*Written while on leave from the Department of Mathematics, University of Auckland, New Zealand.

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RENEWAL THEORY IN TWO DIMENSIONS: ASYMPTOTIC RESULTS

by

Jeffrey J. Hunter*

Department of Statistics
University of North Carolina at Chapel Hill

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ABSTRACT

In an earlier paper (Renewal theory in two dimensions: Basic results) the author developed a unified theory for the study of bivariate renewal processes. In contrast to this aforementioned work where explicit expression were obtained; we develop some asymptotic results concerning the joint distribution of the bivariate renewal counting process $(N_x^{(1)}, N_y^{(2)})$; the distribution of the two dimensional renewal counting process $N_{x,y}$ and the two dimensional renewal function $EN_{x,y}$. A by-product of the investigation is the study of the distribution and moments of the minimum of two correlated normal random variables. A comprehensive bibliography on multidimensional renewal theory is also appended.

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1. INTRODUCTION

In an earlier paper, [8], the following framework for investigating the properties of renewal processes in two dimensions was established.

Let $\{(X_n,Y_n)\}$, $n = 1,2,\ldots$, be a bivariate renewal process, i.e. a sequence of independent and identically distributed bivariate random variables (r.v.'s) with common distribution function (d.f.), say $F(x,y) = P\{X_1 \le x,\ Y_1 \le y\}$. We shall use the following notation for the moments (when they exist): $EX_1 = \mu_1$, $EY_1 = \mu_2$, $\text{var } X_1 = \sigma_1^2$, $\text{var } Y_1 = \sigma_2^2$, $\text{cov }(X_1,Y_1) = \sigma_{12} = \rho\sigma_1\sigma_2$ where $\rho = \text{corr}(X_1,Y_1)$.

Let $\underset{\sim}{S}_n = (S_n^{(1)}, S_n^{(2)}) = (\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i)$.

Define
$$N_x^{(1)} = \max\{n : S_n^{(1)} \le x\},$$
$$N_y^{(2)} = \max\{n : S_n^{(2)} \le y\},$$
$$N_{x,y} = \max\{n : S_n^{(1)} \le x,\ S_n^{(2)} \le y\}.$$

In [8] the accent was on developing explicit expressions whereas in this paper we concentrate on finding asymptotic results. In particular, by restricting attention to regions of the first quadrant of the plane in proximity to the line expectation we are able to develop an asymptotic distribution for the bivariate renewal counting process $(N_x^{(1)}, N_y^{(2)})$, (Section 2), and an asymptotic distribution for the two dimensional renewal counting process $N_{x,y}$, (Section 3). Furthermore, we present some approximate expressions for the two dimensional renewal function $H(x,y) = EN_{x,y}$ which are an improvement on existing results, (Section 4).

In an appendix we discuss the distribution and moments of the minimum of two correlated normal r.v.'s. These results are utilized in discussing both the distribution and moments of $N_{x,y}$ in Sections 3 and 4. We conclude with a discussion on further research problems related to this area.

The list of references is expanded to give a comprehensive bibliography on multidimensional renewal theory.

2. ASYMPTOTIC DISTRIBUTION OF $(N_x^{(1)}, N_y^{(2)})$.

The asymptotic distribution of the marginal r.v.'s $N_x^{(1)}$ and $N_y^{(2)}$ has been discussed by several researchers in the past (e.g., Feller [18] for recurrent events, Takács [21] for renewal processes) and the following results are well known.

If $\sigma_1^2 < \infty$ then

$$\lim_{x\to\infty} P\{ N_x^{(1)} - x/\mu_1 \le \alpha\, \sigma_1\sqrt{x/\mu_1^3}\ \} = \Phi(\alpha) \tag{2.1}$$

and, similarly, if $\sigma_2^2 < \infty$ then

$$\lim_{y\to\infty} P\{ N_y^{(2)} - y/\mu_2 \le \beta\, \sigma_2\sqrt{y/\mu_2^3}\ \} = \Phi(\beta) \ , \tag{2.2}$$

where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} \exp[-t^2/2]dt$, is the d.f. of a standard normal (0,1) r.v.

In other words, $N_x^{(1)}$ and $N_y^{(2)}$ are both asymptotically normal r.v.'s. It seems natural, therefore, to conjecture that their joint distribution should be asymptotically bivariate normal; i.e.

$$\lim_{\substack{x\to\infty\\ y\to\infty}} P\left\{\frac{N_x^{(1)} - x/\mu_1}{\sigma_1\sqrt{x/\mu_1^3}} \le \alpha,\quad \frac{N_y^{(2)} - y/\mu_2}{\sigma_2\sqrt{y/\mu_2^3}} \le \beta\right\} = \Phi_\omega(\alpha,\beta)$$

where $\Phi_\omega(x,y) = (1/2\pi\sqrt{1-\omega^2} \int_{-\infty}^{x}\int_{-\infty}^{y} \exp[-\{u^2-2\omega uv + v^2\}/2(1-\omega^2)]\quad dv\, du$

is the joint d.f. of a pair of bivariate normal $(0,0;1,1;\omega)$ r.v.'s and ω is the asymptotic correlation between $N_x^{(1)}$ and $N_y^{(2)}$.

However, we must be carful how we define our limiting operation " $x \to \infty$ and $y \to \infty$ " in order that the limiting distribution does not degenerate to a singular distribution. In effect, we must ensure that the correlation between $N_x^{(1)}$ and $N_y^{(2)}$ does not tend to zero (unless, of course, $N_x^{(1)}$ and $N_y^{(2)}$ are independent) but retains its asymptotic behaviour.

The approach taken in proving results (2.1) and (2.2) was to use the Central Limit Theorem. The two dimensional analog is stated below. Its proof can be found in many standard probability texts.

LEMMA 2.1: (Bivariate Central Limit Theorem)

If (X_n, Y_n), $n = 1, 2, \ldots,$ are independent and identically distributed with means (μ_1, μ_2) and covariance matrix

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

then, for fixed x and y,

$$\lim_{n\to\infty} P\left\{\frac{\overline{X}_n - \mu_1}{\sigma_1/\sqrt{n}} \le x,\ \frac{\overline{Y}_n - \mu_2}{\sigma_2/\sqrt{n}} \le y\right\} = \Phi_\rho(x,y) \;.$$

where

$$\overline{X}_n = (1/n)\sum_{i=1}^{n} X_i \quad \text{and} \quad \overline{Y}_n = (1/n)\sum_{i=1}^{n} Y_i \;.$$

Note that in the above lemma, the correlation between $\overline{X}_n$ and $\overline{Y}_n$ is also ρ, the correlation between X_1 and Y_1. We wish to modify the above lemma so that, as m and n both tend to ∞ in some prescribed manner,

$$P\left\{\frac{\overline{X}_m - \mu_1}{\sigma_1/\sqrt{m}} \le x,\ \frac{\overline{Y}_n - \mu_2}{\sigma_2/\sqrt{n}} \le y\right\} \to \Phi_\rho(x,y).$$

Now, since

$$\mathrm{cov}\left(\sum_{i=1}^{m} a_i X_i,\ \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j\ \mathrm{cov}(X_i, Y_j)\ ,$$

we observe that

$$\mathrm{cov}(\overline{X}_m, \overline{Y}_n) = \sum_{i=1}^{m}\sum_{j=1}^{n} (1/mn)\ \mathrm{cov}(X_i, Y_j) = [\min(m,n)/mn]\ \mathrm{cov}(X_1, Y_1)$$

and thus

$$\mathrm{corr}(\overline{X}_m, \overline{Y}_n) = [\min(m,n)/\sqrt{mn}]\rho = \min[\sqrt{m/n},\ \sqrt{n/m}]\rho\ . \tag{2.3}$$

Consider the following possible limiting operations in the plane. We shall write $(m,n) \to \infty$ if m and n both tend to infinity in such a way that $m \sim n$; (i.e. $\lim\limits_{n\to\infty} m(n)/n = 1$). We shall also write $[m,n] \to \infty$ if m and n both tend to infinity but remain a bounded distance apart; (i.e. for some finite B, $|m-n| \le B$).

