561

RENEWAL THEORY IN TWO DIMENSIONS:

BASIC RESULTS

by

Jeffrey J. Hunter*

University of North Carolina at Chapel Hill

ABSTRACT

In this paper a unified theory for studying renewal processes in two dimensions is developed. Bivariate probability generating functions and bivariate Laplace transforms are the basic tools used in generalizing the standard theory of univariate renewal processes. Two examples involving the use of independent and correlated exponential distributions are presented. These are used to illustrate the general theory and explicit expressions for the two dimensional renewal density, the two dimensional renewal function, the correlation between the marginal univariate renewal counting processes, and other related quantities are derived.

---

* Written while on leave from the Department of Mathematics, University of Auckland, New Zealand. The research in this report was partially supported by the National Science Foundation under Grant GU-2059 and by the Office of Naval Research under Contract N00014-67-A-0321-002.

Mimeo Series #861.

1. INTRODUCTION

Let $\{(X_n,Y_n)\}$, $n = 1,2,\ldots$, be a sequence of independent and identically distributed non-negative bivariate random variables (r.v.'s) with common distribution function $F(x,y) = P\{X_n \le x,\ Y_n \le y\}$.

Let $\underline{S}_n = (S_n^{(1)},\ S_n^{(2)}) = (\sum_{i=1}^{n} X_i,\ \sum_{i=1}^{n} Y_i)$.

We shall call the sequence of bivariate r.v.'s $\{(X_n,Y_n)\}$ a *bivariate renewal process* and observe that the marginal sequences, $\{X_n\}$ and $\{Y_n\}$ are each (univariate) renewal processes.

In order that we distinguish between the different renewal processes, we shall say that an X-renewal occurs at the point x on the X-axis if $S_n^{(1)} = x$ for some x, a Y-renewal occurs at the point y on the Y-axis if $S_n^{(2)} = y$ for some y, and an (X,Y)-renewal occurs at the point (x,y) in (X,Y) plane if $S_n^{(1)} = x$ and $S_n^{(2)} = y$ for some n.

$$
\begin{aligned}
\text{Define}\quad N_x^{(1)} &= \max\{n:\ S_n^{(1)} \le x\},\\
N_y^{(2)} &= \max\{n:\ S_n^{(2)} \le y\},\\
N_{x,y} &= \max\{n:\ S_n^{(1)} \le x,\ S_n^{(2)} \le y\}\ .
\end{aligned}
$$

Thus, associated with a bivariate renewal process we have various counting processes. Firstly, $N_x^{(1)}$ and $N_y^{(2)}$ are the (univariate) renewal counting processes for the X-renewals and the Y-renewals. We call the random pair $(N_x^{(1)},\ N_y^{(2)})$ the *bivariate renewal counting process*. Secondly, N_{xy} records the number of (X,Y)-renewals that occur in the closed region of the positive quadrant of the (X,Y) plane bounded by the axes and the lines $X = x$ and $Y = y$. We call $N_{x,y}$ the *two dimensional renewal counting process*.

This paper is concerned primarily with the development and derivation of exact results for the distribution, moments and probability generating functions of the bivariate and two-dimensional renewal counting processes. The approach taken in this research is to generalize the univariate theory by using direct two dimensional analogs to derive exact results.

In Section 2, an outline of the properties of bivariate convolutions, bivariate probability generating functions (p.g.f.'s), and bivariate Laplace transforms (L.T.'s) is presented for use in subsequent sections.

Section 3 is devoted exclusively to the properties of bivariate renewal counting processes. The joint distribution of $(N_x^{(1)}, N_y^{(2)})$, its joint p.g.f. and the bivariate L.T. of this p.g.f. are obtained. The moments of $(N_x^{(1)}, N_y^{(2)})$, the marginal renewal functions and renewal densities, and the independence of $N_x^{(1)}$ and $N_y^{(2)}$ are also discussed.

Since it is easily seen that $N_{x,y} = \min\{N_x^{(1)}, N_y^{(2)}\}$, i.e. the minimum of two correlated r.v.'s, the results of Section 3 are utilized in Section 4 in examining the characteristics of two dimensional renewal counting processes. The distribution of $N_{x,y}$, its p.g.f. and the L.T. of this p.g.f. are discussed as is the two dimensional renewal function, $EN_{x,y}$, and its associated renewal density.

The paper concludes with a detailed examination of two bivariate distributions, viz. independent and correlated exponential distributions, and explicit determination of their properties in the above renewal theoretic context.

2. PRELIMINARIES

2.1 Convolutions of distribution functions and density functions.

In this paper $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ will be taken to be bivariate d.f.'s of non-negative r.v.'s. For such functions, or any Stieltjes integrable functions of two variables, we define their double convolution as

$$F ** G(x,y) = \int_0^x \int_0^y F(x-u,\ y-v)dG(u,v). \tag{2.1}$$

This operation is commutative with respect to F and G and the order of integration is immaterial in the work to follow.

For $x,y \ge 0$, we define

$$\begin{aligned} F_0(x,y) &= 1 , \\ F_1(x,y) &= F(x,y) , \\ F_{r+1}(x,y) &= F ** F_r(x,y) ,\ (r \ge 1); \end{aligned}$$

where $F(\cdot,\cdot)$ is taken to be the distribution function of (X_n,Y_n). Then $F_r(\cdot,\cdot)$ is the distribution function of the random pair $(S_r^{(1)}, S_r^{(2)})$, i.e.,

$$F_r(x,y) = P\{S_r^{(1)} \le x ,\ S_r^{(2)} \le y\} .$$

We shall also use the notation that, for d.f.'s $F(\cdot,\cdot)$, the marginal d.f.'s are given by

$$F^1(x,y) \equiv \lim_{y\to\infty} F(x,y) \equiv F(x,\infty) \equiv F^1(x) ,$$

$$F^2(x,y) \equiv \lim_{x\to\infty} F(x,y) \equiv F(\infty,y) \equiv F^2(y).$$

Note that either of the upper limits of integration in (2.1) may be replaced by $+\infty$, since $F(x-u,\ y-v) = 0$ when $u > x$ or $v > y$.

Furthermore,

$$F^1 ** G(x,y) = \int_0^x \int_0^y F(x-u,\infty)dG(u,v)$$

$$= \int_0^\infty \int_0^y F(x-u,\infty)dG(u,v) \ .$$

By virtue of this result (and the bounded convergence theorem) we may easily deduce the following:

<u>Lemma 2.1</u>:

(i) $(F^1 ** G)^1(x,y) = (F ** G)^1(x,y)$.

(ii) $(F^2 ** G)^2(x,y) = (F ** G)^2(x,y)$.

(iii) $(F^1 ** G)^2(x,y) = G(\infty,y)$.

(iv) $(F^2 ** G)^1(x,y) = G(x,\infty)$.

The next Lemma is used in simplifying some of the results in Section 3.

<u>Lemma 2.2</u>:

For any d.f. $G(x,y)$,

$$G = F_0 ** G = F_0^1 ** G = F_0^2 ** G \ .$$

In later parts of this paper we assume that $F(\cdot,\cdot)$ is an absolutely continuous d.f. of the non-negative r.v.'s (X_n,Y_n) with joint probability density function (j.p.d.f.) $f(\cdot,\cdot)$. Thus

$$F(x,y) = \int_0^x \int_0^y f(u,v)\ du\ dv.$$

For any pair of j.p.d.f.'s, f and g we define their double convolution as

$$f \circledast \circledast g(x,y) = \int_0^x \int_0^y f(x-u,\ y-v)\ g(u,v)\ du\ dv \ .$$

Thus, if we define, for $x,y \geq 0$,

$$f_1(x,y) = f(x,y)$$

$$f_{r+1}(x,y) = f_r \circledast\circledast f(x,y), \quad (r \geq 1) ,$$

then $f_r(x,y)$ is the j.p.d.f. of $(S_r^{(1)}, S_r^{(2)})$.

