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RENEWAL THEORY FOR *M*-DEPENDENT VARIABLES

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Let S_n, $n = 1, 2, \cdots$, denote the partial sums of a stationary m-dependent sequence of random variables with positive expectation. The first passage times $\min\{n: S_n > t\}$ are investigated. Several results are extended from the case of independent variables.

0. Introduction. Let $\{X_n\}_1^\infty$ be a stationary m-dependent sequence of random variables. (X will be used to denote a generic variable in the sequence.) The nonnegative integer m will be fixed throughout the paper. The purpose of this paper is to study the sequence of partial sums $S_n = \sum_1^n X_i$, and the stopping times $\tau(t) = \min\{n: S_n > t\}$, when X has positive expectation. In particular, we are interested in the existence and the asymptotic behaviour of the moments of $\tau(t)$ and the overshoot $S_{\tau(t)} - t$ for various integrability conditions on X.

For independent variables (the special case $m = 0$) such problems have been extensively studied, see e.g. [7] and the references listed therein. Some of these results are generalized in this paper.

Related problems, including a version of Blackwell's renewal theorem, are treated for dependent sequences by Berbee [2].

Section 1 uses martingale theory to establish some basic results for the sequence $\{S_n\}$ and arbitrary stopping times, most notably a version of Wald's lemma (Theorem 1.1).

Section 2 applies these results to the renewal times $\tau(t)$. Using these more complicated martingale results, the proofs for the independent case hold with only minor modifications. One of the main results is $E\tau(t) = t/EX + o(t^{1/r})$ as $t \to \infty$, provided $E|X|^r < \infty$.

Section 3 contains a refinement for the case $X \geq 0$. Then $E\tau(t) = t/EX + O(1)$, provided $EX^2 < \infty$.

Section 4 is devoted to applications.

1. Martingales and stopping times. In this section we make the following assumptions.

(i) $\{X_n\}_1^\infty$ is a stationary sequence of random variables adapted to an increasing sequence of σ-fields $\{\mathscr{F}_n\}_1^\infty$ on a probability space $(\Omega, \mathscr{F}_\infty, P)$.
(ii) m is an integer such that $\{X_{n+i}\}_{i=m+1}^\infty$ is independent of $\mathscr{F}_n$ for every n.
(iii) $\mu = EX$ exists (in this section not necessarily positive).
(iv) τ is a stopping time relative to $\{\mathscr{F}_n\}$.

Let $\mathscr{F}_n = \{\varnothing, \Omega\}$ for $n \leq 0$. We define $S_n = \sum_1^n X_k$ and $U_n = E(S_{n+m} - (n+m)\mu \mid \mathscr{F}_n)$, $n \geq 0$. (For $n \leq 0$ let $U_n = 0$.)

Let $V_n = U_n - (S_n - n\mu) = E(\sum_{i=1}^m (X_{n+i} - \mu) \mid \mathscr{F}_n)$ and $\Delta U_n = U_n - U_{n-1} = X_n - \mu + V_n - V_{n-1}$.

If $EX^2 < \infty$, let $\gamma^2 = \mathrm{Cov}(X_n, S_{n+m}) = \mathrm{Var}\, X_n + 2\sum_{i=1}^m \mathrm{Cov}(X_n, X_{n+i}) (n > m)$. Note that $\gamma^2 = \lim_{n\to\infty} (1/n)\, \mathrm{Var}\, S_n \geq 0$.

Lemma 1.1. *$\{U_n\}_0^\infty$ is a martingale.*

Proof. $E(U_n \mid \mathscr{F}_{n-1}) = E(S_{n+m} - (n+m)\mu \mid \mathscr{F}_{n-1}) = U_{n-1} + E(X_{n+m} - \mu \mid \mathscr{F}_{n-1}) = U_{n-1}$.

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THEOREM 1.1. *If* $X \geq 0$ a.s. *or* $E\tau < \infty$, *then*

$$ES_{\tau+m} = (E\tau + m)\mu.$$

PROOF. By Lemma 1.1, $E(S_{\tau\wedge N+m} - (\tau \wedge N + m)\mu) = EU_{\tau\wedge N} = EU_0 = 0$. Thus, $ES_{\tau\wedge N+m} = (E(\tau \wedge N + m))\mu$. For positive X, the result follows by monotone convergence as $N \to \infty$. If $E\tau < \infty$, we use the decomposition $X_n = X_n^+ - X_n^-$.

For independent variables ($m = 0$) this theorem reduces to Wald's lemma $ES_\tau = E\tau\mu$. This formula fails for m-dependent variables, but Theorem 1.1 gives a useful substitute.

We next turn to the variance of $S_{\tau+m} - (\tau + m)\mu$ (cf. [4] for $m = 0$). We single out a special case as a lemma.

LEMMA 1.2. *Assume that* $EX^2 < \infty$ *and that* τ *is a bounded stopping time. Then*

$$EU_{\tau+m}^2 = (E\tau + m)\gamma^2 + EV_{\tau+m}^2.$$

PROOF. We may assume that $\mu = 0$. (Otherwise, we replace X_n by $X_n - \mu$.) Since $I(\tau \geq i - m)$ is $\mathscr{F}_{i-m-1}$-measurable,

$$EU_{\tau+m}^2 = E\sum_{i\leq\tau+m}(\Delta U_i)^2 = E\sum_1^\infty I(\tau \geq i - m)(\Delta U_i)^2$$
$$= \sum_1^\infty E(I(\tau \geq i - m)E((\Delta U_i)^2 \mid \mathscr{F}_{i-m-1})).$$

Furthermore $\Delta U_i = X_i + V_i - V_{i-1}$, which by orthogonality yields

$$E((\Delta U_i)^2 \mid \mathscr{F}_{i-1}) = E((X_i + V_i)^2 \mid \mathscr{F}_{i-1}) - E(V_{i-1}^2 \mid \mathscr{F}_{i-1})$$

and

$$E((\Delta U_i)^2 \mid \mathscr{F}_{i-m-1}) = E(X_i^2 + 2X_iV_i + V_i^2 - V_{i-1}^2 \mid \mathscr{F}_{i-m-1}).$$

