold, then function* (5.1) *is bounded and jointly continuous on $S \times \mathbb{R}$.*

PROOF. Since the function $g(x,t) = 0$ for $|t| > L$, we have $g(x, t - v_n) = 0$ for $|t - v_n| > L$. From this we conclude that

$$|g(x, t - v_n)| < g^* \chi_{\{|t - v_n| \le L\}} = g^* \chi_{\{t - L \le v_n \le t + L\}},$$

which implies that $|G(x,t)| < g^* \mathbf{E}_x \sum_{n \ge 1} \chi_{\{t - L \le v_n \le t + L\}}$. By (5.4) this sum is bounded uniformly in x and t by M. From this we conclude that the function $G(x,t)$ is bounded.

Next we show that the function $G(x,t)$ is continuous. For $n \in \mathbb{N}$ and $L > 0$ (given by the assumption), set

$$\tau_N = \inf\{n \ge N : t - L \le v_n \le t + L\}.$$

By this definition and the fact that $g(x,t) = 0$ for $|t| > L$, we have

$$\left| \mathbf{E}_x \sum_{n=N}^{\infty} g(x_n, t - v_n) \right| = \left| \mathbf{E}_x \sum_{n=\tau_N}^{\infty} g(x_n, t - v_n) \right| = \left| \mathbf{E}_x \chi_{\{\tau_N < \infty\}} \Phi(x_{\tau_N}, v_{\tau_N}) \right|,$$

where

$$\Phi(x, v) = \mathbf{E}_x \sum_{n=0}^{\infty} g(x_n, t - v - v_n).$$

Again taking into account that g has bounded t-support,

$$\begin{aligned} |\Phi(x, v)| &\le \mathbf{E}_x \sum_{n=0}^{\infty} |g(x_n, t - v - v_n)| \\ &\le g^* \mathbf{E}_x \sum_{n=0}^{\infty} \chi_{\{|t - v - v_n| \le L\}} \\ &\le g^* \mathbf{E}_x \sum_{n=0}^{\infty} \chi_{\{-2L \le v_n \le 2L\}}, \end{aligned}$$

where the last inequality holds for $|t - v| \le L$. We use this estimate for $v = v_{\tau_N}$, that is, $|t - v| = |t - v_{\tau_N}| \le L$. This yields finally the estimate

$$\begin{aligned} \left| \mathbf{E}_x \sum_{n=N}^{\infty} g(x_n, t - v_n) \right| &\le \mathbf{E}_x \chi_{\{\tau_N < \infty\}} \left| \Phi\big(x_{\tau_N}, v_{\tau_N}\big) \right| \\ &\le g^* \mathbf{E}_x \chi_{\{\tau_N < \infty\}} \sup_{z \in S} \mathbf{E}_z \sum_{n=0}^{\infty} \chi_{\{-2L \le v_n \le 2L\}} \\ &\le g^* M \mathbf{P}_x(\tau_N < \infty), \end{aligned}$$

where we have used inequality (5.4) in the last step.

Further, for some $0 < \varepsilon \le \beta$,

$$\mathbf{P}_x(\tau_N < \infty) \le \mathbf{P}_x\Big(\tau_N < \infty, \sup_{n\ge N} |\delta_n| \le \varepsilon\Big) + \mathbf{P}_x\Big(\tau_N < \infty, \sup_{n\ge N} |\delta_n| > \varepsilon\Big),$$

where

$$\delta_n = \frac{v_n}{n} - \beta.$$

Therefore, for the first probability on the right-hand side we have for $N > (t+L)/(\beta-\varepsilon)$ that $v_{\tau_N} \ge N(\beta-\varepsilon) \ge t+L$, hence

$$\mathbf{P}_x\Big(\tau_N < \infty, \sup_{n\ge N} |\delta_n| \le \varepsilon\Big) = \mathbf{P}_x\Big(t - L < v_{\tau_N} < t+L, \sup_{n\ge N} |\delta_n| \le \varepsilon\Big) = 0.$$

This means that for this N,

$$\left|\mathbf{E}_x \sum_{n=N}^{\infty} g(x_n, t-v_n)\right| \le g^* M \mathbf{P}_x\Big(\sup_{n\ge N} |\delta_n| > \varepsilon\Big). \tag{5.7}$$

By C2, the probability on the right-hand side tends to zero as $N \to \infty$ for each $x \in S$. Notice now that for $N > l$ by C1,

$$\mathbf{P}_x\left(\sup_{n\ge N}\left|\frac{v_n}{n} - \beta\right| > \varepsilon\right) = \mathbf{P}_x\Big(\sup_{n\ge N} |\delta_n| > \varepsilon\Big) = \mathbf{E}_x \Phi_1(x_l, v_l),$$

where

$$\Phi_1(x, v) = \mathbf{P}_x\left(\sup_{n\ge N-l}\left|\frac{v_n + v}{n+l} - \beta\right| > \varepsilon\right).$$

Therefore, by C4, the function $\mathbf{P}_x(\sup_{n\ge N} |\delta_n| > \varepsilon)$ is continuous for each $N > l$. Since the function $g(x,t)$ is jointly continuous, for every $N < \infty$ by the condition C1 the function

$$G_N(x,t) = \mathbf{E}_x \sum_{n=0}^{N-1} g(x_n, t - v_n)$$

is jointly continuous on $S \times \mathbb{R}$. Now we have that for $N > (2|t_0| + L)/(\beta - \varepsilon) + l$,

$$\begin{aligned}
&|G(x,t) - G(x_0,t_0)| \\
&\quad \le |G_N(x,t) - G_N(x_0,t_0)| + g^* M \left|\mathbf{P}_x\Big(\sup_{n\ge N} |\delta_n| > \varepsilon\Big) - \mathbf{P}_{x_0}\Big(\sup_{n\ge N} |\delta_n| > \varepsilon\Big)\right| \\
&\qquad + 2g^* M \mathbf{P}_{x_0}\Big(\sup_{n\ge N} |\delta_n| > \varepsilon\Big).
\end{aligned}$$

By letting first $x \to x_0$ and $t \to t_0$, and then $N \to \infty$ we obtain joint continuity for $G(x,t)$. $\square$

REMARK 5.4. (a) The main simplification of Kesten's approach takes place in the proof of Proposition 5.3. Comparing our construction of the C_j in (5.2) to (3.56) in [14] we see immediately the advantage of the compact state space S. Kesten starts with compact subsets of a general state space, whereas we start with a compact S and define open sets (allowing for continuity arguments), where finitely many C_j are sufficient by compactness of S.