We shall denote the marginal p.d.f.'s of X_n and Y_n as $f^1(x)$ and $f^2(y)$ respectively. Note that

$$f^1(x) = \int_0^\infty f(x,v)\, dv ,$$

$$f^2(y) = \int_0^\infty f(u,y)\, du .$$

2.2 Generating functions for "tail probabilities."

Feller [3, p. 265] investigates the relationship between the probability generating function (p.g.f.) of a non-negative integer valued r.v. and the g.f. of its "tail probabilities." In this section we develop similar relationships for bivariate, non-negative, integer valued r.v.'s (X,Y). It should be noted that we define our "tail probabilities" slightly differently in order that we obtain complete correspondence between the joint p.g.f. and the g.f. of the "tail probabilities."

Lemma 2.3:

Let $$P(s_1,s_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij} s_1^i s_2^j ,$$

and

$$Q(s_1,s_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_{ij} s_1^i s_2^j ;$$

where $p_{ij} = P\{X = i, Y = j\}$ and $q_{ij} = P\{X \geq i, Y \geq j\}$.

Then, for $|s_1| < 1$ and $|s_2| < 1$,

(2.2) $$(1-s_1)(1-s_2)Q(s_1,s_2) = 1 - s_1P(s_1,1) - s_2P(1,s_2) + s_1s_2P(s_1,s_2) ,$$

and

(2.3) $$s_1s_2P(s_1,s_2) = 1 - (1-s_1)Q(s_1,0) - (1-s_2)Q(0,s_2) ,$$
$$+ (1-s_1)(1-s_2)Q(s_1,s_2) .$$

Proof:

By expanding $(1-s_1)(1-s_2)Q(s_1,s_2)$ and collecting the terms involving $s_1^i\ s_2^j$ together we obtain

$$(1-s_1)(1-s_2)Q(s_1,s_2) = q_{00} - s_1 \sum_{i=0}^{\infty}(q_{i0} - q_{i+1,0})s_1^i$$
$$- s_2 \sum_{j=0}^{\infty}(q_{0j} - q_{0,j+1})s_2^j$$
$$+ s_1s_2 \sum_{i=0}^{\infty}\ \sum_{j=0}^{\infty}(q_{i+1,j+1} - q_{i,j+1} - q_{i+1,j} + q_{ij})\,s_1^i\ s_i^j .$$

Equation (2.2) then follows by observing that for $i,j = 0,1,2,\ldots,$

$$q_{i+1,j+1} - q_{i,j+1} - q_{i+1,j} + q_{ij} = p_{ij} ,$$
$$q_{i0} - q_{i+1,0} = P\{X=i\} = \sum_{j=0}^{\infty} p_{ij} ,$$
$$q_{0j} - q_{0,j+1} = P\{Y=j\} = \sum_{i=0}^{\infty} p_{ij} ,$$

and

$$q_{00} = 1 .$$

Equation (2.3) is derived by putting $s_1 = 0$ and $s_2 = 0$ in (2.2) to obtain, respectively,

(2.4) $$(1-s_2)Q(0,s_2) = 1 - s_2P(1,s_2) ,$$

(2.5) $$(1-s_1)Q(s_1,0) = 1 - s_1P(s_1,1) .$$

Substitution for $P(1,s_2)$ and $P(s_1,1)$ yields the required result. □

Let $P_1(s_1)$ and $P_2(s_2)$ denote the marginal p.g.f's of X and Y, respectively. Then

$$P_1(s_1) = \sum_{i=0}^{\infty} P\{X = i\}s_1^i = \sum_{i=0}^{\infty} p_{i.}s_1^i = P(s_1,1) ,$$

$$P_2(s_2) = \sum_{j=0}^{\infty} P\{Y = j\}s_2^j = \sum_{j=0}^{\infty} p_{.j}s_2^j = P(1,s_2) .$$

Similarly, let $Q_1(s_1)$ and $Q_2(s_2)$ denote the g.f.'s of the "tail probabilities" of the marginal distribution of X and Y, respectively. Then

$$Q_1(s_1) = \sum_{i=0}^{\infty} P\{X \geq i\}s_1^i = \sum_{i=0}^{\infty} q_{i0}s_1^i = Q(s_1,0) ,$$

$$Q_2(s_2) = \sum_{j=0}^{\infty} P\{Y \geq j\}s_2^j = \sum_{j=0}^{\infty} q_{0j}s_2^j = Q(0,s_2) .$$

<u>Corollary 2.3.1</u>:

(2.6) For $|s_1| < 1$, $$Q_1(s_1) = \frac{1 - s_1P_1(s_1)}{1 - s_1} ,$$

(2.7) For $|s_2| < 1$, $$Q_2(s_2) = \frac{1 - s_2P_2(s_2)}{1 - s_2} .$$

<u>Proof</u>: Equation (2.6) (resp. (2.7)) follows directly from (2.4) (resp. (2.5)). □

The following lemma is useful in determining the independence of the random variables X and Y.

<u>Lemma 2.4</u>:

The following three conditions are equivalent:

(i) X and Y are independent,

(i.e., $p_{ij} = p_{i.}p_{.j}$ for all $i,j = 0,1,2,\ldots$) ;

(ii) $P(s_1,s_2) = P(s_1,1)\,P(1,s_2)$, $|s_1| \le 1$, $|s_2| \le 1$;

(iii) $Q(s_1,s_2) = Q(s_1,0)\,Q(0,s_2)$, $|s_1| < 1$, $|s_2| < 1$.

Proof:

The equivalence of (i) and (ii) is well known, e.g. Feller [3, p. 279].

The equivalence of (ii) and (iii) follows upon observing that from (2.2), (2.4) and (2.5) we may obtain the result that

$$(1-s_1)(1-s_2)[Q(s_1,s_2) - Q(s_1,0)Q(0,s_2)]$$

$$= s_1 s_2[P(s_1,s_2) - P(s_1,1)P(1,s_2)] \; . \qquad \square$$

The moments of X and Y are, especially for low orders, easily obtainable from the q_{ij} or from $Q(s_1,s_2)$.

Lemma 2.5:

(2.8) $$EX = \sum_{i=1}^{\infty} q_{i0} = Q(1,0) - 1 = Q_1(1) - 1 \quad ,$$

(2.9) $$EY = \sum_{j=1}^{\infty} q_{0j} = Q(0,1) - 1 = Q_2(1) - 1 \quad ,$$

(2.10) $$EXY = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} q_{ij} = Q(1,1) - Q(1,0) - Q(0,1) + 1 \; ,$$

(2.11) $$Cov(X,Y) = Q(1,1) - Q(0,1)Q(1,0) = Q(1,1) - Q_1(1)Q_2(1).$$

Proof:

Equation (2.8) (and analogously (2.9) follows upon noting

$$EX = \sum_{i=1}^{\infty} i p_{i.} = \sum_{i=1}^{\infty} i(q_{i0} - q_{i+1,0}) = \sum_{i=0}^{\infty} q_{i0} - q_{00} \; .$$

To obtain (2.10), differentiate (2.2) with respect to s_1 and then with respect to s_2 to give

$$Q(s_1,s_2) - (1-s_1)\frac{\partial Q(s_1,s_2)}{\partial s_1} - (1-s_2)\frac{\partial Q(s_1,s_2)}{\partial s_2} + (1-s_1)(1-s_2)\frac{\partial^2 Q(s_1,s_2)}{\partial s_1 \partial s_2}$$

$$= P(s_1,s_2) + s_1\frac{\partial P(s_1,s_2)}{\partial s_1} + s_2\frac{\partial P(s_1,s_2)}{\partial s_2} + s_1 s_2\frac{\partial^2 P(s_1,s_2)}{\partial\ \partial s_1 \partial s_2} .$$

Taking the limits $s_1 \uparrow 1$ and $s_2 \uparrow 1$ yields $Q(1,1) = 1 + EX + EY + EXY$, and the results follow upon using (2.8) and (2.9).