Using $V_i = E(\sum_{j=1}^m X_{i+j} \mid \mathscr{F}_i)$ we obtain $X_iV_i = E(\sum_1^m X_iX_{i+j} \mid \mathscr{F}_i)$ and (by assumption (ii))

$$E(X_i^2 + 2X_iV_i \mid \mathscr{F}_{i-m-1}) = E(X_i^2 + 2\sum_1^m X_iX_{i+j} \mid \mathscr{F}_{i-m-1}) = E(X_i^2 + 2\sum_1^m X_iX_{i+j}) = \gamma^2.$$

Consequently,

$$E((\Delta U_i)^2 \mid \mathscr{F}_{i-m-1}) = \gamma^2 + E(V_i^2 - V_{i-1}^2 \mid \mathscr{F}_{i-m-1})$$

and

$$EU_{\tau+m}^2 = \sum_1^\infty E(I(\tau \geq i - m)(\gamma^2 + E(V_i^2 - V_{i-1}^2 \mid \mathscr{F}_{i-m-1})))$$
$$= \sum_1^\infty E(I(\tau \geq i - m)(\gamma^2 + V_i^2 - V_{i-1}^2)) = E\sum_1^{\tau+m}(\gamma^2 + V_i^2 - V_{i-1}^2)$$
$$= (E\tau + m)\gamma^2 + EV_{\tau+m}^2.$$

THEOREM 1.2. *Assume that* $EX^2 < \infty$ *and* $E\tau < \infty$. *Then*

(i) $EU_{\tau+m}^2 = (E\tau + m)\gamma^2 + EV_{\tau+m}^2$

(ii) $E(S_{\tau+m} - (\tau + m)\mu)^2 = (E\tau + m)\gamma^2 - 2E(S_{\tau+2m} - S_{\tau+m} - m\mu)(S_{\tau+m} - S_\tau - m\mu)$

(iii) $E(S_{\tau+2m} - (\tau + 2m)\mu)^2 = E\tau\cdot\gamma^2 + E(S_{2m} - 2m\mu)^2$.

PROOF. Again, for convenience, we assume that $\mu = 0$. By Hölder's and Cauchy-Schwarz' inequalities,

$$V_{\tau\wedge N+m}^2 \leq E((\sum_1^m X_{\tau\wedge N+i})^2 \mid \mathscr{F}_{\tau\wedge N}) \leq mE(\sum_1^m X_{\tau\wedge N+i}^2 \mid \mathscr{F}_{\tau\wedge N}) \leq mE(\sum_1^{\tau+m} X_n^2 \mid \mathscr{F}_{\tau\wedge N}).$$

Theorem 1.1 applied to $\{X_n^2\}$ shows that $\sum_1^{\tau+m} X_n^2$ is integrable. Hence, the family $\{E(\sum_1^{\tau+m} X_n^2 \mid \mathscr{F}_{\tau\wedge N})\}_{N=1}^\infty$ is uniformly integrable [11, Lemma IV-2-4]. Thus, $\{V_{\tau\wedge N+m}^2\}$ is uniformly integrable and, since $V_{\tau\wedge N+m} \to V_{\tau+m}$ a.s., $EV_{\tau\wedge N+m}^2 \to EV_{\tau+m}^2$ as $N \to \infty$.

Since, by Lemma 2, $E\sum_1^{\tau\wedge N+m}(\Delta U_i)^2 = (E(\tau \wedge N) + m)\gamma^2 + EV_{\tau\wedge N+m}^2$, monotone convergence yields $E\sum_1^{\tau+m}(\Delta U_i)^2 = (E\tau + m)\gamma^2 + EV_{\tau+m}^2 < \infty$ and (i) follows from [4, Theorem 1].

Since $U_{\tau+m} = S_{\tau+m} + V_{\tau+m}$ and $U_{\tau+m} = E(S_{\tau+2m} | \mathscr{F}_{\tau+m})$,

$$\begin{aligned}(E\tau + m)\gamma^2 &= EU_{\tau+m}^2 - EV_{\tau+m}^2 = E(U_{\tau+m} - V_{\tau+m})(U_{\tau+m} + V_{\tau+m}) \\ &= ES_{\tau+m}(2U_{\tau+m} - S_{\tau+m}) \\ &= 2ES_{\tau+m}S_{\tau+2m} - ES_{\tau+m}^2 = ES_{\tau+m}^2 + 2ES_{\tau+m}(S_{\tau+2m} - S_{\tau+m}) \\ &= ES_{\tau+m}^2 + 2E(S_{\tau+2m} - S_{\tau+m})(S_{\tau+m} - S_\tau) + 2E(S_{\tau+2m} - S_{\tau+m})S_\tau.\end{aligned}$$

(i) follows from $E(S_{\tau+2m} - S_{\tau+m})S_\tau = E(E(S_{\tau+2m} - S_{\tau+m} | \mathscr{F}_\tau)S_\tau) = 0$.

To prove (iii), note that by the computation above

$$ES_{\tau+2m}^2 - E\tau\gamma^2 = E(S_{\tau+2m}^2 - 2S_{\tau+m}S_{\tau+2m} + S_{\tau+m}^2) + m\gamma^2 = E(E(S_{\tau+2m} - S_{\tau+m})^2 | \mathscr{F}_\tau) + m\gamma^2.$$

However, $E((S_{\tau+2m} - S_{\tau+m})^2 | \mathscr{F}_\tau)$ is a constant independent of τ. Hence $ES_{\tau+2m}^2 - E\tau\gamma^2$ does not depend on τ, and the trivial case $\tau = 0$ shows that it equals ES_{2m}^2, which proves (iii).