(b) In the next step one has to extend Proposition 5.3 to arbitrary functions g. For this extension Kesten needs the notion of direct Riemann integrability (1.11) in [14], whereas for us the simpler condition (2.3) in combination with the continuity condition C4 suffices. This will be seen in the next section.

6. Proof of the renewal theorem 2.2. First we prove this theorem for a function g satisfying the conditions of Proposition 5.1. In this case, using (2.1),

$$\begin{aligned} G(x,t) &= \mathbf{E}_x \sum_{n=0}^{\infty} g(x_n, t - v_n) \\ &= \mathbf{E}_x \sum_{n=N(t-L)}^{\infty} g(x_n, t - v_n) \\ &= \mathbf{E}_x \mathbf{E}\Bigg[\sum_{n=N(t-L)}^{\infty} g(x_n, t - v_n) \Big| \mathcal{F}_{N(t-L)} \Bigg] \\ &= \mathbf{E}_x G\big(Z(t-L), t - v_{N(t-L)}\big) \\ &= \mathbf{E}_x f\big(Z(t-L), W(t-L)\big), \end{aligned}$$

where $f(z,w) = G(z, L-w)$. By Theorem 4.1 and Proposition 5.1 the following limit exists and is independent of x:

$$\lim_{t\to\infty} G(x,t) = \lim_{t\to\infty} \mathbf{E}_x f\big(Z(t-L), W(t-L)\big) = G_\infty. \tag{6.1}$$

Then by the boundedness of $G(\cdot,\cdot)$ and the dominated convergence theorem,

$$G_\infty = \lim_{T\to\infty} \int_S \pi(dx) \frac{1}{T} \int_0^T G(x,t)\,dt.$$

Further, we have

$$\begin{aligned} \frac{1}{T}\int_0^T G(x,t)\,dt &= \mathbf{E}_x \sum_{n=0}^{\infty} \frac{1}{T} \int_0^T g(x_n, t - v_n)\,dt \\ &= \frac{1}{T} \mathbf{E}_x \sum_{n=M_1(T)}^{M_2(T)} \int_0^T g(x_n, t - v_n)\,dt + \mathbf{E}_x \Delta_1(T) \qquad (6.2) \\ &= \frac{1}{T} \mathbf{E}_x \sum_{n=M_1(T)}^{M_2(T)} \int_{-\infty}^{\infty} g(x_n, t)\,dt + \mathbf{E}_x \Delta_1(T) - \mathbf{E}_x \Delta_2(T), \end{aligned}$$

where

$$\Delta_1(T) = \frac{1}{T}\sum_{n=0}^{M_1(T)-1}\int_0^T g(x_n, t - v_n)\,dt + \frac{1}{T}\sum_{n>M_2(T)}\int_0^T g(x_n, t - v_n)\,dt,$$

$$\Delta_2(T) = \frac{1}{T}\sum_{n=M_1(T)}^{M_2(T)}\biggl(\int_T^\infty g(x_n, t - v_n)\,dt + \int_{-\infty}^0 g(x_n, t - v_n)\,dt\biggr).$$

We set

$$M_1(T) = \biggl[\frac{\varepsilon}{\beta}T\biggr], \qquad M_2(T) = \biggl[\frac{1-\varepsilon}{\beta}T\biggr],$$

where $[a]$ denotes the integer part of a. By substituting the first term of (6.2) in the integral with respect to the stationary measure $\pi(\cdot)$ we have

$$\begin{aligned}
&\int_S \pi(dx)\frac{1}{T}\mathbf{E}_x\sum_{n=M_1(T)}^{M_2(T)}\int_{-\infty}^\infty g(x_n, t)\,dt \\
&\quad = \int_S \pi(dx)\int_{-\infty}^\infty g(x, t)\,dt\,\frac{M_2(T) - M_1(T)}{T} \\
&\quad \to \frac{1-2\varepsilon}{\beta}\int_S \pi(dx)\int_{-\infty}^\infty g(x, t)\,dt, \qquad T \to \infty.
\end{aligned}$$

Further, since $g(x, t) = 0$ for $|t| \ge L$ the last term in (6.2) is bounded by

$$\mathbf{E}_x|\Delta_2(T)| \le g_1^*\frac{1}{T}\sum_{n=M_1(T)}^{M_2(T)}\bigl(\mathbf{P}_x(v_n > T - L) + \mathbf{P}_x(v_n < L)\bigr),$$

where $g_1^* = \sup_{x\in S}\int_{-\infty}^\infty |g(x, t)|\,dt$. By condition C2, for every $\varepsilon \in]0, 1[$,

$$\lim_{T\to\infty}\mathbf{E}_x|\Delta_2(T)| = 0, \qquad x \in S. \tag{6.3}$$

Moreover, concerning $\mathbf{E}_x\Delta_1(T)$ we have

$$\mathbf{E}_x\Biggl|\frac{1}{T}\sum_{n=0}^{M_1(T)-1}\int_0^T g(x_n, t - v_n)\,dt\Biggr| \le \frac{g_1^*}{\beta}\varepsilon$$

and

$$\begin{aligned}
&\mathbf{E}_x\Biggl|\frac{1}{T}\sum_{n>M_2(T)}\int_0^T g(x_n, t - v_n)\,dt\Biggr| \\
&\qquad \le \biggl(\frac{2\varepsilon}{\beta} + \frac{1}{T}\biggr)g_1^* + \mathbf{E}_x\Biggl|\frac{1}{T}\sum_{n>M_3(T)}\int_0^T g(x_n, t - v_n)\,dt\Biggr|,
\end{aligned}$$

where $M_3(T) = [(1+\varepsilon)T/\beta]$. Now, by (5.7),

$$\frac{1}{T}\int_0^T \mathbf{E}_x \sum_{n>M_3(T)} |g(x_n, t - v_n)|\,dt \le g^* M \mathbf{P}_x\Big(\sup_{n\ge M_3(T)} |\delta_n| > \rho\Big)$$

for $\rho < \beta\varepsilon/(1+\varepsilon)$ and sufficiently large $T \to \infty$. Since $\mathbf{E}_x|\Delta_1(T)|$ is bounded by condition C2 we obtain that

$$\limsup_{T\to\infty} \int_S \pi(dx)\mathbf{E}_x|\Delta_1(T)| \le \frac{2g_1^*}{\beta}\varepsilon.$$