We shall make use of the limiting operation $(m,n) \to \infty$ which is not as restrictive as the limiting operation $[m,n] \to \infty$. (It is easily seen that $[m,n] \to \infty$ implies $(m,n) \to \infty$ but the converse does not necessarily hod, e.g. $m = n\ (1+1/\sqrt{n})$.)

Mode [10] uses a limiting operation analogous to $[m,n] \to \infty$ in discussing the asymptotic behaviour of a multi-dimensional renewal density.

Note that as $(m,n) \to \infty$ we see from (2.3) that $\mathrm{corr}(\overline{X}_m, \overline{Y}_n) \to \rho$. This result motivates the following lemma.

LEMMA 2.2: If (X_n, Y_n), $n = 1, 2, \ldots$ are independent and identically distributed with means (μ_1, μ_2) and covariance matrix

$$\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

then, for fixed x and y,

$$\lim_{(m,n)\to\infty} P\left\{\frac{\overline{X}_m - \mu_1}{\sigma_1/\sqrt{m}} \le x, \frac{\overline{Y}_n - \mu_2}{\sigma_2/\sqrt{n}} \le y\right\} = \Phi_\rho(x,y) \ .$$

Proof: We use one of the Mann-Wald theorems [19]; viz. that if $\underset{\sim}{U}_n$ converges in law to $\underset{\sim}{U}$ and $\underset{\sim}{V}_n$ converges in probability to $\underset{\sim}{0}$ then $\underset{\sim}{U}_n + \underset{\sim}{V}_n$ converges in law to $\underset{\sim}{U}$. Without loss of generality we assume $m > n$.

Let
$$\underset{\sim}{U}_n = \left(\frac{\overline{X}_n - \mu_1}{\sigma_1/\sqrt{n}}, \frac{\overline{Y}_n - \mu_2}{\sigma_2/\sqrt{n}}\right) ,$$

$$\underset{\sim}{V}_n = \left(\frac{\overline{X}_m - \mu_1}{\sigma_1/\sqrt{m}} - \frac{\overline{X}_n - \mu}{\sigma_2/\sqrt{n}} , 0\right) \equiv (V_{m,n}, 0),$$

then
$$\underset{\sim}{U}_n + \underset{\sim}{V}_n = \left(\frac{\overline{X}_m - \mu_1}{\sigma_1/\sqrt{m}} , \frac{\overline{Y}_n - \mu_2}{\sigma_2/\sqrt{n}}\right) .$$

The result will follow from Lemma 2.1 if we can show $\underset{\sim}{V}_n \to \underset{\sim}{0}$ in probability, or equivalently for all $\varepsilon > 0$.

$$\lim_{(m,n)\to\infty} P\{|V_{m,n}| \ge \varepsilon\} = 0 \ .$$

Now $EV_{m,n} = 0$ and

$$\mathrm{Var}(V_{m,n}) = (1/\sigma_1^2)\ \mathrm{var}[(1/\sqrt{m}) \sum_{i=1}^{m} X_i - (1/\sqrt{n} \sum_{i=1}^{n} X_i] = (1/\sigma_1^2)\ \mathrm{var}\left(\sum_{i=1}^{n} Z_i\right)$$

where
$$Z_i = \begin{cases} (1/\sqrt{m} - 1/\sqrt{n})\ X_i , & i = 1,\dots, n, \\ (1/\sqrt{m})\ X_i & , i = n+1,\dots, m \ . \end{cases}$$

Thus $\quad \operatorname{var}(V_{m,n}) = \sigma_1^{-2}\left\{\sum_{i=1}^{n} (1/\sqrt{m} - 1/\sqrt{n})^2 \sigma_1^2 + \sum_{i=n+1}^{m} (1/\sqrt{m})^2 \sigma_1^2\right\}$

$$= 2\{1-\sqrt{n/m}\} .$$

Since $EV_{m,n} = 0$, we can use Chebyshev's inequality to deduce that

$$P\{|V_{m,n}| \geq \varepsilon\} \leq \varepsilon^{-2} \operatorname{var}(V_{m,n}) = 2\varepsilon^{-2}(1-\sqrt{n/m}) \to 0 \quad \text{as} \quad (m,n) \to \infty. \quad \square .$$

We make use of the following lemma in the proof of Theorem 2.4.

LEMMA 2.3: If $\lambda_x^{(1)}, \lambda_y^{(2)}, \nu_x^{(1)}, \nu_y^{(2)}$ are functions such that $\lambda_x^{(1)} \to 1$, $\lambda_y^{(2)} \to 1$, $\nu_x^{(1)} \to 0$, $\nu_y^{(2)} \to 0$ as $(x,y) \to \infty$ and if $(Z_x^{(1)}, Z_y^{(2)})$ is a family of random vectors such that as $(x,y) \to \infty$,

$$P\{\lambda_x^{(1)} Z_x^{(1)} + \nu_x^{(1)} \leq \alpha, \lambda_y^{(2)} Z_y^{(2)} + \nu_y^{(2)} \leq \beta\} \to \Phi_\rho(\alpha,\beta)$$

for each fixed α and β, then

$$\lim_{(x,y)\to\infty} P\{Z_x^{(1)} \leq \alpha, Z_y^{(2)} \leq \beta\} = \Phi_\rho(\alpha,\beta) . \tag{2.4}$$

<u>Proof</u>: Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be arbitrarily chosen. Then for x and y both sufficiently large

$$\lambda_x^{(1)} \alpha + \nu_x^{(1)} \leq \alpha + \varepsilon_1 \quad \text{and} \quad \lambda_y^{(2)} \beta + \nu_y^{(2)} \leq \beta + \varepsilon_2 .$$

Thus $\quad P\{Z_x^{(1)} \leq \alpha, Z_y^{(2)} \leq \beta\}$

$$= P\{\lambda_x^{(1)} Z_x^{(1)} + \nu_x^{(1)} \leq \lambda_x^{(1)} \alpha + \nu_x^{(1)}, \lambda_y^{(2)} Z_y^{(2)} + \nu_y^{(2)} \leq \lambda_y^{(2)} \beta + \nu_y^{(2)}\}$$

$$\leq P\{\lambda_x^{(1)} Z_x^{(1)} + \nu_x^{(1)} \leq \alpha + \varepsilon_1, \lambda_y^{(2)} Z_y^{(2)} + \nu_y^{(2)} \leq \beta + \varepsilon_2\} .$$

Therefore, $\limsup_{(x,y)\to\infty} P\{Z_x^{(1)} \leq \alpha, Z_y^{(2)} \leq \beta\} \leq \Phi_\rho(\alpha+\varepsilon_1, \beta+\varepsilon_2)$.