In the next section we shall study a family of stopping times, keeping $\{X_n\}$ fixed. Therefore, we shall use $o(E\tau)$ to denote quantities $\delta(\tau) = \delta(\tau, \{X_n\})$ such that, *for fixed* $\{X_n\}$,

$$\sup\{ |\delta(\tau)| / E\tau : \tau \text{ is a stopping time such that } A < E\tau < \infty\}$$

is finite for any $A > 0$ and converges to 0 as $A \to \infty$. $o((E\tau)^{1/r})$ etc. are defined similarly.

Next, we collect some important consequences of Theorems 1.1 and 1.2.

COROLLARY 1.1. *Assume that* $E|X|^r < \infty$, *where* $1 \le r < \infty$. *For stopping times* τ *such that* $E\tau < \infty$,

(i) $E|X_\tau|^r = o(E\tau)$
(ii) $E|X_\tau| = o((E\tau)^{1/r})$
(iii) $E|S_{\tau+m} - S_\tau|^r = o(E\tau)$
(iv) $E|S_{\tau+m} - S_\tau| = o((E\tau)^{1/r})$
(v) $ES_\tau = E\tau \cdot \mu + o((E\tau)^{1/r})$.

Furthermore, if $r \ge 2$,

(vi) $\| S_\tau - \tau\mu \|_2 = \sqrt{E\tau}\gamma + o((E\tau)^{1/r})$
(vii) $E(S_\tau - \tau\mu)^2 = E\tau\gamma^2 + o((E\tau)^{1/2+1/r})$.

If $r = \infty$ (*X is bounded*), (ii) (*trivially*) *and* (iv) – (vii) *hold with o replaced by O.*

PROOF. By Theorem 1.1 applied to $\{|X_n|^r\}$,

$$E|X_\tau|^r \le E\textstyle\sum_1^{\tau+m} |X_i|^r = (E\tau + m)E|X|^r.$$

This estimate is improved to "o" in the following standard way. Let $\varepsilon > 0$ and let $X_n' = X_n \cdot I(|X_n| > A)$ and $X_n'' = X_n - X_n'$ where A is so large such that $E|X_n'|^r < \varepsilon$. Then

$$E|X_\tau|^r = E|X_\tau'|^r + E|X_\tau''|^r \le (E\tau + m)E|X'|^r + A^r < \varepsilon E\tau + C_\varepsilon.$$

This proves (i). (ii) follows by Lyapunov's inequality. (iii) and (iv) are proved similarly, using Hölder's inequality $|S_{\tau+m} - S_\tau|^r = |\sum_1^m X_{\tau+i}|^r \le m^{r-1} \sum_1^m |X_{\tau+i}|^r$. (v) follows immediately from (iv) and Theorem 1.1.

If $r \ge 2$, Theorem 1.2 (iii) yields

$$\| S_{\tau+2m} - (\tau + 2m)\mu \|_2 = \sqrt{E\tau}\gamma + O((E\tau)^{-1/2}).$$

Since $\| S_{\tau+2m} - S_{\tau+m} \|_2$ is constant (cf. the proof of Theorem 1.2), and $\| S_{\tau+m} - S_\tau \|_2 \le \| S_{\tau+m} - S_\tau \|_r = o((E\tau)^{1/r})$ by (iii), (vi) follows from Minkowski's inequality. (vii) is obtained by squaring. The modifications in the case $r = \infty$ are easy.

Finally, we give an estimate for arbitrary moments of $S_\tau - \tau\mu$. For the independent case, see [7] and [8].

LEMMA 1.3. *Let $1 \le r < \infty$ and $r_1 = \max(r/2, 1)$. Suppose that $E|X|^r < \infty$, $E\tau^{r_1} < \infty$ and $EX = 0$. Then, for any non-negative integer k,*

$$E\left|\sum_{n=1}^{\tau} E(X_{n+k} \mid \mathscr{F}_n)\right|^r \le C_{m,r,k} E\tau^{r_1} E|X|^r.$$

($C_{m,r,k}$ is a universal constant.)

PROOF. If $k > m$, $E(X_{n+k} \mid \mathscr{F}_n) = EX_{n+k} = 0$ and there is nothing to prove. We perform an induction on k (backwards) and r, assuming the estimate to hold for r, $k+1$ as well as for $r/2$ and any k (if $r \ge 2$).

$$\sum_{n=1}^{\tau} E(X_{n+k} \mid \mathscr{F}_n) = \sum_1^{\tau} (E(X_{n+k} \mid \mathscr{F}_n) - E(X_{n+k} \mid \mathscr{F}_{n-1})) + \sum_1^{\tau} E(X_{n+k+1} \mid \mathscr{F}_n) - E(X_{\tau+k+1} \mid \mathscr{F}_\tau).$$

By the c_r-inequalities, it suffices to estimate the three terms on the right-hand side separately. The estimate holds for the second term by the induction hypothesis, and for the third term by

$$E|E(X_{\tau+k+1} \mid \mathscr{F}_\tau)|^r \le E|X_{\tau+k+1}|^r \le E\sum_1^{\tau+m} |X_i|^r = (E\tau + m)E|X|^r$$

(if $k < m$, otherwise the term vanishes).

The first term above is a stopped martingale and we use the Burkholder-Gundy inequality [6]

$$E\left|\sum_1^{\tau} (E(X_{n+k} \mid \mathscr{F}_n) - E(X_{n+k} \mid \mathscr{F}_{n-1}))\right|^r \le C_r EQ^r,$$

where the square function Q is defined by

$$\begin{aligned} Q^2 &= \sum_1^{\tau} (E(X_{n+k} \mid \mathscr{F}_n) - E(X_{n+k} \mid \mathscr{F}_{n-1}))^2 \le \sum_1^{\tau} 2(E(X_{n+k} \mid \mathscr{F}_n)^2 + E(X_{n+k} \mid \mathscr{F}_{n-1})^2) \\ &\le 2\sum_1^{\tau} E(X_{n+k} \mid \mathscr{F}_n)^2 + 2\sum_1^{\tau} E(X_{n+k+1} \mid \mathscr{F}_n)^2. \end{aligned}$$

We will estimate the $r/2$th moment of the first sum, the second sum is treated the same way. We look at two cases separately.