By recalling convergence (6.3), we find upon letting $\varepsilon \to 0$

$$\lim_{t\to\infty} G(x,t) = \frac{1}{\beta}\int_S \pi(dx)\int_{-\infty}^{\infty} g(x,t)\,dt.$$

Now let g be an arbitrary continuous bounded function on $S \times \mathbb{R}$, satisfying condition (2.3) and let $\lambda : \mathbb{R} \to [0, 1]$ be a continuous function such that $\lambda(t) = 1$ for $|t| \le L - 1$ and $\lambda(t) = 0$ for $|t| > L$. Then, making use of inequality (5.4), we obtain that

$$\begin{aligned}
&\left|\mathbf{E}_x \sum_{n=0}^{\infty} g(x_n, t - v_n) - \mathbf{E}_x \sum_{n=0}^{\infty} g(x_n, t - v_n)\lambda(t - v_n)\right| \\
&\quad \le \sum_{j=-\infty}^{\infty} \mathbf{E}_x \sum_{n=0}^{\infty} |g(x_n, t - v_n)|\big(1 - \lambda(t - v_n)\big)\chi_{\{j \le t - v_n < j+1\}} \\
&\quad \le \sum_{|j|\ge L-2} \sup_{z\in S,\ j\le t\le j+1} |g(z,t)|\mathbf{E}_x \sum_{n=0}^{\infty} \chi_{\{t-j-1\le v_n \le t-j\}} \\
&\quad \le M \sum_{|j|\ge L-2} \sup_{z\in S} \sup_{j\le t\le j+1} |g(z,t)|.
\end{aligned}$$

This last expression tends to zero as $L \to \infty$, because g satisfies inequality (2.3).

APPENDIX

In this appendix we prove the relationship (4.2) by a modification of the proof of Lemma 2 in [14]. First we need to construct a measure σ on $\mathcal{G}$. We imitate the construction in [14] simplifying at the appropriate places for our special situation of a compact state space.

A.1. Construction of the measure σ. Consider the measurable space $(X, \mathcal{X})$, where

$$X = \prod_{-\infty}^{\infty} (S \times \mathbb{R}) \quad \text{and} \quad \mathcal{X} = \prod_{-\infty}^{\infty} (\mathcal{G} \times \mathcal{B}),$$

and $\mathcal{B}$ is the Borel σ-field in $\mathbb{R}$. Denote $(\tilde{x}_n, \tilde{u}_n)_{n\in\mathbb{Z}}$ the coordinate process in this space. We also define $\tilde{\mathcal{F}}_k = \sigma\{\tilde{x}_i, \tilde{u}_i : -\infty < i \le k\}$.

By Kolmogorov's theorem we can construct the stationary measure on this space, which has the following finite-dimensional distributions:

$$
\begin{aligned}
&\tilde{\mathbf{P}}(\tilde{x}_{k+1} \in \Gamma_1, \tilde{u}_{k+1} \in A_1, \dots, \tilde{x}_{k+m} \in \Gamma_m, \tilde{u}_{k+m} \in A_m) \\
&\qquad = \int_S \pi(dy)\mathbf{P}_y(x_1 \in \Gamma_1,\, u_1 \in A_1, \dots, x_m \in \Gamma_m,\, u_m \in A_m)
\end{aligned} \tag{A.1}
$$

for every $-\infty < k < \infty$.

It follows from (A.1) that for every bounded measurable function $f : \prod_{i=1}^{\infty}(S \times \mathbb{R}) \to \mathbb{R}$

$$
\tilde{\mathbf{E}}(f(\tilde{x}_k, \tilde{u}_k, \dots, \tilde{x}_{k+m}, \tilde{u}_{k+m}, \dots)|\tilde{\mathcal{F}}_k) = f_1(\tilde{x}_k, \tilde{u}_k), \tag{A.2}
$$

where

$$
f_1(x, u) = \mathbf{E}_x f(x, u, x_1, u_1, \dots, x_m, u_m, \dots).
$$

To construct the functions of this process we set

$$
\tilde{v}_n = \begin{cases} \sum_{i=1}^{n} \tilde{u}_i, & \text{if } n \ge 1, \\ 0, & \text{if } n = 0, \\ -\sum_{i=0}^{n+1} \tilde{u}_i, & \text{if } n < 0. \end{cases} \tag{A.3}
$$

Set

$$
\tilde{\nu}_0 = \max\Big\{n \le 0 : \tilde{v}_n > \sup_{j<n} \tilde{v}_j\Big\} \qquad (= -\infty,\ \text{if no such } n \text{ exists}).
$$

Now we define the measure σ on $\mathcal{G}$ by

$$
\sigma(\Gamma) = \tilde{\mathbf{P}}(\tilde{\nu}_0 = 0, \tilde{x}_0 \in \Gamma), \qquad \Gamma \in \mathcal{G}. \tag{A.4}
$$

Further we need to introduce a sequence of stopping times by

$$
\begin{aligned}
&\nu_0 = 0, \qquad \nu_j = \inf\{i > \nu_{j-1} : v_n > v_{\nu_{j-1}}\}, \qquad j \in \mathbb{N}, \\
&(\nu_j = \infty \text{ if no such } n \text{ exists}).
\end{aligned} \tag{A.5}
$$

Notice that condition C1 implies that for every bounded measurable function f, for $i \ge 0$ and for any $\mathcal{F}_{\nu_i}$ measurable random variable η

$$
\mathbf{E}(f(\eta,\, \nu_{i+1} - \nu_i, v_{\nu_{i+1}} - v_{\nu_i}, x_{\nu_{i+1}})|\mathcal{F}_{\nu_i}) = \pi(f)(\eta,\, x_{\nu_i}), \tag{A.6}
$$

where $\pi(f)(a, x) = \mathbf{E}_x\, f(a, \nu_1,\, v_{\nu_1},\, x_{\nu_1})$. It means that the process $(x_{\nu_i}, v_{\nu_i})_{i\in\mathbb{N}}$ is a homogeneous Markov chain, which satisfies the condition (5.2). Now we need the following lemma, which is proved in [14], (page 368) under different conditions.