But both $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are arbitrary and $\Phi_\rho(\cdot,\cdot)$ is continuous and thus

$$\limsup_{(x,y)\to\infty} P\{Z_x^{(1)} \le \alpha,\ Z_y^{(2)} \le \beta\} \le \Phi_\rho(\alpha,\beta) .$$

Similarly

$$\liminf_{(x,y)\to\infty} P\{Z_x^{(1)} \le \alpha,\ Z_y^{(2)} \le \beta\} \ge \Phi_\rho(\alpha,\beta)$$

and (2.4) follows. □

It should be remarked that the result also holds if $\Phi_\rho(\alpha,\beta)$ is replaced by any continuous joint d.f.

Furthermore, the result also holds if the continuous variables x and y are replaced by positive integers m and n with $(m,n)\to\infty$.

We are now in the position to be able to state and prove the main theorem of this section.

THEOREM 2.4: If α and β are any two fixed real numbers then, provided both $\sigma_1^2 < \infty$ and $\sigma_2^2 < \infty$,

$$(2.5)\qquad \lim_{(x/\mu_1,y/\mu_2)\to\infty} P\left\{\frac{N_x^{(1)} - x/\mu_1}{\sigma_1\sqrt{x/\mu_1^3}} \le \alpha,\ \frac{N_y^{(2)} - y/\mu_2}{\sigma_2\sqrt{y/\mu_2^3}} \le \beta\right\} = \Phi_\rho(\alpha,\beta) .$$

<u>Proof:</u>[1] Let α and β be given. For any α (positive or negative) the function $x\mu_1 + \alpha\sigma_1\sqrt{x}$ is increasing in x for all sufficiently large x. Similarly,

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[1]The proofs of both Lemma 2.3 and Theorem 2.4 are generalizations of the analogous univariate results as proved by W.L. Smith in unpublished lecture notes on Stochastic Processes presented in the Department of Statistics, University of North Carolina at Chapel Hill. See also [12, p. 254].

$y\mu_2 + \beta\sigma_2\sqrt{y}$ is ultimately an increasing function of y.

Thus, for large x and y, we can define two positive, single valued, functions $n_x^{(1)}$ and $n_y^{(2)}$ by the equations

$$\mu_1 n_x^{(1)} + \alpha\sigma_1\sqrt{n_x^{(1)}} = x \ , \tag{2.6}$$

$$\mu_2 n_y^{(2)} + \beta\sigma_2\sqrt{n_y^{(2)}} = y \ . \tag{2.7}$$

We first show that $(x/\mu_1, y/\mu_2) \to \infty$ iff $(n_x^{(1)}, n_y^{(2)}) \to \infty$. From (2.6) and (2.7) it is obvious that $x/\mu_1 \to \infty$ iff $n_x^{(1)} \to \infty$ and that $y/\mu_2 \to \infty$ iff $n_y^{(2)} \to \infty$. It is easily seen that $n_x^{(1)} + \alpha\sigma_1\sqrt{n_x^{(1)}}\ \mu_1 \sim n_x^{(1)}$ and that $n_y^{(2)} + \beta\sigma_2\sqrt{n_y^{(2)}}\ \mu_2 \sim n_y^{(2)}$ and thus from (2.6) and (2.7) we see that $x/\mu_1 \sim n_x^{(1)}$ and $y/\mu_2 \sim n_y^{(2)}$. From this result we can deduce that $x/\mu_1 \sim y/\mu_2$ iff $n_x^{(1)} \sim n_y^{(2)}$.

Let $[n_x^{(1)}]$ denote the integer part of $n_x^{(1)}$ and $\{n_x^{(1)}\} = [n_x^{(1)}] + 1$. Similarly for $[n_y^{(2)}]$ and $\{n_y^{(2)}\}$.

Suppose $(x/\mu_1, y/\mu_2) \to \infty$, or equivalently $(\{n_x^{(1)}\}, \{n_y^{(2)}\}) \to \infty$, then Lemma 2.2 gives

$$P\left\{\frac{S^{(1)}_{\{n_x^{(1)}\}} - \mu_1\{n_x^{(1)}\}}{\sigma_1\sqrt{\{n_x^{(1)}\}}} \le \alpha,\ \frac{S^{(2)}_{\{n_y^{(2)}\}} - \mu_2\{n_y^{(2)}\}}{\sigma_2\sqrt{\{n_y^{(2)}\}}} \le \beta\right\} \to \Phi_\rho(\alpha,\beta) \ .$$

Rearrangement of this expression gives

$$P\left\{\left(\frac{S^{(1)}_{\{n_x^{(1)}\}} - \mu_1 n_x^{(1)}}{\sigma_1\sqrt{n_x^{(1)}}}\right)\sqrt{\frac{n_x^{(1)}}{\{n_x^{(1)}\}}} - \left(\frac{\mu_1\{n_x^{(1)}\} - \mu_1 n_x^{(1)}}{\sigma_1\sqrt{\{n_x^{(1)}\}}}\right) \le \alpha \ ,\right.$$
$$\left.\left(\frac{S^{(2)}_{\{n_y^{(2)}\}} - \mu_2 n_y^{(2)}}{\sigma_2\sqrt{n_y^{(2)}}}\right)\sqrt{\frac{n_y^{(2)}}{\{n_y^{(2)}\}}} - \left(\frac{\mu_2\{n_y^{(2)}\} - \mu_2 n_y^{(2)}}{\sigma_2\sqrt{\{n_y^{(2)}\}}}\right) \le \beta\right\}$$

$$\to \Phi_\rho(\alpha,\beta) \quad \text{as} \quad (n_x^{(1)}, n_y^{(2)}) \to \infty \ .$$

Application of Lemma 2.3 with $\lambda_x^{(i)} = \sqrt{n_x^{(i)}/\{n_x^{(i)}\}}$, which tends to 1, and $\nu_x^{(i)} = \mu_i(\{n_x^{(i)}\}-n_x^{(i)})/\sigma_i\sqrt{\{n_x^{(i)}\}}$, which tends to 0 as $(n_x^{(1)},n_y^{(2)}) \to \infty$; gives

$$P\left\{S^{(1)}_{\{n_x^{(1)}\}} \le \mu_1 n_x^{(1)} + \alpha\sigma_1\sqrt{n_x^{(1)}},\ S^{(2)}_{\{n_y^{(2)}\}} \le \mu_2 n_y^{(2)} + \beta\sigma_2\sqrt{n_y^{(2)}}\right\}$$

$$\to \Phi_\rho(\alpha,\beta), \quad \text{as} \quad (n_x^{(1)},n_y^{(2)}) \to \infty .$$

Thus by (2.6) and (2.7) we deduce that

$$P\left\{S^{(1)}_{\{n_x^{(1)}\}} \le x,\ S^{(2)}_{\{n_y^{(2)}\}} \le y\right\} \to \Phi_\rho(\alpha,\beta) . \tag{2.8}$$

Now, by the definition of $N_x^{(1)}$ and $N_y^{(2)}$, we have that

$$P\left\{N_x^{(1)} \ge m,\ N_y^{(2)} \ge n\right\} = P\left\{S_m^{(1)} \le x,\ S_n^{(2)} \le y\right\} . \tag{2.9}$$

Thus, as $(x/\mu_1,y/\mu_2) \to \infty$ (and equivalently $(n_x^{(1)},n_y^{(2)}) \to \infty$) we deduce from (2.8 and (2.9) that

$$P\left\{N_x^{(1)} \ge \{n_x^{(1)}\},\ N_y^{(2)} \ge \{n_y^{(2)}\}\right\} \to \Phi_\rho(\alpha,\beta) ,$$

and consequently

$$P\left\{N_x^{(1)} > n_x^{(1)},\ N_y^{(2)} > n_y^{(2)}\right\} \to \Phi_\rho(\alpha,\beta) ; \tag{2.10}$$

since $\quad \left[N_x^{(1)} \ge \{n_x^{(1)}\}\right] \equiv \left[N_x^{(1)} > n_x^{(1)}\right]$,

and $\quad \left[N_y^{(2)} \ge \{n_y^{(2)}\}\right] \equiv \left[N_y^{(2)} > n_y^{(2)}\right]$.