If $r \le 2$,

$$\begin{aligned} E\left(\sum_1^{\tau} E(X_{n+k} \mid \mathscr{F}_n)^2\right)^{r/2} &\le E\sum_1^{\tau} |E(X_{n+k} \mid \mathscr{F}_n)|^r \le E\sum_1^{\tau} E(|X_{n+k}|^r \mid \mathscr{F}_n) \\ &\le \sum_1^{\infty} E(I(\tau \ge n - m)E(|X_{n+k}|^r \mid \mathscr{F}_n)) \\ &= \sum_1^{\infty} EI(\tau \ge n - m)E|X_{n+k}|^r = (E\tau + m)E|X|^r. \end{aligned}$$

If $r > 2$, let $\sigma^2 = EX^2$. Then

$$\sum_1^{\tau} E(X_{n+k} \mid \mathscr{F}_n)^2 \le \sum_1^{\tau} E(X_{n+k}^2 \mid \mathscr{F}_n) = \tau\sigma^2 + \sum_1^{\tau} E(X_{n+k}^2 - \sigma^2 \mid \mathscr{F}_n).$$

$E(\tau\sigma^2)^{r/2} \le E\tau^{r/2} E|X|^r$ and by the induction hypothesis applied to $\{X_n^2 - \sigma^2\}$,

$$E\left|\sum_1^{\tau} E(X_{n+k}^2 - \sigma^2 \mid \mathscr{F}_n)\right|^{r/2} \le C_{m,r/2,k} E\tau^{\max(r/4,1)} E|X^2 - \sigma^2|^{r/2}.$$

Thus, for any r, $EQ^r = E(Q^2)^{r/2} \le C_{m,r,k} E\tau^{r_1} E|X|^r$. The proof is complete.

THEOREM 1.3. *Suppose that $E|X|^r < \infty$, where $1 \le r < \infty$.*
(i) *If $E\tau$, $E\tau^{r/2} < \infty$, then $E|S_\tau - \tau\mu|^r \le C_{m,r}(E\tau + E\tau^{r/2})E|X|^r < \infty$.*
(ii) *If $E\tau^r < \infty$, then $E|S_\tau|^r \le C_{m,r} E\tau^r E|X|^r < \infty$.*

PROOF. (i) follows by taking $k = 0$ in Lemma 3 (applied to $\{X_n - \mu\}$) and (ii) is an immediate consequence.

2. Renewal theory. From now on, we assume that $\mu = EX > 0$. We shall study the stopping times $\tau(t)$ defined in the introduction. (We may take $\mathscr{F}_n = \mathscr{F}(X_1, \cdots, X_n)$.) By the law of large numbers $\tau(t) < \infty$ a.s. Also, $\tau(t) \to \infty$ and thus $E\tau(t) \to \infty$ as $t \to \infty$.

For completeness, we include the following theorem which is proved exactly as for independent variables.

THEOREM 2.1.
(i) $\tau(t)/t \to 1/\mu$ a.s.
(ii) $S_{\tau(t)}/t \to 1$ a.s.
(iii) *If* $E|X|^r < \infty$, $1 \le r < 2$, *then* $\tau(t) - t/\mu = o(t^{1/r})$ a.s. *and* $S_{\tau(t)} - t = o(t^{1/r})$ a.s.
(iv) *If* $EX^2 < \infty$ *and* $\gamma^2 \ne 0$, *then* $\dfrac{\tau(t) - t/\mu}{\sqrt{\gamma^2 t/\mu^3}} \to_d N(0, 1)$.

PROOF. By the law of large numbers $S_n/n \to \mu$ and $X_n/n \to 0$ a.s. as $n \to \infty$. Hence $S_{\tau(t)}/\tau(t) \to \mu$ and $X_{\tau(t)}/\tau(t) \to 0$ a.s. as $t \to \infty$, which imply $t/\tau(t) \to \mu$ a.s., from which (i) and (ii) follow.

(iii) is proved similarly, cf. [7; Theorem 28].

(iv) follows from the central limit theorem, Anscombe's theorem [1] and the estimate $(S_{\tau(t)} - t)/\sqrt{t} \to_P 0$ (which follows e.g. from Theorem 2.2 (iii) below).

In the remainder of this section we shall combine the martingale estimates of Section 1 with arguments from [7] to estimate moments.

LEMMA 2.1. *For any* $t \ge 0$, $E\tau(t) < \infty$.

PROOF. Define $X'_n = X_n \wedge A$, where A is a large constant such that $\mu' = EX' > 0$. Let S'_n and $\tau'(t)$ have the obvious meanings. Then $S'_n \le S_n$ whence $\tau'(t) \ge \tau(t)$. Thus, it suffices to show that $E\tau'(t) < \infty$. Since $S'_n \le t$ for $n < \tau'(t)$, and $X'_i \le A$, $S'_n \le t + (m+1)A$ for $n \le \tau'(t) + m$. Hence, by Theorem 1.1,

$$(E\tau'(t) \wedge N + m)\mu' = ES'_{\tau'(t)\wedge N+m} \le t + (m+1)A.$$

By monotone convergence, $E\tau'(t) \le (t + (m+1)A)/\mu' < \infty$.