LEMMA A.1. *If conditions* C1 *and* C2 *hold, then*

$$\mathbf{P}_x(\nu_i < \infty) = 1 \qquad \forall x \in S,\ \forall i \geq 1 \tag{A.7}$$

and

$$\tilde{\mathbf{P}}(\tilde{\nu}_0 = 0) > 0. \tag{A.8}$$

Further, σ *is an invariant measure of the Markov chain* $(x_{\nu_i})_{i\geq 0}$, *that is,*

$$\int_S \sigma(dy)\mathbf{E}_y f(x_{\nu_i}) = \int_S f(y)\sigma(dy) \tag{A.9}$$

for every bounded measurable function f *on* S. *Finally,*

$$\int_S \sigma(dy)\mathbf{E}_y v_{\nu_1} = \beta. \tag{A.10}$$

PROOF. First notice that condition C2 implies (A.7). Since the process $(\tilde{u}_j)_{j\in\mathbb{Z}}$ is stationary with respect to measure $\tilde{\mathbf{P}}$, by the Birkhoff–Khinchine theorem [6] the following limit exists $\tilde{\mathbf{P}}$-a.s.:

$$\lim_{n\to\infty} \frac{\tilde{v}_{-n}}{n} = -\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{1-n} \tilde{u}_j.$$

Moreover, by definition of the distribution $\tilde{\mathbf{P}}$ in (A.1) and condition C2, we have, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \tilde{\mathbf{P}}\left(\left|\frac{1}{n}\sum_{j=0}^{1-n}\tilde{u}_j - \beta\right| > \varepsilon\right) = \lim_{n\to\infty}\int_S \pi(dy)\mathbf{P}_y\left(\left|\frac{v_n}{n} - \beta\right| > \varepsilon\right) = 0,$$

which means that

$$\lim_{n\to\infty} \frac{\tilde{v}_{-n}}{n} = -\beta, \qquad \tilde{\mathbf{P}}\text{-a.s.} \tag{A.11}$$

From this we conclude $\tilde{\mathbf{P}}(\tilde{v}_0 > -\infty) = 1$. Moreover, taking (A.1) into account, we obtain

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} \tilde{\mathbf{P}}(\tilde{\nu}_0 = -n) \leq \sum_{n=0}^{\infty} \tilde{\mathbf{P}}\Big(\tilde{v}_{-n} > \sup_{j<-n} \tilde{v}_j\Big) \\ &= \sum_{n=0}^{\infty} \tilde{\mathbf{P}}\Big(\tilde{v}_0 > \sup_{j<0} \tilde{v}_j\Big) = \sum_{n=0}^{\infty} \tilde{\mathbf{P}}(\tilde{\nu}_0 = 0). \end{aligned}$$

From this we get (A.8).

Furthermore, from (A.4) we obtain that, for every bounded measurable function $f : S \to \mathbb{R}$,

$$\int_S \sigma(dy)\mathbf{E}_y f(x_{\nu_1}) = \tilde{\mathbf{E}}\chi_{\{\tilde{\nu}_0 = 0\}}\Phi(\tilde{x}_0),$$

where $\Phi(y) = \mathbf{E}_y f(x_{\nu_1}) = \mathbf{E}_y f_1(x_1, u_1, \ldots, x_m, u_m, \ldots)$,

$$f_1(x_1, u_1, \ldots, x_m, u_m, \ldots) = \sum_{k=1}^{\infty} f(x_k)\chi_{\{v_1 \le 0, \ldots, v_{k-1} \le 0, v_k > 0\}}.$$

Therefore, by (A.2) we have that

$$\begin{aligned}\int_S \sigma(dy)\mathbf{E}_y f(x_{\nu_1}) &= \sum_{k=1}^{\infty} \tilde{\mathbf{E}} f(\tilde{x}_k)\chi_{\{\sup_{j<0} \tilde{v}_j < 0\}}\chi_{\{\tilde{v}_1 \le 0, \ldots, \tilde{v}_{k-1} \le 0, \tilde{v}_k > 0\}} \\ &= \sum_{k=1}^{\infty} \tilde{\mathbf{E}} f(\tilde{x}_0)\chi_{\{\sup_{j<-k} \tilde{v}_j < \tilde{v}_{-k}\}}\chi_{\{\tilde{v}_{-k+1} \le \tilde{v}_{-k}, \ldots, \tilde{v}_{-1} \le \tilde{v}_{-k}, \tilde{v}_{-k} < 0\}} \\ &= \sum_{k=1}^{\infty} \tilde{\mathbf{E}} f(\tilde{x}_0)\chi_{\{\tilde{v}_0 = 0\}}\chi_{\{\lambda = -k\}},\end{aligned}$$

where $\lambda = \inf\{l < 0 : \tilde{v}_l = \sup_{j<0} \tilde{v}_j\}$. Notice from (A.11) that $\tilde{\mathbf{P}}(\lim_{n\to-\infty} \tilde{v}_n = -\infty) = 1$, which implies that $\tilde{\mathbf{P}}(\lambda > -\infty) = 1$. Therefore we have

$$\begin{aligned}\int_S \sigma(dy)\mathbf{E}_y f(x_{\nu_1}) &= \tilde{\mathbf{E}} f(\tilde{x}_0)\chi_{\{\tilde{v}_0 = 0\}} \sum_{k=1}^{\infty} \chi_{\{\lambda = -k\}} \\ &= \tilde{\mathbf{E}} f(\tilde{x}_0)\chi_{\{\tilde{v}_0 = 0\}} = \int_S \sigma(dy) f(y).\end{aligned}$$

Taking (A.6) into account as well, we obtain (A.9).