If we now substitute in (2.10) the expressions for $n_x^{(1)}$ and $n_y^{(2)}$ given by (2.6) and (2.7) we obtain

$$P\left\{\frac{N_x^{(1)} - x/\mu_1}{\sigma_1\sqrt{x/\mu_1^3}}\left(\frac{\sqrt{x/\mu_1}}{\sqrt{n_x^{(1)}}}\right) > -\alpha,\ \frac{N_y^{(2)} - y/\mu_2}{\sigma_2\sqrt{y/\mu_2^2}}\left(\frac{\sqrt{y/\mu_2}}{\sqrt{n_y^{(2)}}}\right) > -\beta,\right\} \to \Phi_\rho(\alpha,\beta) .$$

An additional appeal to Lemma 2.3 with $\lambda_x^{(i)} = \sqrt{x/\mu_i}\ /\ \sqrt{n_x^{(i)}} \to 1$ (by (2.6) and (2.7)) and $\nu_x^{(i)} = 0$, gives

$$\lim_{(x/\mu_1,y/\mu_2)\to\infty} P\left\{\frac{N_x^{(1)} - x/\mu_1}{\sigma_1\sqrt{x/\mu_1^3}} > -\alpha,\ \frac{N_y^{(2)} - y/\mu_2}{\sigma_2\sqrt{y/\mu_2^3}} > -\beta\right\} = \Phi_\rho(\alpha,\beta) .$$

The final result (2.5) now follows upon noting that $\Phi_\rho(\alpha,\beta) = P\{Z_1\le\alpha, Z_2\le\beta\}$ where (Z_1,Z_2) is bivariate normal $(0,0;1,1;\rho)$. But $(-Z_.,-Z_2)$ is also bivariate normal $(0,0;1,1;\rho)$ and thus $\Phi_\rho(\alpha,\beta) = P\{-Z,\le\alpha,-Z_2\le\beta\} = P\{Z_1\ge-\alpha,Z_2\ge-\beta\}$ □.

As a corollarly to Theorem 2.4 we take $x/\mu_1 = y/\mu_2 = z$ and deduce that along the line of expectation $N_x^{(1)}$ and $N_y^{(2)}$ is asymptotically bivariate normal, viz.

COROLLARLY 2.4.1: For fixed α and β, $\sigma_1^2 < \infty$ and $\sigma_2^2 < \infty$,

$$\lim_{z\to\infty} P\left\{N_{\mu_1 z}^{(1)} - z \le \alpha\sigma_1\sqrt{z}/\mu_1,\ N_{\mu_2 z}^{(2)} - z \le \beta\sigma_2\sqrt{z}/\mu_2\right\} = \Phi_\rho(\alpha,\beta) .$$

3. ASYMPTOTIC DISTRIBUTION OF $N_{x,y}$.

In an earlier paper ([8]) we showed that

$$(3.1) \qquad N_{x,y} = \min(N_x^{(1)}, N_y^{(2)})$$

and that

$$(3.2) \qquad P\left\{N_{x,y} \geq n\right\} = P\left\{N_x^{(1)} \geq n, N_y^{(2)} \geq n\right\},$$

$$(3.3) \qquad = P\left\{S_n^{(1)} \leq x, S_n^{(2)} \leq y\right\}.$$

It appears that we have two approaches to finding an asymptotic distribution for $N_{x,y}$; one using (3.2) and proceeding via the results of Section2 on the asymptotic distribution of $(N_x^{(1)}, N_y^{(2)})$ and one using (3.3) and the Bivariate Central Limit Theorem. Both of these approaches lead effectively to the same result when we restrict attention to limits along the line of expectation. In fact, in order that we obtain a satisfactory proof we need to make this restriction and use the second approach suggested above.

From Lemma 2.1 (the Bivariate Central Limit Theorem), for fixed α and β we have

$$(3.4) \qquad \lim_{n\to\infty} P\left\{S_n^{(1)} \leq n\mu_1 + \alpha\sigma_1\sqrt{n},\ S_n^{(2)} \leq n\mu_2 + \beta\sigma_2\sqrt{n}\right\} = \Phi_\rho(\alpha,\beta) .$$

Thus we need to define $n (= n_{x,y})$ such that

$$(3.5) \qquad n\mu_1 + \alpha\sigma_1\sqrt{n} = x$$

$$(3.6) \qquad n\mu_2 + \beta\sigma_2\sqrt{n} = y$$

Solving equations (3.5) and (3.6) simultaneously for n we see that

$$(3.7) \qquad n = (x/\mu_1 - y/\mu_2)^2 / (\alpha\sigma_1/\mu_1 - \beta\sigma_2/\mu_2)^2$$

provided $\alpha\sigma_1/\mu_1 \neq \beta\sigma_2/\mu_2$.

To use (3.4) we require $n \to \infty$. But if $[x/\mu_1, y/\mu_2] \to \infty$ then $|x/\mu_1 - y/\mu_2| \le B$ together with the condition $\alpha\sigma_1/\mu_1 \ne \beta\sigma_2/\mu_2$ implies, from (3.7) that n is bounded and hence cannot become arbitrarily large. Thus, if we use (3.4), (3.5) and (3.6) with x and y in close proximity to the line of expectation, we require $\alpha\sigma_1/\mu_1 = \beta\sigma_2/\mu_2$. This condition is, in fact, equivalent to $x/\mu_1 = y/\mu_2$. [Since, if $x/\mu_1 = y/\mu_2 = z$ then (3.5) and (3.6) become, respectively $n + \alpha\sigma_1\sqrt{n}/\mu_1 = z$ and $n + \beta\sigma_2\sqrt{n}/\mu_2 = z$ which imply $(\alpha\sigma_1/\mu_1 - \beta\sigma_2/\mu_2)\sqrt{n} = 0$, which in turn implies $\alpha\sigma_1/\mu_1 = \beta\sigma_2/\mu_2$ since $n \sim z$. Conversely, $\alpha\sigma_1/\mu_1 = \beta\sigma_2/\mu_2$ implies $x/\mu_1 = y/\mu_2$.]

As a consequence of the above observations, we state and prove the following theorem concerning the asymptotic distribution of $N_{x,y}$ for x and y constrained to lie on the line of expectation.