THEOREM 2.2. *Suppose that* $E|X|^r < \infty$, *where* $1 \le r < \infty$. *Then, as* $t \to \infty$,
(i) $E\tau(t) = t/\mu + o(t^{1/r})$
(ii) $E(S_{\tau(t)} - t)^r = o(t)$
(iii) $ES_{\tau(t)} - t = o(t^{1/r})$.
Furthermore, if $r > 2$,
(iv) $\operatorname{Var} \tau(t) = t\gamma^2/\mu^3 + o(t^{1/2+1/r})$
(v) $E(\tau(t) - t/\mu)^2 = t\gamma^2/\mu^3 + o(t^{1/2+1/r})$.
If $r = \infty$ *(X is bounded) these estimates (except* (ii)) *holds with o replaced by O.*

REMARK 2.1. For positive variables, (i) and (iii) will be sharpened, but we postpone this to the next section.

PROOF. Observe that

$$0 < S_{\tau(t)} - t \le X_{\tau(t)}.$$

Hence, using Corollary 1.1 (ii) and (v),

$$t = ES_{\tau(t)} + o((E\tau(t))^{1/r}) = E\tau(t)\mu + o((E\tau(t))^{1/r}).$$

Thus, $t \to \infty$ implies

$$t/E\tau(t) \to \mu, \quad \text{i.e.} \quad E\tau(t)/t \to 1/\mu \quad \text{as} \quad t \to \infty.$$

Hence, all terms $o((E\tau(t))^{1/r})$ may be replaced by $o(t^{1/r})$ and (i) follows from the computations above. (ii) and (iii) follow from Corollary 1.1 (i) and (ii). If $r \ge 2$, (ii) yields $\| S_{\tau(t)} - t \|_2 \le \| S_{\tau(t)} - t \|_r = o(t^{1/r})$. By Corollary 1.1 (vi),

$$\| t - \tau(t)\mu \|_2 = \sqrt{E\tau(t)}\gamma + o(t^{1/r}) = \sqrt{t/\mu}\gamma + o(t^{1/r})$$

and (v) follows by squaring. (v) and (i) yield (iv). The modifications for $r = \infty$ are omitted.

THEOREM 2.3. *Let* $1 \le r < \infty$.
(i) *If* $E(X^+)^r < \infty$ *then* $ES^r_{\tau(t)} < \infty$ *and* $ES^r_{\tau(t)}/t^r \to 1$ *as* $t \to \infty$.
(ii) *If* $E(X^-)^r < \infty$ *then* $E\tau(t)^r < \infty$ *and* $E\tau(t)^r/t^r \to \mu^{-r}$ *as* $t \to \infty$.

PROOF. (i) $E(X^+_{\tau(t)})^r = o(E\tau(t)) = o(t)$ by Corollary 1.1 and Theorem 2.2. Thus, $\| S_{\tau(t)} - t \|_r \le \| X^+_{\tau(t)} \|_r = o(t^{1/r})$ and $\| S_{\tau(t)} \|_r = t + o(t^{1/r})$. (ii) We assume that $E|X|^r < \infty$. (Otherwise truncate as in the proof of Lemma 2.1. Cf. [7; Theorem 2.3].) For $r = 1$ there is nothing new to prove. If $r > 2$ we assume, by induction on r, that the theorem holds for $r/2$. Thus, with $r_1 = \max(r/2, 1)$, $E\tau(t)^{r_1} = O(t^{r_1})$. By Theorem 1.3 (i).

$$E\,|\,S_{\tau(t)} - \tau(t)\mu\,|^r \le CE\tau(t)^{r_1}E\,|\,X\,|^r = O(t^{r_1}).$$

Thus

$$\| S_{\tau(t)} - \tau(t)\mu \|_r = O(t^{r_1/r}).$$

Minkowski's inequality and (i) finally yield

$$\| \tau(t)\mu \|_{L^r} = t + O(t^{r_1/r}) = t + o(t).$$

REMARK 2.2 Under the hypothesis of Theorem 2.3, $\{S_{\tau(t)}/t\}_{t\ge1}$ and $\{\tau(t)/t\}_{t\ge1}$, respectively, are uniformly integrable. This follows from a combination with Theorem 2.1.

REMARK 2.3. (i) may be compared to Theorem 3.1 which shows that (for $X \ge 0$) $E(S_{\tau(t)} - t)^r$ is bounded as $t \to \infty$ if and only if $EX^{r+1} < \infty$.

REMARK 2.4. Since $X_1^+ \le S_{\tau(t)}$, the converse of (i) holds, viz. if $ES^r_{\tau(t)} < \infty$ for some $t \ge 0$, then $E(X^+)^r < \infty$. For independent summands, the converse of (ii) also holds [7], but for m-dependent variables this is not true, as is shown by the following example.

EXAMPLE 2.1. Let $r > 1$. Let $\{\xi_n\}_0^\infty$ be positive i.i.d. random variables such that $P(\xi_n < 1) > 0$ and $E\xi_n^r = \infty$. Define

$$X_n = \begin{cases} -\xi_n & \text{if } \xi_{n-1} < 1 < \xi_n \\ 1 + \xi_{n-1} & \text{otherwise.} \end{cases}$$

Then $\{X_n\}$ is 1-dependent, $E(X^-)^r = \infty$, but $E\tau(t)^r < \infty$. In fact, since every negative X_n is more than cancelled by the next one, $\tau(t) < 2(t + 1)$.

REMARK 2.5. Several of the results may be extended to the case $\mu = +\infty$, i.e. $EX^- < \infty$, $EX^+ = \infty$, by truncation. For instance, with $X' = X \wedge t$, where t is large enough, $(E\tau'(t) + m)EX' = ES'_{\tau'(t)+m} \le (m + 2)t$. Hence $E\tau(t) \le E\tau'(t) \le (m + 2)t/E(X \wedge t)$. Thus, $E\tau(t)/t \to 0 = 1/\mu$. Some results can be extended to $0 < r < 1$, e.g. $E(X^+)^r < \infty \Leftrightarrow ES^r_{\tau(t)} < \infty$.