We show now that

$$\int_S \sigma(dy)\mathbf{E}_y \nu_1 = 1. \tag{A.12}$$

Indeed, by the same method as above one can obtain that

$$\begin{aligned}&\int_S \sigma(dy)\mathbf{E}_y \nu_1 \\ &\quad = \sum_{k=1}^{\infty} k\tilde{\mathbf{P}}(\tilde{v}_0 = 0, \tilde{v}_1 \le 0, \ldots, \tilde{v}_{k-1} \le 0, \tilde{v}_k > 0) \\ &\quad = \sum_{k=1}^{\infty} k\tilde{\mathbf{P}}\Big(\sup_{j<0} \tilde{v}_j < 0, \tilde{v}_1 \le 0, \ldots, \tilde{v}_{k-1} \le 0, \tilde{v}_k > 0\Big) \\ &\quad = \sum_{k=1}^{\infty} \sum_{r=0}^{k-1} \tilde{\mathbf{P}}\Big(\sup_{j<-r} \tilde{v}_j < \tilde{v}_{-r}, \tilde{v}_{1-r} \le \tilde{v}_{-r}, \ldots, \tilde{v}_{k-r-1} \le \tilde{v}_{-r}, \tilde{v}_{k-r} > \tilde{v}_{-r}\Big) \\ &\quad = \sum_{k=1}^{\infty} \sum_{r=0}^{k-1} \tilde{\mathbf{P}}(\tilde{\nu}_0 = -r,\ \tilde{\nu}_1 = k - r),\end{aligned}$$

where $\tilde{\nu}_1 = \inf\{l > \tilde{\nu}_0 : \tilde{v}_l > \tilde{v}_{\tilde{\nu}_0}\}$ ($= \infty$ if no such n exists). Notice that the definition of $\tilde{\nu}_1$ implies that $\tilde{\nu}_1 \geq 1$ a.s. Indeed, if $\tilde{\nu}_0 = 0$, then $\tilde{\nu}_1 \geq 1$ by definition. Now assume that $\tilde{\nu}_0 = l < 0$. Then

$$\sup_{j<l} \tilde{v}_j < \tilde{v}_l, \qquad \tilde{v}_{l+1} \leq \tilde{v}_l, \dots, \tilde{v}_{-1} \leq \tilde{v}_l$$

and $\tilde{v}_l \geq 0$. This follows from the fact that $\tilde{v}_l < 0$ implies $\sup_{j<0} \tilde{v}_j < 0 = \tilde{v}_0$. But this means that $\tilde{\nu}_0 = 0$ which contradicts $\tilde{\nu}_0 = l < 0$. Hence, $\tilde{\nu}_1 \geq 1$ a.s.

Condition C2 implies that $\tilde{\mathbf{P}}(\tilde{\nu}_1 < \infty) = 1$. Therefore,

$$\int_S \sigma(dy)\mathbf{E}_y \nu_1 = \sum_{r=0}^{\infty} \sum_{j=1}^{\infty} \tilde{\mathbf{P}}(\tilde{\nu}_0 = -r,\ \tilde{\nu}_1 = j) = 1.$$

Now let $Q(\cdot)$ be the probability measure on $(\Omega, \mathcal{F})$ defined by

$$Q(A) = \int_S \tilde{\sigma}(dy)\mathbf{P}_y(A),$$

where $\tilde{\sigma}$ is a probability measure on S, that is,

$$\tilde{\sigma}(\Gamma) = \frac{\sigma(\Gamma)}{\sigma(S)} = \frac{\sigma(\Gamma)}{\tilde{\mathbf{P}}(\tilde{\nu}_0 = 0)}, \qquad \Gamma \in \mathcal{G}.$$

Define the stochastic process

$$Y_j = (\nu_j - \nu_{j-1},\ v_{\nu_j} - v_{\nu_{j-1}}, x_{\nu_j}), \qquad j \in \mathbb{N}. \tag{A.13}$$

By (A.6) this process is a homogeneous Markov chain in $\mathbb{R}^3$ with respect to the measure Q, which is strictly stationary by Lemma A.2. Hence, by the Birkhoff–Khinchine theorem [6] we have

$$\lim_{n\to\infty} \frac{\nu_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\nu_j - \nu_{j-1}) = \nu_\infty = \mathbf{E}^Q(\nu_1 | \mathfrak{J}), \qquad Q\text{-a.s.},$$

$$\lim_{n\to\infty} \frac{v_{\nu_n}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(v_{\nu_j} - v_{\nu_{j-1}}\right) = v_\infty = \mathbf{E}^Q(v_{\nu_1} | \mathfrak{J}), \qquad Q\text{-a.s.},$$

where $\mathfrak{J}$ is the Borel field of invariant sets, $\mathbf{E}^Q(\cdot)$ denotes the expectation with respect to the measure Q.

Condition C2 implies that $v_\infty = \beta \nu_\infty$. Therefore,

$$\begin{aligned}\int_S \sigma(dy)\mathbf{E}_y v_{\nu_1} &= \tilde{\mathbf{P}}(\tilde{\nu}_0 = 0)\mathbf{E}^Q v_{\nu_1} = \tilde{\mathbf{P}}(\tilde{\nu}_0 = 0)\,\mathbf{E}^Q v_\infty \\ &= \beta\tilde{\mathbf{P}}(\tilde{\nu}_0 = 0)\mathbf{E}^Q \nu_\infty = \beta\tilde{\mathbf{P}}(\tilde{\nu}_0 = 0)\mathbf{E}^Q \nu_1 = \beta \int_S \sigma(dy)\mathbf{E}_y \nu_1.\end{aligned}$$

Taking (A.12) into account we obtain (A.10). □

LEMMA A.2. *If conditions* C1 *and* C2 *hold, then the process* $(Y_j)_{j\in\mathbb{N}}$ *is a strictly stationary process, that is, for all* $m,k\in\mathbb{N}$ *and* $i_1<\cdots<i_k$,

$$\mathbf{E}^Q f_1(Y_{i_1+m})\cdots f_k(Y_{i_k+m})=\mathbf{E}^Q f_1(Y_{i_1})\cdots f_k(Y_{i_k}), \tag{A.14}$$

for all measurable bounded functions $(f_i)_{1\le i\le k}$.