THEOREM 3.1: If $\sigma_1^2 < \infty$ and $\sigma_2^2 < \infty$, then for any fixed c,

$$\lim_{z\to\infty} P\left\{\frac{N_{\mu_1 z,\mu_2 z} - z}{\sqrt{z}} \le c\right\} = 1 - \Phi_\rho\left(\frac{-\mu_1 c}{\sigma_1}, \frac{-\mu_2 c}{\sigma_2}\right) . \tag{3.8}$$

Proof: Let c (positive or negative) be given and define $\alpha = \mu_1 c/\sigma_1$ and $\beta = \mu_2 c/\sigma_2$. For sufficiently large z we can define n_z, a positive single valued function, by the equation

$$n_z + c\sqrt{n_z} = z . \tag{3.9}$$

Observe that $n_z \sim z$, and $n_z \to \infty$ iff $z \to \infty$. Define $\{n_z\} = [n_z] + 1$. From (3.4) we may write

$$\lim_{n_z\to\infty} P\left\{S^{(1)}_{\{n_z\}} \le \{n_z\}\mu_1 + \mu_1 c\sqrt{\{n_z\}},\ S^{(2)}_{\{n_z\}} \le \{n_z\}\mu_2 + \mu_2 c\sqrt{\{n_z\}}\right\} = \Phi_\rho(\mu_1 c/\sigma_1, \mu_2 c/\sigma_2) .$$

We can now rearrange this above expression, apply Lemma 2.3, and use equation (3.9), (analogous to the technique used in the proof of Theorem 2.4) to obtain

$$\lim_{n_z\to\infty} P\left\{S^{(1)}_{\{n_z\}} \le \mu_1 z,\ S^{(2)}_{\{n_z\}} \le \mu_2 z\right\} = \Phi_\rho(\mu_1 c/\sigma_1, \mu_2 c/\sigma_2) .$$

From equation (3.3) we deduce that

$$\lim_{z\to\infty} P\left\{N_{\mu_1 z,\mu_2 z} \ge \{n_z\}\right\} = \Phi_\rho(\mu_1 c/\sigma_1, \mu_2 c/\sigma_2)$$

Now $P\left\{N_{\mu_1 z,\mu_2 z} \ge \{n_z\}\right\} = P\left\{N_{\mu_1 z,\mu_2 z} > n_z\right\} = P\left\{\frac{N_{\mu_1 z,\mu_2 z} - z}{\sqrt{z}}\left(\frac{\sqrt{z}}{\sqrt{n_z}}\right) > -c\right\}$

and using a univariate version of Lemma 2.3 we may deduce that, since (3.9) implies $\sqrt{z/n_z} \to 1$,

$$\lim_{z\to\infty} P\left\{\frac{N_{\mu_1 z,\mu_2 z} - z}{\sqrt{z}} > -c\right\} = \Phi_\rho(\mu_1 c/\sigma_1, \mu_2 c/\sigma_2)$$

Equation (3.8) follows by replacing c by $-c$ and taking the complementary event. □

The limit distribution given by (3.8) seems to be a little unusual in that we have come to expect normal limit laws. However, from Theorem A.1 of the appendix, we can conclude that the limit distribution arises as the distribution of the minimum of two bivariate normal $(0,0;\ \sigma_1^2/\mu_1^2, \sigma_2^2/\mu_2^2; \rho)$ r.v.'s.

This serves as a means of reconciling the two approaches mentioned at the beginning of this section. Let

$$Z_x^{(1)} = (N_x^{(1)} - x/\mu_1)/\sigma_1\sqrt{x/\mu_1^3} , \tag{3.10}$$

$$Z_y^{(2)} = (N_y^{(2)} - y/\mu_2)/\sigma_2\sqrt{y/\mu_2^3} . \tag{3.11}$$

Using (3.10) and (3.11) in conjunction with (3.1) we can deduce that

$$(N_{\mu_1 z,\mu_2 z} - z)/\sqrt{z} = \min[\sigma_1 Z^{(1)}_{\mu_1 z}/\mu_1, \sigma_2 Z^{(2)}_{\mu_2 z}/\mu_2] . \tag{3.12}$$

Cor 2.4.1 implies that $(Z^{(1)}_{\mu_1 z}, Z^{(2)}_{\mu_2 z})$ is asymptotically bivariate normal $(0,0;1,1;\rho)$. Thus from (3.12) we can infer that $(N_{\mu_1 z, \mu_2 z} - z)/\sqrt{z}$ is distributed as the minimum of two bivariate normal $(0,0;\sigma_1^2/\mu_1^2, \sigma_2^2/\mu_2^2,\rho)$ r.v.'s in accordance with our observation above.

More generally, Theorem 2.4 implies that, as $(x/\mu_1, y/\mu_2) \to \infty$, $(N_x^{(1)}, N_y^{(2)})$ is asymptotically bivariate normal $(x/\mu_1, y/\mu_2;\ \sigma_1^2 x/\mu_1^2, \sigma_2^2 y/\mu_2^2;\rho)$. Hence, by virtue of (3.1) and Theorem A.1, we would expect to conclude that

$$P\left\{N_{x,y} \le c\right\} \sim 1 - \Phi_\rho\left(\frac{x/\mu_1 - c}{\sigma_1\sqrt{x/\mu_1^3}}, \frac{y/\mu_2 - c}{\sigma_2\sqrt{y/\mu_2^3}}\right),$$

as $(x/\mu_1, y/\mu_2) \to \infty$. This result is of limited use in that in order that we obtain any useful approximations we require c to be of the same order as x/μ_1 and y/μ_2. Theorem 3.1 provides us with a more satisfactory result.

The only work related to limit distributions of the type considered in this section is that of Farrell [5], [7] where he considers a class of stopping variables for multidimensional random walks. However his results cannot be applied to finding the asymptotic distribution of $N_{x,y}$. In terms of our notation he defines $g_n = g\,(S_n^{(1)}, S_n^{(2)})$ which is assumed to be a homogeneous function of degree one in the two variables. Given $t > 0$, let $M(t)$ be the least integer n such that $g_n > t$ with $M(t) = \infty$ if for all $n \ge 1$, $g_n \le t$. Under the assumption that g has continuous first partial derivatives in the open first quadrant Farrell, [7], shows

$$\lim_{t\to\infty} P\left[\{M(t) - t/g(\underset{\sim}{\mu})\}t^{1/2} \le \sigma t\{g(\underset{\sim}{\mu})\}^{-3/2}\right] = \Phi(t), \tag{3.13}$$

where $\sigma^2 = \underset{\sim}{\alpha}' \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \underset{\sim}{\alpha}$ and $\underset{\sim}{\alpha}$ is the column vector of first partial derivatives of g evaluated at $\underset{\sim}{\mu}' = (\mu_1, \mu_2)$.

If we define $g(x,y) = \max(x/\mu_1, y/\mu_2)$, then $M(t) = N_{\mu_1 t, \mu_2 t} + 1$ and we would expect a normal limit law for $N_{\mu_1 t, \mu_2 t}$. However, it is easily seen that, for this particular choice of g, (3.13) does not hold since the first partial derivatives of g are discontinuous along the line of expectation $x/\mu_1 = y/\mu_2$.

4. THE TWO DIMENSIONAL RENEWAL FUNCTION

In this section we investigate the determination of asymptotic expressions for $H(x,y) = EN_{x,y}$.

In univariate renewal theory, the renewal function $H_1(x) = EN_x^{(1)}$ has been well studied, (see Smith [12]). The simplest result is the so called "elementary renewal theorem":

$$\lim_{t\to\infty} H_1(t)/t = \mu_1^{-1} , \tag{4.1}$$

where $\mu_1 = EX_i \le \infty$, with the limit μ_1^{-1} being interpreted as zero if $\mu_1 = \infty$.