3. Further results. Theorem 2.2 is not completely satisfactory since the error terms are coarser than what presumably is required. For independent variables it is e.g. known that (in the non-lattice case) $E\tau(t) - t/\mu$ converges to a finite limit, provided only that $EX^2 < \infty$. We have not been able to prove this for m-dependent variables, but with the extra assumption $X \ge 0$ we now will prove a somewhat weaker result (Theorem 3.1). Cf. [2, Chapter 3] and [10] for related material.

LEMMA 3.1. *Assume that* $X \ge 0$ a.s. *and that* $EX^r < \infty$, *where* $1 < r < \infty$. *Then* $E(S_{\tau(t)+m} - t)^{r-1} = O(1)$.

PROOF. Extend $\{X_n\}_1^\infty$ to a doubly infinite stationary sequence $\{X_n\}_{-\infty}^\infty$. We may assume that $\Omega = R^\infty$ and that $\{X_n\}_{-\infty}^{\infty}$ are the coordinate functions. Let T denote the measure preserving shift mapping $\Omega \to \Omega$ defined by $X_n(T\omega) = X_{n+1}(\omega)$.

We extend the definition $\tau(t, \omega) = \inf\{n \ge 0 : S_n(\omega) > t\}$ to all real t. Thus $\tau(t, \omega) = 0$ for $t < 0$. Let $\varphi(t, \omega) = (S_{\tau(t,\omega)+m}(\omega) - t)^{r-1}$. Thus, for $t < 0$, $\varphi(t) = (S_m - t)^{r-1}$. Since $S_n(\omega) = X_1(\omega) + S_{n-1}(T\omega)$, $n > 0$, it follows that

$$\tau(t, \omega) = \tau(t - X_1(\omega), T\omega) + 1, \qquad t \ge 0,$$

and

$$\varphi(t, \omega) = \varphi(t - X_1(\omega), T\omega) = \varphi(t - X_0(T\omega), T\omega), \qquad t \ge 0.$$

Thus,

$$\int \varphi(t, \omega)\, dP = \int \varphi(t - X_0(T\omega), T\omega)\, dP = \int \varphi(t - X_0(\omega), \omega)\, dP, \qquad t \ge 0.$$

Let ν be the measure $I(0 \le s < X_0)\, ds\, dP$ on $R \times \Omega$. The total mass of ν equals $\int X_0\, dP = \mu$. If $t \ge 0$,

$$\int \varphi(t - s, \omega)\, d\nu = \iint_{0 \le s < X_0} \varphi(t - s, \omega)\, ds\, dP = \iint_{t \ge u > t - X_0} \varphi(u, \omega)\, du\, dP$$

and

$$\int \varphi(t - s, \omega)\, d\nu - \int \varphi(-s, \omega)\, d\nu$$

$$= \iint (I(t \ge u > t - X_0) - I(0 \ge u > -X_0))\varphi(u, \omega)\, du\, dP$$

$$= \iint (I(t \ge u > 0) - I(t - X_0 \ge u > -X_0))\varphi(u, \omega)\, du\, dP$$

$$= \iint I(t \ge u > 0)(\varphi(u, \omega) - \varphi(u - X_0, \omega))\, du\, dP$$

$$= \int_0^t \left(\int \varphi(u, \omega)\, dP - \int \varphi(u - X_0, \omega)\, dP \right) du = 0.$$

Thus,

$$\int \varphi(t - s, \omega)\, d\nu = \int \varphi(-s, \omega)\, d\nu = \iint_{0 \le s < X_0} (S_m + s)^{r-1}\, ds\, dP$$

$$\le \int \frac{1}{r} (X_0 + S_m)^r\, dP = \frac{1}{r} ES_{m+1}^r < \infty, \qquad t \ge 0.$$

Let A denote this common value of $\int \varphi(t - s, \omega)\, d\nu$, $t \ge 0$. $\tau(t)$ and $S_{\tau(t)+m}$ are non-decreasing functions of t. Thus

$$S_{\tau(t)+m} - t \le S_{\tau(t+h)+m} - (t + h) + h, \qquad h > 0.$$

For simplicity, we assume that $r \le 2$, in which case $\varphi(t, \omega) \le \varphi(t + h, \omega) + h^{r-1}$. (The proof for $r > 2$ is similar, using $(a + b)^r \le (1 + \varepsilon)a^r + C_{r,\varepsilon} b^r$, $\varepsilon > 0$.)

Choose $\delta > 0$ such that $P(X > \delta) > 0$. Then, if $t \ge 0$,

$$\delta \int \varphi(t, \omega) I(X_0 \geq \delta)\, dP \leq \int_0^\delta \int \varphi(t+h, \omega) I(X_0 \geq \delta)\, dP\, dh + \delta^r$$

$$= \int_0^\delta \int \varphi(t+\delta-s, \omega) I(X_0 \geq \delta)\, dP\, ds + \delta^r$$

$$\leq \int\int I(X_0 > s > 0) \varphi(t+\delta-s, \omega)\, dP\, ds + \delta^r = A + \delta^r.$$

Furthermore, if $t \geq 0$,

$$\varphi(t, \omega) = \varphi(t - X_1(\omega), T\omega) \leq \varphi(t, T\omega) + X_1(\omega)^{r-1}.$$

By iteration,

$$\varphi(t, \omega) \leq \varphi(t, T^m\omega) + X_1(\omega)^{r-1} + \cdots + X_m(\omega)^{r-1}.$$

Thus, since X_0 is independent of $\varphi(t, T^m\omega)$,

$$\int \varphi(t, \omega) I(X_0 < \delta)\, dP \leq \int \varphi(t, T^m\omega) I(X_0 < \delta)\, dP + mEX^{r-1}$$

$$= P(X_0 < \delta) \int \varphi(t, T^m\omega)\, dP + mEX^{r-1} = P(X < \delta) \int \varphi(t, \omega)\, dP + mEX_1^{r-1}.$$

Combining these two estimates we obtain

$$\int \varphi(t, \omega)\, dP \leq A/\delta + \delta^{r-1} + P(X < \delta) \int \varphi(t, \omega)\, dP + mEX^{r-1}.$$