PROOF. First notice that (A.6) implies that for all $n\in\mathbb{N}$

$$\mathbf{E}\big(f(Y_{j+n})|Y_0,\dots,Y_j\big)=T^n(f)(Y_j),$$

where $T^n(f)(\cdot)$ is nth power of the operator $T(f)$ which is defined for every bounded measurable function $f:\mathbb{R}^3\to\mathbb{R}$ as

$$\begin{aligned} T(f)(Y)&=\pi_0(f)(y_3),\\ Y&=(y_1,y_2,y_3)\in\mathbb{R}^3,\\ \pi_0(f)(y)&=\mathbf{E}_y f(Y_1). \end{aligned}$$

We prove (A.14) by induction. For $k=2$ we have

$$\begin{aligned} &\mathbf{E}^Q f_1(Y_{i_1+m})f_2(Y_{i_2+m})\\ &\quad=\int_S\tilde\sigma(dy)\mathbf{E}_y f_1(Y_{i_1+m})f_2(Y_{i_2+m})\\ &\quad=\int_S\tilde\sigma(dy)\mathbf{E}_y f_1(Y_{i_1+m})\mathbf{E}\big(f_2(Y_{i_2+m})|Y_0,\dots,Y_{i_1+m}\big)\\ &\quad=\int_S\tilde\sigma(dy)\mathbf{E}_y\tilde f_1(Y_{i_1+m}), \end{aligned}$$

where $\tilde f_1(Y)=f_1(Y)T^{i_2-i_1}(f_2)(Y)$. Thus by (A.6) and (A.9) we obtain

$$\begin{aligned} \mathbf{E}^Q f_1(Y_{i_1+m})f_2(Y_{i_2+m})&=\int_S\tilde\sigma(dy)\mathbf{E}_y\pi(\tilde f_1)(x_{\nu_{i_1+m-1}})\\ &=\int_S\tilde\sigma(dy)\mathbf{E}_y\pi(\tilde f_1)(x_{\nu_{i_1-1}})\\ &=\int_S\tilde\sigma(dy)\mathbf{E}_y\tilde f_1(Y_{i_1})\\ &=\mathbf{E}^Q f_1(Y_{i_1})f_2(Y_{i_2}). \end{aligned}$$

We assume now that (A.14) is true for some fixed $k\in\mathbb{N}$ and prove that it is true for $k+1$. By the same method as above we obtain

$$\begin{aligned} &\mathbf{E}^Q f_1(Y_{i_1+m})\cdots f_k(Y_{i_k+m})f_{k+1}(Y_{i_{k+1}+m})\\ &\quad=\mathbf{E}^Q f_1(Y_{i_1+m})\cdots\tilde f_k(Y_{i_k+m}), \end{aligned}$$

where $\tilde{f}_k(Y) = f_k(Y)T^{i_{k+1}-i_k}(f_{k+1})(Y)$. By assumption we obtain that

$$
\begin{aligned}
&\mathbf{E}^Q f_1(Y_{i_1+m}) \cdots f_k(Y_{i_k+m}) f_{k+1}(Y_{i_{k+1}+m}) \\
&\quad = \mathbf{E}^Q f_1(Y_{i_1}) \cdots \tilde{f}_k(Y_{i_k}) \\
&\quad = \int_S \tilde{\sigma}(dy) \mathbf{E}_y f_1(Y_{i_1}) \cdots f_k(Y_{i_k}) T^{i_{k+1}-i_k}(f_{k+1})(Y_{i_k}) \\
&\quad = \int_S \tilde{\sigma}(dy) \mathbf{E}_y f_1(Y_{i_1}) \cdots f_k(Y_{i_k}) f_{k+1}(Y_{i_{k+1}}),
\end{aligned}
$$

which gives (A.14) for $k+1$. □

A.2. Invariant measure for the process in continuous time (3.2). In this section we show that the measure μ as defined in (4.2) [with σ defined in (A.4)] is an invariant measure for the process $(Z(t), W(t))_{t \ge 0}$.

Recall the definition of ν_i in (A.5) and consider the function

$$
\Theta(z, \Gamma, t) = \sum_{i=0}^{\infty} \mathbf{P}_z(x_{\nu_i} \in \Gamma, v_{\nu_i} \le t), \qquad z \in S, \Gamma \in \mathcal{G}, t > 0,
$$

and $\Theta(z, \Gamma, t) = 0$ for $t \le 0$. Furthermore, from (A.5) it follows directly that $v_{\nu_i} \ge v_i$ therefore by Proposition 5.1 for any $\Gamma \in \mathcal{G}$, $z \in S$ and $t \ge 0$,

$$
\begin{aligned}
\Theta(z, \Gamma, t) &\le \sum_{i=0}^{\infty} \mathbf{P}_z(v_{\nu_i} \le t) \\
&= \sum_{i=0}^{\infty} \mathbf{P}_z(0 \le \widehat{v}_i \le t) < \infty.
\end{aligned}
$$

For every measurable bounded function $f : S \times \mathbb{R} \to \mathbb{R}$ we define the operator

$$
\begin{aligned}
\Theta(f)(z, t) &= \int_0^t f(x, t-\tau) \Theta(z, dx, d\tau) \\
&= \sum_{i=0}^{\infty} \mathbf{E}_z f^+(x_{\nu_i}, t - v_{\nu_i}) \\
&= \sum_{i=0}^{\infty} \mathbf{E}_z f^+(\widehat{x}_i, t - \widehat{v}_i),
\end{aligned}
$$

where $f^+(x, t) = f(x, t)\chi_{\{t \ge 0\}}$.