Equation (2.11) follows by the definition of cov(X,Y). □

2.3 Laplace-Stieltjes transforms and Laplace transforms.

By direct analogy with univariate transforms we define, for functions which vanish identically for negative values of their arguments, the following bivariate transforms.

(i) If $F(x,y)$ is any function of bounded variation in every finite rectangle then we shall write

$$F^*(p,q) = \int_0^\infty \int_0^\infty e^{-px-qy}\, dF(x,y)$$

for the bivariate Laplace-Stieltjes transform (L.S.T.) of $F(x,y)$.

(ii) If $F(x,y)$ is any function which is integrable in every finite rectangle then we shall write

$$F^O(p,q) = L^2\{F(x,y)\}$$

$$= \int_0^\infty \int_0^\infty e^{-px-qy}\, F(x,y)\, dx\, dy ,$$

for the bivariate Laplace transform (L.T.) of $F(x,y)$.

We do not discuss the region of convergence of these transforms other than to point out that this is discussed in some detail in both [2] and [10] for bivariate L.T.'s. It should be remarked, however, that if the bivariate L.T. is absolutely convergent for some complex valued $(p,q) = (p_0,q_0)$, then the bivariate L.T. exists for all p,q such that $\mathcal{R}(p) \geq \mathcal{R}(p_0)$, and $\mathcal{R}(q) \geq \mathcal{R}(q_0)$.

Note that when F(x,y) has both a bivariate L.S.T. and a bivariate L.T. then

$$(2.12) \qquad F^*(p,q) = pq\, F^{\circ}(p,q) \ .$$

For the univariate transforms we write

$$F^*(p) = \int_0^{\infty} e^{-px}\, dF(x) \ ,$$

and

$$F^{\circ}(p) = L\{F(x)\} = \int_0^{\infty} e^{-px}\, F(x)\, dx$$

for the L.S.T. and L.T. (when they exist) of the function $F(x)$.

Let us now assume in what follows that $F(x,y)$ is the d.f. of an absolutely continuous distribution with j.p.d.f. $f(x,y)$. As introduced in §2.1 we let $F^1(x)$ and $F^2(y)$ denote the marginal d.f.'s and $f^1(x)$ and $f^2(y)$ the marginal p.d.f.'s. We also tacitly assume that all the transforms exist. (In fact, $f^{\circ}(p,q)$ will exist for all p,q such that $R(p) \geq 0$, $R(q) \geq 0$).

The first observation we make is that

$$(2.13) \qquad F^*(p,q) = f^{\circ}(p,q) \ ,$$

$$(2.14) \qquad F^{\circ}(p,q) = \frac{1}{pq}\, f^{\circ}(p,q) \ .$$

The following lemma gives connections between the univariate and bivariate transforms for the marginal d.f.'s and p.d.f.'s.

<u>Lemma 2.6</u>:

$$(2.15) \qquad F^*(p,0) = F^{1*}(p) = f^{1\circ}(p) = f^{\circ}(p,0) \ ,$$

$$(2.16) \qquad F^*(0,q) = F^{2*}(q) = f^{2\circ}(q) = f^{\circ}(0,q) \ .$$

Also,

$$(2.17) \qquad f^{1\circ}(p,q) = \frac{1}{q}\, f^{\circ}(p,0) \ ,$$

$$(2.18) \qquad f^{2\circ}(p,q) = \frac{1}{p}\, f^{\circ}(0,q) \ .$$

Also,

$$(2.19) \qquad F^{1\circ}(p,q) = \frac{1}{q}\, F^{1\circ}(p) = \frac{1}{pq}\, f^{\circ}(p,0) \ ,$$

$$(2.20) \qquad F^{2\circ}(p,q) = \frac{1}{p}\, F^{2\circ}(q) = \frac{1}{pq}\, f^{\circ}(0,q) \ .$$

Proof:

Equation (2.15) (and analogously (2.16)) follows from the fact that

$$\begin{aligned} F^*(p,0) &= \int_0^\infty \int_0^\infty e^{-px}\, dF(x,y) \\ &= \int_0^\infty e^{-px}\, dF^1(x) \qquad (= F^{1*}(p)) , \\ &= \int_0^\infty e^{-px}\, f^1(x) dx \qquad (= f^{1\circ}(p)), \\ &= \int_0^\infty \int_0^\infty e^{-px}\, f(x,y)\, dx\, dy \quad (= f^\circ(p,0)). \end{aligned}$$

For (2.17) (and similarly (2.18)) note that

$$\begin{aligned} f^{1\circ}(p,q) &= \int_0^\infty \int_0^\infty e^{-px-qy}\, f^1(x)\, dx\, dy \\ &= \frac{1}{q} \int_0^\infty e^{-px}\, f^1(x)\, dx = \frac{1}{q}\, f^{1\circ}(p) \end{aligned}$$

and the result follows from (2.15).

Finally (2.19) (similarly (2.20)) follows on noting that

$$\begin{aligned} F^{1\circ}(p,q) &= \int_0^\infty \int_0^\infty e^{-px-qy} F(x,\infty)\, dx\, dy \\ &= \frac{1}{q} \int_0^\infty e^{-px}\, F^1(x)\, dx \qquad (= \frac{1}{q}\, F^{1\circ}\ (p)) \\ &= \frac{1}{q} \int_0^\infty e^{-px}\, \{\int_0^x f^1(u)\, du\}\, dx \\ &= \frac{1}{pq}\, F^{1\circ}(p) = \frac{1}{pq}\, f^\circ(p,0)\ . \qquad \square \end{aligned}$$

(Note that the results concerning L.S.T.'s do not require the assumption of absolute continuity).

Concerning bivariate L.S.T.'s and L.T.'s of convolutions of bivariate d.f.'s and their p.d.f.'s we have the following: (See [10, p. 36])

Lemma 2.7:

Let F and G be bivariate d.f.'s, then

(2.21) $$(F ** G)^*(p,q) = F^*(p,q)G^*(p,q) .$$

Furthermore:

(i) If G is absolutely continuous with p.d.f. g then

(2.22) $$(F ** G)^\circ(p,q) = F^\circ(p,q)g^\circ(p,q) .$$

(ii) If both F and G are absolutely continuous with p.d.f.'s f and g, respectively, then

(2.23) $$(F ** G)^\circ(p,q) = \frac{1}{pq} f^\circ(p,q)g^\circ(p,q) ,$$

(2.24) $$(F^1 ** G)^\circ(p,q) = \frac{1}{pq} f^\circ(p,0)g^\circ(p,q) ,$$

(2.25) $$(F^2 ** G)^\circ(p,q) = \frac{1}{pq} f^\circ(0,q)g^\circ(p,q) .$$

Also,

(2.26) $$(f \circledast\circledast g)^\circ(p,q) = f^\circ(p,q)g^\circ(p,q) .$$

As an obvious corollary to (2.26) we may use induction to show that

(2.27) $$f_r^\circ(p,q) = L^2\{f_r(x,y)\} = [f^\circ(p,q)]^r .$$

We conclude this section with a lemma that is used directly in Theorem 3.4.