Several authors have considered generalizations of (4.1) to more than one dimension. In Section 3 we mentioned the studies by Farrell of the asymptotic behavior of some multidimensional random walks. One of the results obtained in his paper [5] is that, if $g(x,y) = \min(x,y)$ then $\lim_{t\to\infty} EM(t)/t = 1/\min(\mu_1,\mu_2)$. This result is related to the "elementary renewal theorem for the plane" given by Bickel and Yahav [1]: Under the assumption of non negative, non arithmetic r.v.'s with finite means,

$$\lim_{t\to\infty} H(t,t)/t = 1/\max(\mu_1,\mu_2) , \tag{4.2}$$

where the limit $= \infty$ if both $\mu_1 = \mu_2 = 0$. (This is actually a special case of their theorem, using the L_∞ norm.)

In this paper we have found it natural to consider limits in the plane along the line of expectation. The following theorem, which we derive from (4.2), provides a result in this direction.

THEOREM 4.1:

If the bivariate renewal process is non arithmetic and $0 < \mu_1 < \infty$, $0 < \mu_2 < \infty$, then

$$\lim_{t\to\infty} H(\mu_1 t\ ,\ \mu_2 t)/t = 1\ . \tag{4.3}$$

Proof: From the bivariate renewal process $\{(X_n,Y_n)\}$ form a new renewal process $\{(U_n,V_n)\}$ where $U_n = X_n/\mu_1$, and $V_n = Y_n/\mu_2$. Let $T_n^{(1)} = \sum_{i=1}^{n} U_i = S_n^{(1)}/\mu_1$ and $T_n^{(2)} = \sum_{i=1}^{n} V_i = S_n^{(2)}/\mu_2$.

Since $EU_n = EV_n = 1$ we have from (4.2) $\lim_{t\to\infty} EN'_{t,t}/t = 1$, where

$$N'_{t,t} = \max\{n : T_n^{(1)} \le t,\ T_n^{(2)} \le t\}$$

$$= \max\{n : S_n^{(1)} \le \mu_1 t,\ S_n^{(2)} \le \mu_2 t\}$$

$= N_{\mu_1 t,\mu_2 t}$, and the result follows. □

In the univariate case, if one makes the additional assumption that $EX_i < \infty$ even more specific information can be obtained about $H_1(t)$, viz..

$$H_1(t) = t/\mu_1 + (\sigma_1^2 - \mu_1^2)/2\mu_1^2 + o(1)\ . \tag{4.4}$$

Both of the results (4.1) and (4.4) are satisfied for all t, without any limiting operation, when the underlying renewal process is sampled from an exponential (μ_1) distribution with p.d.f. given by $f_1(x) = (1/\mu_1)\exp(-x/\mu_1)$; $x > 0$.

Prepatory to giving a general discussion of the two dimensional renewal function we obtain explicit expressions for H(x,y) when the bivariate renewal process is sampled from a bivariate exponential distribution. The results we obtain will be special cases of any general asymptotic expression, hopefully with some of the properties satisfied in the univariate case.

Let (X_1, Y_1) have a bivariate exponential distribution with j.p.d.f. given by

$$f(x,y) = f_1(x)\ f_2(y) \sum_{n=0}^{\infty} \rho^n g_n^{(1)}(x) g_n^{(2)}(y)\ ; \quad x \geq 0,\ y \geq 0\ ; \tag{4.5}$$

where $f_1(x)$ and $f_2(y)$ are the marginal p.d.f.'s of X_1 and Y_1 which are both exponential distributions with means μ_1 and μ_2 respectively;

$$g_n^{(1)}(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j!} \left(\frac{x}{\mu_1}\right)^j, \quad g_n^{(2)}(y) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j!} \left(\frac{y}{\mu_2}\right)^j ,$$

and $\rho = \text{corr}(X_1, Y_1)$, $(0 \leq \rho < 1)$. (Other alternative forms for $f(x,y)$ are also given in [8].)

When $\rho = 0$, $f(x,y) = f_1(x) f_2(y)$, and (X_1, Y_1) are two independent exponential distributions.

Let $H_\rho(x,y)$ be the two dimensional renewal function for the bivariate renewal process $\{(X_n, Y_n)\}$ with (X_1, Y_1) having the j.p.d.f. given by (4.5).

LEMMA 4.2:

$$H_\rho(x,y) = (1-\rho)\ H_0(x/(1-\rho),\ y/(1-\rho))\ . \tag{4.6}$$

Proof: In [8] we showed that

$$H_0(x,y) = \sum_{k=1}^{\infty} P(k, x/\mu_1)\ P(k, y/\mu_2)$$

and

$$H_\rho(x,y) = (1-\rho) \sum_{k=1}^{\infty} P(k, x/(1-\rho)\mu_1)\ P(k, y/(1-\rho)\mu_2)$$

where $P(k,x)$ is the incomplete gamma function which can be expressed as $\sum_{r=k}^{\infty} e^{-x} x^r/r!$. (4.6) follows from these two results. □

For any two dimensional renewal function we have from (3.1)

$$E\,N_{x,y} = E\min(N_x^{(1)}, N_y^{(2)}) \,,$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \min(i,j)p_{i,j}(x,y) \,,$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \{\tfrac{1}{2}(i+j) - \tfrac{1}{2}|i-j|\}p_{i,j}(x,y) \,,$$

$$= \tfrac{1}{2}E\,N_x^{(1)} + \tfrac{1}{2}E\,N_y^{(2)} - \tfrac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |i-j|p_{i,j}(x,y) \,, \tag{4.7}$$

where $p_{i,j}(x,y) = P\{N_x^{(1)} = i,\ N_y^{(2)} = j\}$.

This result enables us to establish the following lemma:

LEMMA 4.3: For $0 \le \rho < 1$, $t > 0$,

$$H_\rho(\mu_1 t,\mu_2 t) = t - t\exp\{-2t/(1-\rho)\}[I_0(2t/(1-\rho)) + I_1(2t/(1-\rho))] \tag{4.8}$$

where $I_n(x)$ is the modified Bessel function of the first kind of the n-th order. Furthermore, as $t \to \infty$,

$$H_\rho(\mu_1 t,\mu_2 t) = t - (t(1-\rho)/\pi)^{1/2} + o(t^{1/2}), \quad \text{as} \quad t \to \infty. \tag{4.9}$$

Proof: From Lemma (4.2) it is sufficient to determine only $H_0(\mu_1 t,\mu_2 t)$. For the independent exponential case

$$p_{ij}(x,y) = \exp[-\frac{x}{\mu_1} - \frac{y}{\mu_2}](\frac{x}{\mu_1})^i(\frac{y}{\mu_2})^j / i!\, j! \,,$$

$$E\,N_x^{(1)} = x/\mu_1 \,, \quad \text{and} \quad E\,N_y^{(2)} = y/\mu_2 \,;$$

thus if we let $x/\mu_1 = y/\mu_2 = t$ we obtain from (4.7)

$$H_0(\mu_1 t,\mu_2 t) = t - \Delta_1/2 \,, \tag{4.10}$$

where

$$\Delta_1 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |i-j|p_i\, p_j \,,$$

with

$$p_i = e^{-t}\, t^i/i! \,.$$

Ramasubban [20] shows that Δ_1, the "mean difference" for the Poisson distribution, with p_i as above, can be expressed as

$$\Delta_1 = 2t\, e^{-2t}\{I_0(2t) + I_1(2t)\} \ . \tag{4.11}$$

From (4.10) and (4.11) we obtain an expression for $H_0(\mu_1 t,\mu_2 t)$ from which, upon replacing t by $t/(1-\rho)$ and using (4.6), we deduce (4.8).