Since $\int \varphi(t, \omega)\, dP = E(S_{\tau(t)+m} - t)^{r-1}$ is finite by Corollary 1.1,

$$E(S_{\tau(t)+m} - t)^{r-1} \leq (A/\delta + \delta^{r-1} + mEX^{r-1})/P(X \geq \delta).$$

THEOREM 3.1. *Assume that* $X \geq 0$ a.s.
(i) $EX^2 < \infty \Leftrightarrow E\tau(t) = t/\mu + O(1)$
(ii) $EX^r < \infty \Rightarrow E\tau(t) = t/\mu + o(t^{2-r})$ as $t \to \infty$, $1 < r < 2$
(iii) $EX^r < \infty \Leftrightarrow E(S_{\tau(t)} - t)^{r-1} = O(1)$, $1 < r < \infty$.

PROOF. If $EX^2 < \infty$, then $E\tau(t)\mu = ES_{\tau(t)+m} - m\mu = t + O(1)$ by Theorem 1.1 and Lemma 3.1.

If $EX^r < \infty$, $1 < r < 2$, we combine Lemma 3.1, Theorem 2.2 and Corollary 1.1 by Hölder's inequality to obtain

$$E(S_{\tau(t)+m} - t) \leq (E(S_{\tau(t)+m} - t)^{r-1})^{r-1} (E(S_{\tau(t)+m} - t)^r)^{2-r} = o(t^{2-r}),$$

and (ii) follows by Theorem 1.1.

The direct implication of (iii) follows immediately from the lemma and $0 < S_{\tau(t)} - t \leq S_{\tau(t)+m} - t$.

To prove the converse implication of (iii), assume that $E(S_{\tau(t)} - t)^{r-1} \leq M$, $0 < t < \infty$. Then

$$r \int_0^T (S_{\tau(t)} - t)^{r-1}\, dt \geq \int_0^{S_{\tau(T)-1}} r(S_{\tau(t)} - t)^{r-1}\, dt$$

$$= \sum_1^{\tau(T)-1} \int_{S_{n-1}}^{S_n} r(S_n - t)^{r-1}\, dt = \sum_1^{\tau(T)-1} X_n^r \geq \sum_1^{\tau(T)-1} (X_n \wedge A)^r.$$

Theorem 1.1, this estimate and Fubini's theorem yield

$$(E\tau(T) + m)E(X \wedge A)^r = E \sum_1^{\tau(T)+m} (X_n \wedge A)^r$$

$$\leq E\left(r \int_0^T (S_{\tau(t)} - t)^{r-1}\, dt + (m + 1)A^r \right) \leq rTM + (m + 1)A^r.$$

We divide this by T and take the limits as first $T \to \infty$ and then $A \to \infty$ to obtain $\mu^{-1}EX^r \leq rM$ and $EX^r \leq rM\mu$. To show that $\mu < \infty$, we assume temporarily that $EX^p < \infty$ where $p \leq 1$. By Theorem 1.1 $(E\tau(T) + m)EX^p = E \sum_1^{\tau+m} X_n^p \geq E(\sum_1^{\tau+m} X_n)^p \geq T^p$, $T > 0$.

Since $E(X \wedge A)^r \geq A^r P(X > A)$ we obtain, choosing $T = A^r$ in the inequalities above,

$$EX^p(rM + m + 1)A^r \geq EX^p(E\tau(A^r) + m)E(X \wedge A)^r \geq A^{rp}A^rP(X > A).$$

Thus $P(X > A) \leq CA^{-rp}$, which implies that $EX^q < \infty$ for $q < rp$. Since $EX^{r-1} < \infty$ by Remarks 2.4 and 2.5, we may iteratively improve the exponent by this argument until we reach $\mu = EX < \infty$. This completes the proof of $EX^r \to \infty$.

The converse implication of (i) follows from this and Theorem 1.1.

REMARK 3.1. If $EX^2 < \infty$, then

$$\int_0^T (S_{\tau(t)+m} - t)\, dt = \sum_1^{\tau(T)-1} (X_n^2/2 + X_n \sum_{i=1}^m X_{n+i}) + \theta(X_{\tau(T)}^2/2 + X_{\tau(T)} \sum_1^m X_{\tau(T)+i})$$

for some θ, $0 \leq \theta \leq 1$. Using $E(X_n^2/2 + \sum_{i=1}^m X_nX_{n+i} \mid \mathscr{F}_{n-m-1}) = \gamma^2/2 + (m + ½)\mu^2$, one obtains

$$E \int_0^T (S_{\tau(t)+m} - t)\, dt = E \sum_1^{\tau(T)+m} (X_n^2/2 + \sum_{i=1}^m X_nX_{n+i}) + o(T)$$

$$= (E\tau(T) + m)(\gamma^2/2 + (½ + m)\mu^2) + o(T) = (\gamma^2/2\mu + (m + ½)\mu)T + o(T).$$

Hence

$$\frac{1}{T}\int_0^T \frac{E\tau(t) - t}{\mu}\, dt = \frac{1}{\mu T}\int_0^T (ES_{\tau(t)+m} - t)\, dt - m \to \frac{\gamma^2}{2\mu^2} + \frac{1}{2}.$$

We conjecture that $E\tau(t) - t/\mu \to \gamma^2/2\mu^2 + ½$ as $t \to \infty$ (with minor modifications when X has a lattice distribution) (this is true for independent variables), but we have not been able to prove this.

REMARK 3.2. Similarly $(1/T) \int_0^T E(S_{\tau(t)} - t)^{r-1}\, dt \to EX^r/r\mu$ as $T \to \infty$. We conjecture that $E(S_{\tau(t)} - t)^{r-1} \to EX^r/r\mu$ (in the non-lattice case).