Lemma 2.8:

For all $r \geq 0$, $k \geq 0$,

(2.28) $$(F_k ** F_r)^{\circ}(p,q) = \frac{1}{pq}[f^{\circ}(p,q)]^{k+r} ,$$

(2.29) $$(F_k^1 ** F_r)^{\circ}(p,q) = \frac{1}{pq}[f^{\circ}(p,0)]^{k}[f^{\circ}(p,q)]^{r} ,$$

(2.30) $$(F_k^2 ** F_r)^{\circ}(p,q) = \frac{1}{pq}[f^{\circ}(0,q)]^{k}[f^{\circ}(p,q)]^{r} .$$

Proof:

From (2.22) we obtain the result

$$(G ** F_r)^{\circ}(p,q) = G^{\circ}(p,q)f_r^{\circ}(p,q) .$$

By taking G as F_k, F_k^1 and F_k^2 with bivariate L.T.'s

$\frac{1}{pq} f_k^{\circ}(p,q)$, $\frac{1}{pq} f_k^{\circ}(p,0)$, and $\frac{1}{pq} f_k^{\circ}(0,q)$ obtained, respectively, from (2.14), (2.19) and (2.20), the lemma follows for $r \geq 1$, $k \geq 1$ upon utilization of (2.27). The results also hold for $r = 0$ or $k = 0$ since $F_0 = F^1 = F^2$ with bivariate L.T. $\frac{1}{pq}$. □

3. BIVARIATE RENEWAL COUNTING PROCESSES

3.1 The distribution of $(N_x^{(1)}, N_y^{(2)})$.

Theorem 3.1:

If $p_{r,s}(x,y) = P\{N_x^{(1)} = r, N_y^{(2)} = s\}$, $(x,y \geq 0;\ r,s = 0,1,2,\ldots)$,

then

(3.1) $$p_{r,r}(x,y) = [F_0 - F^1 - F^2 + F] ** F_r(x,y) , \quad (r \geq 0),$$

(3.2) $$p_{s+n,s}(x,y) = [F_n^1 - F_{n+1}^1 - F_{n-1}^1 ** F + F_n^1 ** F] ** F_s(x,y),$$

$$(s \geq 0,\ n \geq 1) ,$$

(3.3) $$p_{r,r+n}(x,y) = [F_n^2 - F_{n+1}^2 - F_{n-1}^2 ** F + F_n^2 ** F] ** F_r(x,y)$$

$$(r \geq 0,\ n \geq 1).$$

Proof: To facilitate a direct proof of these results we partition the positive quadrant of the (X,Y) plane into the following four regions. For fixed x and y define

$$E_{xy} = \{(X,Y) : 0 \leq X \leq x,\ 0 \leq Y \leq y\} ,$$

$$E_{\bar{x}y} = \{(X,Y) : X > x,\ 0 \leq Y \leq y\} ,$$

$$E_{x\bar{y}} = \{(X,Y) : 0 \leq X \leq x,\ Y > y\} ,$$

$$E_{\bar{x}\bar{y}} = \{(X,Y) : X > x,\ Y > y\} .$$

For notational ease let $\underline{S}_i = (S_i^{(1)}, S_i^{(2)})$, $(i = 1,2,\ldots)$, and $\underline{S}_0 = (0,0)$.

We derive (3.1) as follows. For $r \geq 0$,

$$p_{rr}(x,y) = P\{\underline{S}_r \in E_{xy}, \underline{S}_{r+1} \in E_{\bar{x}\bar{y}}\} .$$

In particular, making use of Lemma 2.2,

$$p_{00}(x,y) = P\{\underline{S}_1 \in E_{xy}\} = P\{(X_1Y_1) \in E_{\bar{x}\bar{y}}\}$$

$$= 1 - F(x,\infty) - F(\infty,y) + F(x,y)$$

$$= [F_0 - F^1 - F^2 + F] ** F_0(x,y) ;$$

and for $r \geq 1$,

$$p_{rr}(x,y) = \int_0^x \int_0^y P\{u \le S_r^{(1)} \le u + du,\ v \le S_r^{(2)} \le v + dv\}$$

$$\cdot\ P\{X_{r+1} > x - u,\ Y_{r+1} > y - v\}$$

$$= \int_0^x \int_0^y [1 - F(x-u,\infty) - F(\infty,y-v) + F(x-u,y-v)]\,dF_r(u,v)$$

$$= [F_0 - F^1 - F^2 + F] * * F_r(x,y)\ .$$

To derive the expressions given by (3.2) (and analogously (3.3)) note that for $r > s$,

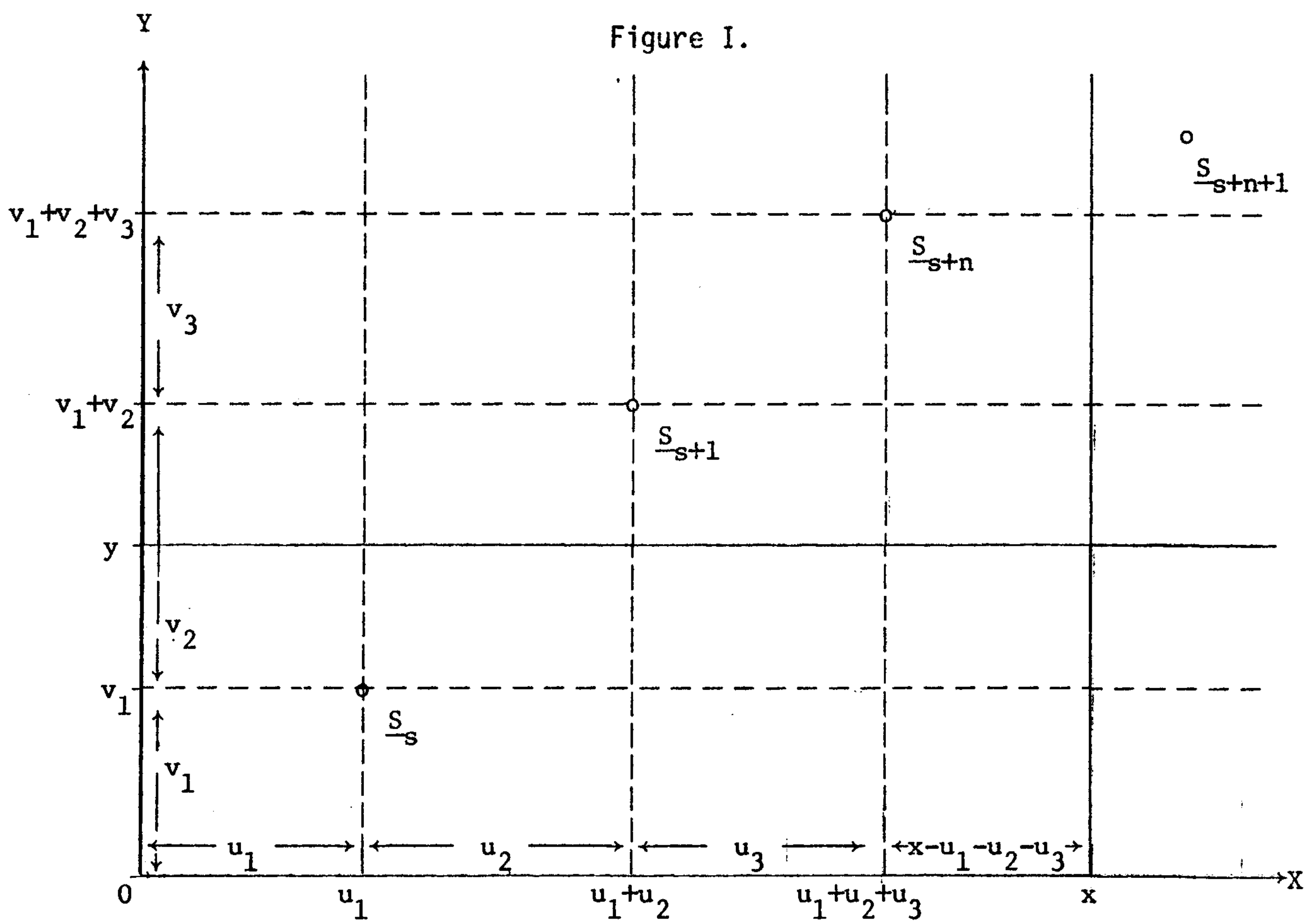
$$p_{rs}(x,y) = P\{\underline{S}_s \in E_{xy},\ \underline{S}_{s+1} \in E_{x\bar{y}},\ \underline{S}_r \in E_{x\bar{y}},\ \underline{S}_{r+1} \in E_{\bar{x}\bar{y}}\}\ .$$