The asymptotic expression for $H_\rho(\mu_1 t,\mu_2 t)$, (4.9), follows upon using the asymptotic expansion, $I_n(z) \sim e^z/\sqrt{2\pi z}$ as $z \to \infty$ (n fixed). $\square$

Let us now consider the general two dimensional function $H(x,y)$. The proof of Theorem 4.1 required the finiteness of the first order moments of (X_1,Y_1). If we are willing to assume the existence of higher order moments it should be possible to obtain more precise estimates, especially along the line of expectation, than those given by Theorem 4.1. To this end we have been able to obtain some approximate formulae that give an indication as to the asymptotic behavior.

Our approximation is based upon the premise that convergence in distribution implies convergence of the moments - a result that is not in general true without "uniform integrability" conditions.

We have seen (Theorem 3.1 and Theorem A.1) that $(N_{\mu_1 t,\mu_2 t} - t)/\sqrt{t}$ is distributed asymptotically as the minimum of two bivariate normal $(0,0;\ \sigma_1^2/\mu_1^2,\ \sigma_2^2/\mu_2^2;\rho)$ r.v.'s. Thus we would expect that $\lim_{t\to\infty} E(N_{\mu_1 t,\mu_2 t} - t)/\sqrt{t}$ would be given by (A.9) with $\theta = 0$ and σ_i replaced by σ_i/μ_i $(i=1,2)$. This leads to the approximation, as $t \to \infty$,

$$H(\mu_1 t,\mu_2 t) - t - D(t/2\pi)^{1/2} + o(t^{1/2}) \ . \tag{4.12}$$

where $D = [(\sigma_1/\mu_1)^2 + (\sigma_2/\mu_2)^2 - 2\rho(\sigma_1\sigma_2/\mu_1\mu_2)]^{\frac{1}{2}}$.

Thus, under the assumption of the existence of second order moments for (X_1, Y_1), the result given by (4.12) is an improvement over (4.3).

Note that when the (X_1, Y_1) are sampled from the bivariate exponential distribution (with j.p.d.f given by (4.5)). $\sigma_1 = \mu_1$ and $\sigma_2 = \mu_2$ and thus $D = \sqrt{2(1-\rho)}$ and the expansion given by (4.12) is the same as that given by (4.9).

As a slight generalization we can obtain an approximation to $E\, N_{x,y}$ as $(x/\mu_1,\ y/\mu_2) \to \infty$. Now $N_{x,y} = \min(N_x^{(1)},\ N_y^{(2)})$ where, from Theorem 3.4 under the stated limiting operations, $(N_x^{(1)},\ N_y^{(2)})$ is asymptotically bivariate normal $(x/\mu_1,\ y/\mu_2;\ \sigma_1^2 x/\mu_1^3,\ \sigma_2^2 y/\mu_2^3;\rho)$. Thus from (A.7) we would expect

$$(4.13)\qquad \lim_{(x/\mu_1, y/\mu_2)\to\infty} E\, N_{x,y} = \frac{x}{\mu_1}\,\Phi\left(\frac{y/\mu_2 - x/\mu_1}{\Delta_{x,y}}\right) + \frac{y}{\mu_2}\,\Phi\left(\frac{x/\mu_1 - y/\mu_2}{\Delta_{x,y}}\right) - \Delta_{x,y}\,\phi\left(\frac{x/\mu_1 - y/\mu_2}{\Delta_{x,y}}\right),$$

where $\Delta_{x,y}^2 = \dfrac{\sigma_1^2 x}{\mu_1^3} + \dfrac{\sigma_2^2 y}{\mu_2^3} - \dfrac{2\rho\sigma_1\sigma_2}{\mu_1\mu_2}\sqrt{\dfrac{xy}{\mu_1\mu_2}}$.

Note that (4.12) also follows from (4.13) with $x/\mu_1 = y/\mu_2 = t$.

5. FURTHER RESEARCH

5.1 Although we presented, in Section 4, an approximate expression, (4.12), for $H(\mu_1 t, \mu_2 t)$ for large t, further research is required to give a rigorous proof of this result. Attempts to generalize the one dimensional techniques have so far proved fruitless.

5.2 If one is interested in examining the behavior of $\mathrm{var}(N_{x,y})$, an approximate expression can be obtained from (A.10) giving, as $t \to \infty$,

$$\mathrm{var}(N_{\mu_1 t, \mu_2 t}) \sim \frac{t}{2}\left[\left(\frac{\sigma_1}{\mu_1}\right)^2 + \left(\frac{\sigma_2}{\mu_2}\right)^2 - \frac{D^2}{\pi}\right] .$$

No detailed examination of this result has been carried out.

5.3 Theorem 2.4 provides a motivation that would lead us to suspect that as $(x/\mu_1, y/\mu_2) \to \infty$, $\mathrm{corr}(N_x^{(1)}, N_y^{(2)})$ tends to the correlation of the limiting bivariate normal r.v.'s, $\rho = \mathrm{corr}(X_1, Y_1)$. Actually, in [8] we showed that if (X_1, Y_1) has the j.p.d.f. given by (4.5) then

$$\mathrm{corr}(N_x^{(1)}, N_y^{(2)}) = \rho\sqrt{\frac{\mu_1\mu_2}{xy}}\; H(x,y) .$$

Thus, for (x,y) lying on the line of expectation, we can conclude from (4.3) that

$$\lim_{t\to\infty} \mathrm{corr}(N_{\mu_1 t}^{(1)}, N_{\mu_2 t}^{(2)}) = \rho$$

Further investigations need to be carried out to determine the validity of this result in general.

APPENDIX

In this appendix we examine the distribution and moments of the minimum of two bivariate normal r.v.'s. The techniques used are based upon those used by Afonja [17] in deriving more general results for the moments of the maximum of correlated normal r.v.'s.

Let (U_1,U_2) have a bivariate normal $(\theta_1,\theta_2;\ \sigma_1^2,\sigma_2^2;\rho)$ distribution; i.e. $E\,U_1 = \theta_1$, $EU_2 = \theta_2$, var $U_1 = \sigma_1^2$, var $U_2 = \sigma_2^2$, $\mathrm{corr}(U_1,U_2) = \rho$.

Let $\Phi_\rho(x,y)$ be the joint d.f., and $\phi_\rho(x,y)$ be the joint p.d.f., of a pair of bivariate normal $(0,0;\ 1,1;\rho)$ random variables.