REMARK 3.3. In the case of independent variables, it is often possible to reduce theorems to the case of positive variables by the introduction of the ladder variables $\{Y_n\}$ defined by $N_0 = 0$, $N_{i+1} = \tau(S_{N_i})$ and $Y_i = S_{N_i} - S_{N_{i-1}}$. This fails for m-dependent variables, at least in its simplest version. The following example shows that, in general, $\{Y_n\}$ is neither stationary nor m-dependent.

EXAMPLE 3.1. Let $\{\xi_n\}_0^\infty$ be i.i.d. r.v. with $P(\xi_n = 0) = P(\xi_n = 1) = ½$. Define $\{X_n\}$ by

$$X_n = \begin{cases} 2 & \text{if } \xi_{n-1} = 0,\ \xi_n = 0 \\ 4 & \text{if } \xi_{n-1} = 0,\ \xi_n = 1 \\ -1 & \text{if } \xi_{n-1} = 1,\ \xi_n = 0 \\ 5 & \text{if } \xi_{n-1} = 1,\ \xi_n = 1. \end{cases}$$

$\{X_n\}$ is 1-dependent.

The ladder variables take the values 1, 2, 3, 4, 5. They constitute a Markov chain with transition matrix

$$P = \begin{Bmatrix} 0 & ½ & 0 & ½ & 0 \\ 0 & ½ & 0 & ½ & 0 \\ ¼ & 0 & ¼ & 0 & ½ \\ ¼ & 0 & ¼ & 0 & ½ \\ ¼ & 0 & ¼ & 0 & ½ \end{Bmatrix}.$$

The initial distribution $\pi_1 = (⅛, ¼, ⅛, ¼, ¼)$ differs from the stationary one $\pi_\infty = (⅙, ⅙, ⅙, ⅙, ⅓)$. Thus, the ladder variables are not identically distributed. Furthermore, if Y_n and Y_{n+m+1} were independent for some $m, n \ge 1$, the rows of P^{m+1} would have to coincide. This would imply that $P^{m+1} = P^\infty$, which is impossible since the entries in P^{m+1} are dyadic rationals, while P^∞ contains ⅙ and ⅓. Hence $\{Y_n\}$ is not m-dependent for any m.

4. Applications. Our first application is due to Carl-Gustav Esseen. For further results, see [5].

EXAMPLE 4.1. Let $\{\xi_n\}_1^\infty$ be independent random variables, uniformly distributed on (0, 1), and set $X_n = I(\xi_{n+1} < \xi_n)$. Then $\{X_n\}$ is 1-dependent. $\tau(k-1) = \inf\{n : S_n = k\}$ equals the position of the kth decrease in the sequence $\{\xi_n\}$. Thus $L_k = \tau(k-1) - \tau(k-2)$ is the length of the kth increasing run. L_k may be interpreted as the length of the kth run of a "very long" random permutation. In this example, $\mu = ½$ and $\gamma^2 = 1/12$ (from $EX^2 = ½$ and $EX_1X_2 = ⅙$). Thus, by Theorems 2.2 and 3.1

$$E \textstyle\sum_1^n L_k = E\tau(n-1) = 2n + O(1)$$

$$\operatorname{Var}(\textstyle\sum_1^n L_k) = \operatorname{Var} \tau(n-1) = 2n/3 + O(n^{1/2}).$$

The following example yields new proofs of some results on how many random letters are required until a given sequence occurs. (See [3], [9] and further references given there.) We only consider the simplest case.

EXAMPLE 4.2. Let $\{\xi_n\}_1^\infty$ be independent, uniformly distributed random letters from an alphabet with N letters. Let $\alpha_1 \cdots \alpha_{m+1}$ be a fixed sequence of $m+1$ letters and define $X_n = I(\xi_n \cdots \xi_{n+m} = \alpha_1 \cdots \alpha_{m+1})$.

Let $L_k = \tau(k-1) - \tau(k-2)$. L_k is the distance from the $(k-1)$th to the kth occurrence of $\alpha_1 \cdots \alpha_{m+1}$. In this case $L_2, L_3, \cdots$ are independent and identically distributed. Thus, Theorem 2.2 implies, with $\varepsilon_k = I(\alpha_1 \cdots \alpha_k = \alpha_{m+2-k} \cdots \alpha_{m+1})$, that $EL_n = 1/\mu = N^{m+1}$, $n \ge 2$

$$\operatorname{Var}(L_n) = \gamma^2/\mu^3 = N^{m+1}(N^{m+1} + s \textstyle\sum_1^m \varepsilon_k N^k - 2m - 1), \qquad n \ge 2.$$

It is easily seen that $EX_{\tau(k)+i} = E(X_{n+i} \mid X_n = 1) = \varepsilon_{m+1-i} N^{-i}$, $i = 0 \cdots m$. Thus

$$ES_{\tau(0)+m} = ES_{\tau(0)} + \textstyle\sum_{i=1}^m EX_{\tau(0)+i} = 1 + \sum_1^m \varepsilon_{m+1-i} N^{-i} = \sum_1^{m+1} \varepsilon_i N^{i-m-1}.$$

By Lemma 1.1 $ES_{\tau(0)+m} = (E\tau(0) + m)N^{-m-1}$ whence $EL_1 = E\tau(0) = \sum_1^{m+1} \varepsilon_i N^i - m$. With some effort one obtains, using Theorem 1.2,

$$\operatorname{Var} L_1 = (\textstyle\sum_1^{m+1} \varepsilon_i N^i)^2 - \sum_1^{m+1} (2i-1)\varepsilon_i N^i.$$

REMARK 4.1. These examples are of the form $X_n = \varphi(\xi_n, \cdots, \xi_{n+m})$, with ξ_n i.i.d.; it is an interesting and apparently open problem whether every stationary m-dependent sequence can be written in this way.

Note added in proof. The lattice variable version of the conjecture in Remark 3.1 is false, see Janson, Runs in m-dependent sequences (to appear).

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