Thus for $s \ge 0,\ n \ge 1$

$$p_{s+n,s}(x,y) = P\{\underline{S}_s \in E_{xy},\ \underline{S}_{s+1} \in E_{x\bar{y}},\ \underline{S}_{s+n} \in E_{x\bar{y}},\ \underline{S}_{s+n+1} \in E_{\bar{x}\bar{y}}\}\ .$$

In the general case when $s > 0$ and $n > 1$, consider Figure 1.

Figure I.



Thus

$$p_{s+n,s}(x,y) = \iint\limits_{\substack{0\le u_1\le x\\ 0\le v_1\le y}} \iint\limits_{\substack{0\le u_2\le x-u_1\\ y-v_1\le v_2<\infty}} \iint\limits_{\substack{0\le u_3\le x-u_1-u_2\\ 0\le v_3<\infty}}$$

$$\cdot\ P\{(u_1 \le S_s^{(1)} \le u_1+du_1,\ v_1 \le S_s^{(2)} \le v_1+dv_1)$$

$$\cap\ (u_2 \le X_{s+1} \le u_2+du_2,\ v_2 \le Y_{s+1} \le v_2+dv_2)$$

$$\cap\ (u_3 \le \sum_{i=s+2}^{s+n} X_i \le u_3+du_3,\ v_3 \le \sum_{i=s+2}^{s+n} Y_i \le v_3+dv_3)$$

$$\cap\ (X_{s+n+1} \ge x-u_1-u_2-u_3,\ Y_{s+n+1} \ge 0)\}\ .$$

i.e.,

$$p_{s+n,s}(y,y) = \int_0^x\int_0^y\int_0^{x-u_1}\int_{y-v_1}^{\infty} \int_0^{x-u_1-u_2} \int_0^{\infty} \{1 - F(x-u_1-u_2-u_3,\infty)\}$$

$$dF_{n-1}(u_3,v_3)dF(u_2,v_2)dF_s(u_1,v_1)\ .$$

Simplification is carried out in stages, namely

$$p_{s+n,s}(x,y) = \int_0^x\int_0^y\int_0^{x-u_1}\int_{y-v_1}^{\infty} \{F_{n-1}(x-u_1-u_2,\infty) - F^1 * * F_{n-1}(x-u_1-u_2,\infty)\}$$

$$dF(u_2,v_2)\ dF_s(u_1,v_1)$$

$$= \int_0^x\int_0^y \{F^1_{n-1} * * F(x-u_1,\infty) - (F^1 * * F_{n-1})^1 * * F(x-u_1,\infty)$$

$$- F^1_{n-1} * * F(x-u_1 y-v_1) - (F^1 * * F_{n-1})^1 * * F(x-u_1,y-v)\}$$

$$dF_s(u_1,v_1)$$

$$= (F^1_{n-1} * * F)^1 * * F_s(x,y)$$

$$- ((F^1 * * F_{n-1})^1 * * F)^1 * * F_s(x,y)$$

$$- (F^1_{n-1} * * F) * * F_s(x,y)$$

$$+ ((F^1 * * F_{n-1})^1 * * F)^1 * * F_s(x,y)\ .$$

Simplifying using Lemma 2.1 yields (3.2). The other special cases (when either $s = 0$ or $n = 1$) follow using Lemma 2.2 (or by special consideration). □

The marginal distributions of $N_x^{(1)}$ and $N_y^{(2)}$ follow from the above theorem.

Corollary 3.1.1:

(3.4) $$P\{N_x^{(1)} = r\} = F_r(x,\infty) - F_{r+1}(x,\infty) , \qquad (r = 0,1,2,\ldots),$$

(3.5) $$P\{N_y^{(2)} = s\} = F_s(\infty,y) - F_{s+1}(\infty,y) , \qquad (s = 0,1,2,\ldots) .$$

Proof:

For $r \geq 1$, we may write

$$\begin{aligned} P\{N_x^{(1)} = r\} &= \sum_{s=0}^{\infty} p_{r,s}(x,y) \\ &= \sum_{n=1}^{r} p_{r,r-n}(x,y) + p_{r,r}(x,y) + \sum_{n=1}^{\infty} p_{r,r+n}(x,y) \\ &= \sum_{n=1}^{r} [F_n^1 - F_{n+1}^1 - F_{n-1}^1 ** F + F_n^1 ** F] ** F_{r-n}(x,y) \\ &\quad + [F_0 - F^1 - F^2 + F] ** F_r(x,y) \\ &\quad + \sum_{n=1}^{\infty} [F_n^2 - F_{n+1}^2 - F_{n-1}^2 ** F + F_n^2 ** F] ** F_r(x,y). \end{aligned}$$

Simplification is possible by noting, for example, that

$$(F_n^1 ** F) ** F_{r-n}(x,y) = F_n^1 ** F_{r-n+1}(x,y).$$

The terms involving the second marginal sum to zero, and the expression reduces to give

$$\begin{aligned} P\{N_x^{(1)} = r\} &= [F_r^1 - F_{r+1}^1] ** F_0(x,y) \\ &= F_r(x,\infty) - F_{r+1}(x,\infty) . \end{aligned}$$

Equation (3.4) follows analogously for the special case $r = 0$, as does (3.5). □

These results are, in fact, well known since each marginal is a univariate renewal process (e.g. [1, p. 36]).

The results of Theorem 3.1 can be derived indirectly by considering the "tail probabilities" of $(N_x^{(1)}, N_y^{(2)})$.

For $m, n = 0,1,2,\ldots$, define

$$q_{m,n}(x,y) = P\{N_x^{(1)} \geq m,\ N_y^{(2)} \geq n\}$$

By the usual renewal theoretic arguments it is easily seen that

$$\{N_x^{(1)} \geq m\} = \{S_m^{(1)} \leq x\}\ ,$$
$$\{N_y^{(2)} \geq n\} = \{S_n^{(2)} \leq y\}\ ,$$

and hence

$$\{N_x^{(1)} \geq m,\ N_y^{(2)} \geq n\} = \{S_m^{(1)} \leq x,\ S^{(2)} \leq y\}\ .$$

Thus

$$q_{m,n}(x,y) = P\{S_m^{(1)} \leq x,\ S_n^{(2)} \leq y\}\ .$$

Theorem 3.2:

For $m,n = 0,1,2,\ldots$

(3.6)
$$q_{m,n}(x,y) = \begin{cases} F_m(x,y) & , \quad (m = n)\ , \\ F_{m-n}^1 ** F_n(x,y) & , \quad (m > n)\ , \\ F_{n-m}^2 ** F_m(x,y) & , \quad (m < n)\ . \end{cases}$$

Proof:

First note that $q_{m,m}(x,y) = P\{S_m^{(1)} \leq x,\ S_m^{(2)} \leq y\} = F_m(x,y)$ and (3.6) holds for $m = n$.