As we have already remarked, the Markov chain $(\widehat{x}_i, \widehat{v}_i)_{i \ge 0}$ satisfies condition (5.2), which implies immediately that the function $\Theta(f)(z, t)$ satisfies the following equation:

$$
\Theta(f)(z, t) = f^+(z, t) + \mathbf{E}_z \Theta(f)(x_{\nu_1}, t - v_{\nu_1}), \qquad z \in S, t \ge 0.
$$

Furthermore, since σ is an invariant measure for $(x_{\nu_i})_{i\ge 0}$, we obtain

$$\begin{aligned}
&\int_{S\times\mathbb{R}} \Theta(f)(y,t-w)\mu(dy,dw)\\
&\quad= \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \int_0^{\upsilon_{\nu_1}\wedge t} \Theta(f)\big(x_{\nu_1}, t-w\big)\,dw\\
&\quad= \frac{1}{\beta}\int_S \sigma(dy)\int_0^t \Theta(f)(y,t-w)\,dw\\
&\qquad - \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \int_{\upsilon_{\nu_1}\wedge t}^t \Theta(f)\big(x_{\nu_1}, t-w\big)\,dw\\
&\quad= \frac{1}{\beta}\int_S \sigma(dy)\int_0^t f(y,w)\,dw\\
&\qquad + \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \chi_{\{\upsilon_{\nu_1}\le t\}} \int_0^{t-\upsilon_{\nu_1}} \Theta(f)\big(x_{\nu_1}, t-w-\upsilon_{\nu_1}\big)\,dw\\
&\qquad - \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \chi_{\{\upsilon_{\nu_1}\le t\}} \int_0^{t-\upsilon_{\nu_1}} \Theta(f)\big(x_{\nu_1}, w\big)\,dw.
\end{aligned}$$

By change of variables, setting $u = t - w - \upsilon_{\nu_1}$ in the second integral in the last equality, we obtain that

$$\begin{aligned}
&\int_{S\times\mathbb{R}} \Theta(f)(y,t-w)\mu(dy,dw)\\
&\qquad= \frac{1}{\beta}\int_S \sigma(dy)\int_0^t f(y,w)\,dw.
\end{aligned}\tag{A.15}$$

We show next that the measure μ is invariant for $(Z(t), W(t))_{z\ge 0}$, that is, for every bounded measurable function $f : S\times\mathbb{R}\to\mathbb{R}$,

$$\begin{aligned}
&\int_{S\times\mathbb{R}} \mathbf{E}_{z,w} f\big((Z(t), W(t)\big)\mu(dz,dw)\\
&\qquad= \int_{S\times\mathbb{R}} f(z,w)\mu(dz,dw), \qquad t\ge 0,
\end{aligned}\tag{A.16}$$

where $\mathbf{E}_{z,w}$ denotes the expectation by the distribution of the process $(Z(t), W(t), t\ge 0)$ under the initial condition $Z(0)=z$, $W(0)=w$. Notice that condition (2.1) and definitions (3.2) imply that for every measurable bounded function f and $t\ge 0$,

$$\mathbf{E}_{z,w} f\big(Z(t), W(t)\big) = \begin{cases} f(z, w-t), & \text{if } 0\le t< w,\\ \mathbf{E}_z f\big(Z(t-w), W(t-w)\big), & \text{if } t\ge w.\end{cases}\tag{A.17}$$

Indeed, taking into account that on the set $\{N(0)=n\}$ for $t \geq w$ we have $v_n = v_{N(0)} = W(0) = w \leq t$, we conclude

$$\begin{aligned}
&\mathbf{E}_{z,w} f\big(Z(t), W(t)\big)\\
&\quad = \mathbf{E}_{z,w}\mathbf{E}\big(f\big(Z(t), W(t)\big)\big|\mathcal{F}_{N(0)}\big)\\
&\quad = \mathbf{E}_{z,w}\sum_{n=1}^{\infty}\chi_{\{N(0)=n\}}\mathbf{E}\big(f\big(Z(t), W(t)\big)\big|\mathcal{F}_n\big)\\
&\quad = \mathbf{E}_{z,w}\sum_{n=1}^{\infty}\chi_{\{N(0)=n\}}\mathbf{E}\Bigg(\sum_{k=n+1}^{\infty} f(x_k, v_k - t)\chi_{\{N(t)=k\}}\Bigg|\mathcal{F}_n\Bigg)\\
&\quad = \mathbf{E}_{z,w}\sum_{n=1}^{\infty}\chi_{\{v_1\leq 0,\dots,v_{n-1}\leq 0, v_n>0\}}\\
&\qquad\quad \times \mathbf{E}\Bigg(\sum_{k=n+1}^{\infty} f(x_k, v_k - t)\chi_{\{v_n\leq t,\dots,v_{k-1}\leq t, v_k>t\}}\Bigg|\mathcal{F}_n\Bigg).
\end{aligned}$$

By condition (2.1) we have also that

$$\begin{aligned}
\mathbf{E}_{z,w} f\big(Z(t), W(t)\big) &= \mathbf{E}_{z,w}\sum_{n=1}^{\infty}\chi_{\{N(0)=n\}}\Phi(x_n, v_n)\\
&= \mathbf{E}_{z,w}\Phi\big(x_{N(0)}, v_{N(0)}\big) = \Phi(z, w),
\end{aligned}$$

where for $t \geq v$

$$\begin{aligned}
\Phi(x, v) &= \mathbf{E}_x \sum_{k=1}^{\infty} f(x_k, v_k + v - t)\chi_{\{v_1\leq t-v,\dots,v_{k-1}\leq t-v, v_k>t-v\}}\\
&= \mathbf{E}_x \sum_{k=1}^{\infty} f(x_k, v_k + v - t)\chi_{\{N(t-v)=k\}}\\
&= \mathbf{E}_x f\big(x_{N(t-v)}, v_{N(t-v)} + v - t\big)\\
&= \mathbf{E}_x f\big(Z(t-v), W(t-v)\big).
\end{aligned}$$

From this we obtain (A.17).