THEOREM A.1:

The d.f., $F(y)$, of $Y = \min(U_1,U_2)$ is given by

$$F(y) = 1 - \Phi_\rho\left(\frac{\theta_1-y}{\sigma_1},\ \frac{\theta_2-y}{\sigma_2}\right). \tag{A.1}$$

Proof:

$1-F(y) = P\{\min(U_1,U_2) > y\} = P\{U_1 > y,\ U_2 > y\}$

$$= (1/2\pi\sigma_1\sigma_2\sqrt{1-\rho^2})\int_y^\infty\int_y^\infty \exp\left[-\left\{\left(\frac{u_1-\theta_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{u_1-\theta_1}{\sigma_1}\right)\left(\frac{u_2-\theta_2}{\sigma_2}\right) + \left(\frac{u_2-\theta_2}{\sigma_2}\right)^2\right\}/2(1-\rho^2)\right]\,du_2du_1\ .$$

The result follows upon making the substitution

$$z_1 = -(u_1-\theta_1)/\sigma_1\ ,\quad z_2 = -(u_2-\theta_2)/\sigma_2\ . \qquad \square$$

THEOREM A.2:

If $Y = \min(U_1,U_2)$ then, for $r \ge 1$, (A.2) $$E\,Y^r = \sum_{i=1}^{2} \sum_{j=0}^{r} \binom{r}{j} \theta_i^{r-j} \sigma_i^j \alpha_{i,j}/\Delta^j ,$$

where

$$\alpha_{i,j} = (1/2\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{(\theta_{3-i}-\theta_i)/\Delta} \{(1-\rho^2)^{\frac{1}{2}} \sigma_{3-i} z_i + (\sigma_i - \rho\sigma_{3-i}) z_{3-i}\}^j$$

$$\exp[-(z_i^2 + z_{3-i}^2)/2] dz_{3-i} dz_i \quad , \qquad (i=1,2;\ j=0,1,\ldots,r),$$

and

$$\Delta = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)^{\frac{1}{2}} .$$

Proof:

From (A.1) we can express the d.f. of Y, $F(y)$, as

$$F(y) = 1 - \int_{b_1(y)}^{\infty} \int_{b_2(y)}^{\infty} \phi_\rho(z_1, z_2) dz_2\, dz_1$$

where $b_1(y) = (y-\theta_1)/\sigma_1$ and $b_2(y) = (y-\theta_2)/\sigma_2$.

The p.d.f. of Y, $f(y)$, is given by

$$f(y) = \frac{dF(y)}{dy} = \sum_{i=1}^{2} \frac{\partial F}{\partial b_i} \cdot \frac{\partial b_i}{\partial y}$$

$$= (1/\sigma_1) \int_{b_2(y)}^{\infty} \phi_\rho((y-\theta_1)/\sigma_1, z_2) dz_2 + (1/\sigma_2) \int_{b_1(y)}^{\infty} \phi_\rho(z_1, (y-\theta_2)/\sigma_2)) dz_1$$

$$= f_1(y) + f_2(y) \text{ , say.}$$

Now,

(A.4) $$E\,Y^r = \int_{-\infty}^{\infty} y^r f(y) dy = \sum_{i=1}^{2} \int_{-\infty}^{\infty} y^r f_i(y) dy = \sum_{i=1}^{2} I_{i,r}$$

where, for example,

$$I_{1,r} = (1/2\pi\sigma_1\sqrt{1-\rho^2}) \int_{-\infty}^{\infty} y^r \int_{(y-\theta_2)/\sigma_2}^{\infty} \exp[-\{(\frac{y-\theta_1}{\sigma_1})^2 - 2\rho(\frac{y-\theta_1}{\sigma_1})z_2 + z_2^2\}/2(1-\rho^2)] \, dz_2 \, dy \;.$$

This can be simplified by a series of transformations. First, put $z_1 = (y-\theta_1)/\sigma_1$ to obtain

$$I_{1,r} = \int_{-\infty}^{\infty} \int_{(\theta_1-\theta_2+\sigma_1 z_1)/\sigma_2}^{\infty} (\theta_1+z_1\sigma_1)^r \phi_\rho(z_1,z_2) dz_2 \, dz_1 \;.$$

Secondly, the transformation $u_1 = z_1$, $u_2 = z_1\sigma_1/\sigma_2 - z_2$ (whose Jacobian is 1) and the binomial expansion of $(\theta_1+u_1\sigma_1)^r$ gives

$$I_{1,r} = \sum_{j=0}^{r} \binom{r}{j} \theta_1^{r-j}\sigma_1^j \beta_{1,j} \tag{A.5}$$

where

$$\beta_{1,j} = (1/2\pi\sqrt{1-\rho^2}) \int_{-\infty}^{\infty} \int_{-\infty}^{(\theta_2-\theta_1)/\sigma_2} u_1^{\,j}$$

$$\exp[-\{u_1^2\Delta^2/\sigma_2^2 - 2u_1u_2(\sigma_1/\sigma_2-\rho) + u_2^2\}/2(1-\rho^2)] \, du_2 \, du_1 \;.$$

Now $\beta_{1,j}$ can be simplified by the transformation

$$u_1\Delta/\sigma_2 - u_2(\sigma_1-\rho\sigma_2)/\Delta = z_1(1-\rho^2)^{\frac{1}{2}}$$

$$u_2\sigma_2/\Delta \qquad = z_2$$

(whose Jacobian is $(1-\rho^2)^{\frac{1}{2}}$), to give

$$\beta_{1,j} = (1/2\pi\Delta^j)\int_{-\infty}^{\infty} \int_{-\infty}^{(\theta_2-\theta_1)/\Delta} \{z_1\sigma_2(1-\rho^2)^{\frac{1}{2}} + z_2(\sigma_1-\rho\sigma_2)\}^j \exp[-(z_1^2+z_2^2)/2] dz_2 \, dz_1$$

$$= \alpha_{1,j}/\Delta^j \;. \tag{A.6}$$

Equation (A.2) follows from (A.4), (A.5), (A.6) and analogous results for $I_{2,r}$. □

Evaluation of the $\alpha_{i,j}$ can now be effected (using integration by parts, where necessary) in terms of the standard normal d.f. $\Phi(x)$ and p.d.f. $\phi(x)$. In particular, for i = 1,2,

$$\alpha_{i,0} = \Phi(a_i) \ ,$$
$$\alpha_{i,1} = -(\sigma_i - \rho\sigma_{3-i})\phi(a_i) \ ,$$
$$\alpha_{i,2} = \Delta^2\Phi(a_i) - a_i(\sigma_i - \rho\sigma_{3-i})^2\phi(a_i) \ ,$$

where $a_i = (\theta_{3-i} - \theta_i)/\Delta$.

The corollary below follows from these results together with the observations that $\phi(x) = \phi(-x)$, $\Phi(0) = \frac{1}{2}$, and $\phi(0) = 1/\sqrt{2\pi}$.

Cor. A.2.1:

(A.7) $E\ Y = \theta_1\Phi((\theta_2-\theta_1)/\Delta) + \theta_2\Phi((\theta_1-\theta_2)/\Delta) - \Delta\phi((\theta_1-\theta_2)/\Delta)$,

(A.8) $E\ Y^2 = (\theta_1^2+\sigma_1^2)\Phi((\theta_2-\theta_1)/\Delta) + (\theta_2^2+\sigma_2^2)\Phi((\theta_1-\theta_2)/\Delta)$
$$- (\theta_1+\theta_2)\Delta\phi((\theta_2-\theta_1)/\Delta) \ .$$

In particular, when $\theta_1 = \theta_2 = \theta$, say,

(A.9) $$E\ Y = \theta - \Delta/\sqrt{2\pi} \ ,$$

(A.10) $$\text{var } Y = (\sigma_1^2 + \sigma_2^2 - \Delta^2/\pi)/2 \ .$$

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