Let us now assume that $m > n$. When $n = 0$, we have

$$q_{m0}(x,y) = P\{S_m^{(1)} \le x\} = F_m(x,\infty) = F_m^1(x,y) = F_m^1 ** F_0(x,y) \ .$$

When $m > n \ge 1$ observe that $\{S_m^{(1)} \le x\}$ implies $\{S_n^{(1)} \le x\}$. Thus

$$\begin{aligned} q_{m,n}(x,y) &= P\{\underline{S}_n \in E_{xy} \ ; \ \underline{S}_m \in E_{x\infty}\} \\ &= \int_0^x \int_0^y P\{(u \le S_n^{(1)} \le u+du,\ v \le S_n^{(2)} \le v+dv) \\ &\qquad \cap ((\sum_{i=n+1}^{m} X_i \ , \ \sum_{i=n+1}^{m} Y_i) \in E_{x-u,\infty})\} \\ &= \int_0^x \int_0^y F_{m-n}(x-u,\infty)\ dF_n(u,v) \\ &= F_{m-n}^1 ** F_n(x,y). \end{aligned}$$

The result when $m < n$ follows analogously. □

The results of Theorem 3.1 follow from Theorem 3.2 (or vice-versa) by noting that, for $m, n = 0,1,\ldots$,

(i) $$q_{m,n}(x,y) = \sum_{r=m}^{\infty} \sum_{s=n}^{\infty} p_{r,s}(x,y) \quad ,$$

(ii) $$p_{m,n}(x,y) = q_{m+1,n+1}(x,y) - q_{m,n+1}(x,y) - q_{m+1,n}(x,y) + q_{m,n}(x,y) \ .$$

3.2 The joint probability generating function of $(N_x^{(1)}, N_y^{(2)})$.

The expressions given in Theorem 3.2 for the "tail probabilities" of $(N_x^{(1)}, N_y^{(2)})$ are much easier to handle than those given by Theorem 3.1 for $p_{r,s}(x,y)$. For this reason we use the results of Lemma 2.3 to find the j.p.g.f. of $(N_x^{(1)}, N_y^{(2)})$.

Define

$$P(x,y;\ s_1,s_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n}(x,y)\ s_1^m\ s_2^n ,$$

and

$$Q(x,y;\ s_1,s_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_{m,n}(x,y)\ s_1^m\ s_2^n .$$

Theorem 3.3:

For $|s_1| < 1,\ |s_2| < 1$,

$$(3.7)\quad Q(x,y;s_1,s_2) = [F_0 + \sum_{i=1}^{\infty} s_1^i\ F_i^1 + \sum_{j=1}^{\infty} s_2^j\ F_j^2] ** [\sum_{r=0}^{\infty} (s_1\ s_2)^r\ F_r(x,y)] ,$$

$$(3.8)\quad s_1 s_2\ P(x,y;s_1,s_2)$$

$$= (1-s_1)(1-s_2)[F_0 + \sum_{i=1}^{\infty} s_1^i\ F_i^1 + \sum_{i=1}^{\infty} s_2^j\ F_j^2] ** [\sum_{r=1}^{\infty} (s_1 s_2)^r\ F_r(x,y)]$$

$$- s_2(1-s_1) \sum_{i=1}^{\infty} s_1^i\ F_i^1\ (x,y) - s_1(1-s_2) \sum_{j=1}^{\infty} s_2^j\ F_j^2(x,y) + s_1 s_2\ F_0(x,y) .$$

Proof:

$$Q(x,y;s_1,s_2) = \sum_{r=0}^{\infty} q_{r,r}(x,y)\ s_1^r\ s_2^r$$

$$+ \sum_{r=0}^{\infty} \sum_{i=1}^{\infty} q_{r+i,r}(x,y)\ s_1^{r+i}\ s_2^r$$

$$+ \sum_{r=0}^{\infty} \sum_{j=1}^{\infty} q_{r,r+j}(x,y)\ s_1^r\ s_2^{r+j} .$$

Equation (3.7) follows immediately on substitution from (3.6).

Equation (3.8) follows from (2.3) following substitution and simplification using the results

$$Q(x,y;s_1,0) = [F_0 + \sum_{i=1}^{\infty} s_1^i\ F_i^1] ** F_0(x,y) ,$$

$$Q(x,y;0,s_2) = [F_0 + \sum_{j=1}^{\infty} s_2^j\ F_j^2] ** F_0(x,y) . \qquad \square$$

Note that

$$P(x,y;s_1,1) = \sum_{m=0}^{\infty} P\{N_x^{(1)} = m\} s_1^m \equiv P_1(x;s_1) ,$$

$$P(x,y;1,s_2) = \sum_{n=0}^{\infty} P\{N_y^{(2)} = n\} s_2^n \equiv P_2(y;s_2)$$

gives the marginal p.g.f.'s for $N_x^{(1)}$ and $N_y^{(2)}$ respectively.

Define, analogously,

$$Q_1(x;s_1) = Q(x,y;s_1,0) \quad \text{and} \quad Q_2(y,s_2) = Q(x,y;0,s_2).$$

From Theorem 3.3 we obtain immediately:

Corollary 3.3.1:

(3.9) $$P_1(x;\ s_1) = 1 + (s_1-1) \sum_{n=1}^{\infty} F_n^1(x)\ s_1^{n-1} ,$$

(3.10) $$P_2(y;\ s_2) = 1 + (s_2-1) \sum_{n=1}^{\infty} F_n^2(y)\ s_2^{n-1} ,$$

(3.11) $$Q_1(x;\ s_1) = \sum_{i=0}^{\infty} s_2^j\ F_i(x,\infty) ,$$

(3.12) $$Q_2(y;\ s_2) = \sum_{j=0}^{\infty} s_2^j\ F_j(\infty,y) .$$

It should be remarked that (3.9) and (3.10) giving the p.g.f.'s for the marginal distribution of $N_x^{(1)}$ and $N_y^{(2)}$ are well known. See, for example, Cox [1, p. 37].

3.3 The Laplace transform of the joint p.g.f. of $(N_x^{(1)}, N_y^{(2)})$.

In the discussion associated with the paper by W.L. Smith [8, p. 285], Bartlett derives a portmanteau formula for univariate renewal processes - basically an expression for the Laplace transform of the p.g.f. of $N_x^{(1)}$.

A similar formula is given by Cox [1, p. 37]. We derive an analogous expression for the bivariate L.T. of the j.p.g.f. of $(N_x^{(1)}, N_y^{(2)})$. Once again a preferable method is via the "tail probabilities."

Let
$$P^{\circ}(p,q;\ s_1,s_2) = L^2\{P(x,y;\ s_1,s_2)\} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p^{\circ}_{m,n}(p,q)\ s_1^m\ s_2^n\ ,$$
and
$$Q^{\circ}(p,q;\ s_1,s_2) = L^2\{Q(x,y;\ s_1,s_2)\}\ .$$

Theorem 3.4:

For $|s_1| < 1$, $|s_2| < 1$,

$$Q^{\circ}(p,q;\ s_1,s_2) = \frac{[1-s_1 s_2 f^{\circ}(p,0)f^{\circ}(0,q)]}{pq[1-s_1 s_2 f^{\circ}(p,q)][1-s_1 f^{\circ}(p,0)][1-s_2 f^{\circ}(0,q)]}\ , \tag{3.13}$$

$$P^{\circ}(p,q;\ s_1,s_2) = \frac{(1-s_1)(1-s_2)[f^{\circ}(p,q) - f^{\circ}(p,0)f^{\circ}(0,q)]}{pq[1-s_1 s_2 f^{\circ}(p,q)][1-s_1 f^{\circ}(p,0)][1-s_2 f^{\circ}(0,q)]} + \frac{[1-f^{\circ}(p,0)][1-f^{\circ}(0,q)]}{pq[1-s_1 f^{\circ}(p,0][1-s_2 f^{\circ}(0,q)]}\ . \tag{3.14}$$

Proof:

Equation (3.13) follows from (3.7) and Lemma 2.8 and simplifying by summing the geometric series.