It remains to show (A.16). Indeed, by definition $N(t) = \nu_i$ for some $i \in \mathbb{N}$. Therefore, for $t \geq 0$,

$$\begin{aligned}
&\mathbf{E}_x f\big(Z(t), W(t)\big)\\
&\qquad = \sum_{i=1}^{\infty}\mathbf{E}_x f\big(x_{\nu_i}, v_{\nu_i} - t\big)\chi_{\{N(t)=\nu_i\}}
\end{aligned}$$

$$= \sum_{i=0}^{\infty} \mathbf{E}_x \chi_{\{v_{\nu_i} \le t\}} f^+\big(x_{\nu_{i+1}}, v_{\nu_{i+1}} - t\big)$$

$$= \mathbf{E}_x \sum_{i=0}^{\infty} \chi_{\{v_{\nu_i} \le t\}} g\big(x_{\nu_i}, t - v_{\nu_i}\big) = \Theta(g)(x, t),$$

where $g(x, u) = \mathbf{E}_x f^+(x_{\nu_1}, v_{\nu_1} - u)$. Therefore, taking (A.15) into account we obtain that

$$\begin{aligned}
&\int_{S\times\mathbb{R}} \mathbf{E}_{z,w} f\big(Z(t), W(t)\big)\mu(dz, dw) \\
&\quad= \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \int_0^{v_{\nu_1}} \mathbf{E}_{x_{\nu_1},w} f\big(Z(t), W(t)\big)\, dw \\
&\quad= \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \int_0^{v_{\nu_1}\wedge t} \mathbf{E}_{x_{\nu_1},w} f\big(Z(t), W(t)\big)\, dw \\
&\qquad+ \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \chi_{\{v_{\nu_1}\ge t\}} \int_{v_{\nu_1}\wedge t}^{v_{\nu_1}} \mathbf{E}_{x_{\nu_1},w} f\big(Z(t), W(t)\big)\, dw \\
&\quad= \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \int_0^{v_{\nu_1}\wedge t} \mathbf{E}_{x_{\nu_1}} f\big(Z(t-w), W(t-w)\big)\, dw \\
&\qquad+ \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \chi_{\{v_{\nu_1}\ge t\}} \int_t^{v_{\nu_1}} f\big(x_{\nu_1}, w - t\big)\, dw \\
&\quad= \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \int_0^{v_{\nu_1}\wedge t} \Theta(g)\big(x_{\nu_1}, t - w\big)\, dw \\
&\qquad+ \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \chi_{\{v_{\nu_1}\ge t\}} \int_0^{v_{\nu_1}-t} f\big(x_{\nu_1}, w\big)\, dw \\
&\quad= \int_{S\times\mathbb{R}} \Theta(g)(y, t-w)\mu(dy, dw) \\
&\qquad+ \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \chi_{\{v_{\nu_1}\ge t\}} \int_0^{v_{\nu_1}-t} f\big(x_{\nu_1}, w\big)\, dw \\
&\quad= \frac{1}{\beta}\int_S \sigma(dy) \int_0^t g(y, w)\, dw + \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \chi_{\{v_{\nu_1}\ge t\}} \int_0^{v_{\nu_1}-t} f\big(x_{\nu_1}, w\big)\, dw \\
&\quad= \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \int_0^t f^+\big(x_{\nu_1}, v_{\nu_1} - w\big)\, dw \\
&\qquad+ \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \chi_{\{v_{\nu_1}\ge t\}} \int_0^{v_{\nu_1}-t} f\big(x_{\nu_1}, w\big)\, dw \\
&\quad= \frac{1}{\beta}\int_S \sigma(dy)\mathbf{E}_y \bigg(\int_{v_{\nu_1}-v_{\nu_1}\wedge t}^{v_{\nu_1}} f\big(x_{\nu_1}, w\big)\, dw + \chi_{\{v_{\nu_1}\ge t\}} \int_0^{v_{\nu_1}-t} f\big(x_{\nu_1}, w\big)\, dw\bigg)
\end{aligned}$$

$$= \frac{1}{\beta} \int_S \sigma(dy) \mathbf{E}_y \int_0^{v_{\nu_1}} f(x_{\nu_1}, w)\, dw$$

$$= \int_{S \times \mathbb{R}} f(y, w) \mu(dy, dw).$$

From this equality (A.16) follows.

A.3. Proof of (4.3). For the function (4.2) in the definition of H for $t > \eta$ we have

$$\int_{S \times \mathbb{R}} H(z, t - w) \mu(dz, dw)$$

$$= \int_{-\infty}^{\infty} \theta(s)$$

$$\times \left(\int_{S \times \mathbb{R}} \mathbf{E}_z f\big(Z(t + s - w), W(t + s - w)\big) \chi_{\{t+s \geq w\}} \mu(dz, dw) \right) ds$$

$$= \int_{-\infty}^{\infty} \theta(s) \left(\int_{S \times \mathbb{R}} \mathbf{E}_{z,w} f\big(Z(t + s), W(t + s)\big) \mu(dz, dw) \right) ds - \delta(t),$$

where

$$\delta(t) = \int_{-\infty}^{\infty} \theta(s) \int_{S \times \mathbb{R}} f(z, w - t - s) \chi_{\{t+s \leq w\}} \mu(dz, dw)\, ds.$$

Taking (A.16) and the definition of the function $\theta(\cdot)$ in (4.2) into account we obtain that the first integral in the last equality is equal to the right-hand side of (4.3), and by the dominated convergence theorem we get $\delta(t) \to 0$ as $t \to \infty$. Hence the relationship (4.3) holds.

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CENTER FOR MATHEMATICAL SCIENCES
MUNICH UNIVERSITY OF TECHNOLOGY
D-80290 MUNICH
GERMANY
E-MAIL: cklu@ma.tum.de
URL: www.ma.tum.de/stat/

LABORATOIRE DE MATHÉMATIQUES
RAPHAËL SALEM
UMR CNRS 6085, SITE COLBERT
UNIVERSITÉ DE ROUEN
F76 821 MONT SAINT AIGNAN CEDEX
FRANCE
E-MAIL: Serge.Pergamenchtchikov@univ-rouen.fr