To obtain (3.14) first note that from (2.3) we may write

$$s_1 s_2\ P^{\circ}(p,q;\ s_1,s_2) = (1-s_1)(1-s_2)\ Q^{\circ}(p,q;\ s_1,s_2) - (1-s_1)\ Q^{\circ}(p,q;\ s_1,0) - (1-s_2)Q^{\circ}(p,q;0,s_2) + 1/pq.$$

Substitution using (3.13) gives the required result. □

Concerning the univariate L.T. of the p.g.f.'s of $N_x^{(1)}$ and $N_y^{(2)}$ note that

$$P^o(p,q;\ s_1,1) = L^2\{P_1(x;s_1)\}$$
$$\frac{1}{q}L\{P_1(x;s_1)\} = \frac{1}{q}\ P_1^o(p;s_1)$$

Similarly,

$$P^o(p,q;\ 1,s_2) = \frac{1}{p}P_2^o(q;s_2)\ .$$

Thus from (2.14), and (2.15) and (2.16) of Lemma 2.6, we obtain the following:

Corollary 3.4.1:

(3.15) $$P_1^o(p;\ s_1) = \frac{1 - f^{1o}(p)}{p[1 - s_1 f^{1o}(p)]}\ , \quad |s_1| \leq 1\ ;$$

(3.16) $$P_2^o(q;\ s_2) = \frac{1 - f^{2o}(q)}{q[; - s_2 f^{2o}(q)]}\ , \quad |s_2| \leq 1\ .$$

These are the well known portmanteau formulae referred to previously, e.g. [1, p. 37].

We find it convenient to consider the univariate L.T. of the marginal g.f.'s for the tail probabilities.

$$Q^o(p,q;\ s_1,0) = L^2\{Q(x,y;\ s_1,0)\} = L^2\{Q_1(x;\ s_1)\}$$

$$= \frac{1}{q}\ L\{Q_1(x;\ s_1)\} = \frac{1}{q}\ Q_1^o(p;\ s_1)$$

Similarly

$$Q_0(p,q;0,s_2) = \frac{1}{p}\ Q_2^o(p;s_2)\ .$$

Equation (2.13) and Lemma 1.6 now yield:

Corollary 3.4.2:

(3.17) $$Q_1^{o}(p;\ s_1) = \frac{1}{p[1 - s_1 f^{1o}(p)]} \quad , \quad |s_1| < 1 \ ;$$

(3.18) $$Q_2^{o}(q;\ s_2) = \frac{1}{q[1 - s_2 f^{2o}(q)]} \quad , \quad |s_2| < 1 \ .$$

3.4 The marginal renewal functions and renewal densities.

The moments of $(N_x^{(1)}, N_y^{(2)})$ for low orders are given by the following:

Theorem 3.5: For all $x \geq 0$, $y \geq 0$

(3.19) $$EN_x^{(1)} = \sum_{i=1}^{\infty} F_i(x,\infty) \quad ,$$

(3.20) $$EN_y^{(2)} = \sum_{j=1}^{\infty} F_j(\infty,y) \quad ,$$

(3.21) $$EN_x^{(1)}N_y^{(2)} = [F_0 + \sum_{i=1}^{\infty} F_i^1 + \sum_{j=1}^{\infty} F_j^2] ** [\sum_{r=1}^{\infty} F_r(x,y)] \quad ,$$

(3.22) $$\mathrm{Cov}(N_x^{(1)},N_y^{(2)}) = [F_0 + \sum_{i=1}^{\infty} F_i^1 + \sum_{j=1}^{\infty} F_j^2] ** [\sum_{r=1}^{\infty} F_r(x,y)] - [\sum_{i=1}^{\infty} F_i(x,\infty)][\sum_{j=1}^{\infty} F_j(\infty,y)] \ .$$

Proof:

The results of this theorem follow from Lemma 2.5 and Theorem 3.3.

In particular note that

$$EN_x^{(1)} = \sum_{i=1}^{\infty} i p_{ij}(x,y) = \sum_{i=1}^{\infty} q_{i0}(x,y)$$

and

$$EN^{(2)} = \sum_{j=1}^{\infty} j p_{ij}(x,y) = \sum_{j=1}^{\infty} q_{0j}(x,y) \ . \qquad \square$$

The functions $H_1(x) \equiv EN_x^{(1)}$ and $H_2(y) \equiv EN_y^{(2)}$ are both univariate *renewal functions* and have been studied extensively. (cf. Smith, [8]).

From (3.19) one can derive the well known "*integral equation of renewal theory*" (cf. Smith [8, p. 252], viz.

$$H_1(x) = F^1(x) + \int_0^x H_1(x-u)dF^1(u) \quad , \tag{3.23}$$

and also an expression for the L.S.T. of $H_1(x)$, viz.

$$H_1^*(p) = \frac{F^{1*}(p)}{1 - F^{1*}(p)} \quad . \tag{3.24}$$

When $F^1(x)$ is absolutely continuous one can differentiate $H_1(x)$ and obtain $h_1(x) = \frac{d}{dx} H_1(x) = \sum_{n=1}^{\infty} f_n^1(x)$, the *renewal density* for the X-renewals. From (3.23) one can infer the *renewal density integral equation*

$$h_1(x) = f^1(x) + \int_0^x h_1(x-u)f^1(u)\, du \tag{3.25}$$

from which one obtains an expression for the L.T. of $h_1(x)$,

$$h_1^{\circ}(p) = \frac{f^{1\circ}(p)}{1 - f^{1\circ}(p)} \quad . \tag{3.26}$$

Analogous results follow for the Y-renewals, in particular

$$H_2^*(q) = \frac{F^{2*}(q)}{1 - F^{2*}(q)} \quad , \tag{3.27}$$

and

$$h_2^{\circ}(q) = \frac{f^{2\circ}(q)}{1 - f^{2\circ}(q)} \quad . \tag{3.28}$$

Since the marginal renewal functions $H_1(x)$ and $H_2(y)$ are concerned with the first order moments of $N_x^{(1)}$ and $N_y^{(2)}$ there is no joint renewal function for $N_x^{(1)}$ and $N_y^{(2)}$. However, as a measure of the dependence of $N_x^{(1)}$ and

$N_y^{(2)}$ we could consider, for $x,y \geq 0$,

$$K(x,y) \equiv \text{cov}(N_x^{(1)}, N_y^{(2)}) .$$

The result given by (3.22) for $K(x,y)$ was obtained from (2.11) and (3.7), i.e.

$$K(x,y) = Q(x,y;\ 1,1) - Q_1(x;1)Q_2(y;1)$$

and thus

$$K^o(p,q) = L^2\{K(x,y)\} = Q^o(p,q;1,1) - Q_1^o(p;1)Q_2^o(q;1)$$

(3.29) $$= \frac{f^o(p,q) - f^o(p,0)f^o(0,q)}{pq[1-f^o(p,q)][1-f^o(p,0)][1-f^o(0,q)]}$$

3.5 Independence of $N_x^{(1)}$ and $N_y^{(2)}$.

We conclude this chapter with an interesting result concerning the independence of the r.v.'s $N_x^{(1)}$ and $N_y^{(2)}$. (In the proof of the following theorem we have assumed that $F(x,y)$, the joint distribution of each (X_n,Y_n) is absolutely continuous. This restriction may be removed by using bivariate L.S.T.'s rather than bivariate L.T.'s).

Theorem 3.6:

The following conditions are equivalent.

(i) X_1 and Y_1 are independent.

(ii) $\text{cov}(N_x^{(1)}, N_y^{(2)}) = 0$ for all $x \geq 0$, $y \geq 0$.

(iii) $N_x^{(1)}$ and $N_y^{(2)}$ are independent for all $x \geq 0$, $y \geq 0$.

Proof:

X_1 and Y_1 are independent

iff $F(x,y) = F^1(x)F^2(y)$, for all x,y ;

iff $f(x,y) = f^1(x)f^2(y)$, for all x,y, (a.e.);

iff $f^o(p,q) = f^{1o}(p)f^{2o}(q) = f^o(p,0)f^o(0,q)$.

Now $K(x,y) = \text{cov}(N_x^{(1)}, N_y^{(2)}) = 0$, for all $x \geq 0$, $y \geq 0$,

iff $K^o(p,q) = 0$,

iff $f^o(p,q) = f^o(p,0)f^o(0,q)$, (from (3.29)).

Similarly, $N_x^{(1)}$ and $N_y^{(2)}$ are independent for all $x \geq 0$, $y \geq 0$,

iff $Q(x,y,s_1,s_2) = Q_1(x;s_1)Q_2(y;s_2)$ for $|s_1| < 1$, $|s_2| < 1$ (from Lemma 2.4),

iff $Q^o(p,q;s_1,s_2) - Q_1^o(p;s_1)Q_2^o(q;s_2) = 0$,

iff
$$\frac{s_1s_2[f^o(p,q) - f^o(p,0)f^o(0,q)]}{pq[1-s_1s_2f^o(p,q)][1-s_1f^o(p,0)][1-s_2f^o(0,q)]} = 0 ,$$

iff
$$f^o(p,q) = f^o(p,0)f^o(0,q) \quad ;$$

and the results follow. □

4. TWO DIMENSIONAL RENEWAL COUNTING PROCESSES

4.1 The distribution of N_{xy}.

Theorem 4.1:

For $x,y \geq 0$, and $k \geq 0$,

(4.1) $$P\{N_{x,y} = k\} = F_k(x,y) - F_{k+1}(x,y).$$

Proof:

This result can be proven directly from the joint distribution of $N_x^{(1)}$ and $N_y^{(2)}$, since

$$P\{N_{x,y} = k\} = P\{\min(N_x^{(1)}, N_y^{(2)}) = k\}$$

$$= \sum_{\min(i,j)=k}\sum p_{ij}(x,y)$$

$$= p_{k,k}(x,y) + \sum_{n=1}^{\infty} p_{k,k+n}(x,y) + \sum_{n=1}^{\infty} p_{k+n,n}(x,y) \ .$$

Equation (4.1) follows upon substitution for the $p_{i,j}(x,y)$ from Theorem 3.1.

Alternatively, we may consider the "tail probabilities" of $N_{x,y}$.

(4.2) $$\{N_{xy} \geq n\} = \{\min(N_x^{(1)}, N_y^{(2)}) \geq n\}$$

$$= \{N_x^{(1)} \geq n, N_y^{(2)} \geq n\}$$

$$= \{S_n^{(1)} \leq x, S_n^{(2)} \leq y\}$$

Thus

(4.3) $$P\{N_{x,y} \geq n\} = q_{n,n}(x,y) = F_n(x,y) \ ,$$

and (3.1) follows. □

4.2 The probability generating function of $N_{x,y}$.

As was the case for the joint p.g.f. of $N_x^{(1)}$ and $N_x^{(2)}$ we find it expedient to consider also the (univariate) g.f. for the tail probabilities. Define

$$\Pi(x,y;s) = \sum_{k=0}^{\infty} P\{N_{x,y} = k\}s^k$$

and

$$\Theta(x,y;s) = \sum_{k=0}^{\infty} P\{N_{x,y} \geq k\}s^k \ .$$

Theorem 4.2:

For $|s| < 1$, and $x,y \geq 0$,

(4.4) $$\Theta(x,y;s) = \sum_{k=0}^{\infty} s^k F_k(x,y) \ ,$$

(4.5) $$\Pi(x,y;s) = 1 + (s-1)\sum_{k=1}^{\infty} s^{k-1} F_k(x,y) .$$

Proof:

Equation (4.4) comes directly from (4.3) while (4.5) is derived by utilizing (2.6) and (4.4). □

One should note the similarity between the expressions for the p.g.f.'s of $N_x^{(1)}$ and $N_y^{(2)}$ given by equations (3.9) and (3.10) and the p.g.f. of N_{xy} given by (4.4). In fact, we note that

$$\Pi(x,\infty;s) = P_1(x;s) \quad \text{and} \quad \Pi(\infty,y;s) = P_2(y;s).$$

Thus from the p.g.f. of N_{xy} we can obtain the p.g.f.'s of $N_x^{(1)}$ and $N_y^{(2)}$.

Similarly, for tail probabilities of N_{xy} we observe that

$$\Theta(x,\infty;s) = Q_1(x;s) \quad \text{and} \quad \Theta(\infty,y;s) = Q_2(y;s) .$$

4.3 The Laplace transform of the p.g.f. of N_{xy}.

Let assume $F(x,y)$ has a bivariate p.d.f. $f(x,y)$ and define

$$\Pi^{\circ}(p,q;s) = L^2\{\Pi(x,y;s)\}$$

and

$$\Theta^{\circ}(p,q;s) = L^2\{\Theta(x,y;s)\}.$$

Theorem 4.3:

For $|s| < 1$

(4.6) $$\Theta^{\circ}(p,q;s) = \frac{1}{pq[1 - sf^{\circ}(p,q)]} ,$$

(4.7) $$\Pi^{\circ}(p,q;s) = \frac{1 - f^{\circ}(p,q)}{pq[1 - sf^{\circ}(p,q)]} .$$

Proof:

$$\Theta^{\circ}(p,q;s) = \sum_{k=0}^{\infty} s^k L^2\{F_k(x,y)\}, \quad \text{(from (4.4))}$$

$$= \frac{1}{pq}\sum_{k=0}^{\infty} s^k[f^{\circ}(p,q)]^k, \quad \text{(from (2.28))}$$

and (4.6) follows. Equation (4.7) follows analogously, or may be derived by observing that (2.6) gives

$$(1-s)\Theta^{o}(p,q;s) = 1/pq - s\Pi^{o}(p,q;s) \ . \qquad \square$$

Equation (4.7) is the portmanteau formula for two dimensional renewal processes and one should note its similarity with (3.15) and (3.16), its one dimensional analogs.

One can also show that

$$\lim_{q\to 0} [q\Pi^{o}(p,q;s)] = P_1^{o}(p;s)$$

and, similarly

$$\lim_{p\to 0} [p\Pi^{o}(p,q;s)] = P_2^{o}(q;s) \ .$$

4.4 <u>The two dimensional renewal function and renewal density</u>.

$N_{x,y}$ is the number of two dimensional (X,Y)-renewals in the closed rectangular region in the plane with corners at the points (0,0), (x,0), (x,y) and (0,y). In analogy with the univariate theory we define the *two dimensional renewal function*, $H(x,y) = EN_{x,y}$.

<u>Theorem 4.4</u>:

For all $x \geq 0$, $y \geq 0$,

$$H(x,y) = \sum_{k=1}^{\infty} F_k(x,y) \ . \tag{4.8}$$

<u>Proof</u>:

There are various ways of establishing this result, in particular note that

$$\begin{aligned} EN_{x,y} &= \sum_{k=1}^{\infty} k \, P\{N_{x,y} = k\} \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \min(i,j) p_{ij}(x,y) \\ &= \sum_{k=1}^{\infty} q_{k,k}(x,y) \ . \qquad \square \end{aligned}$$

Just as the p.g.f. of $N_{x,y}$ provides information concerning $N_x^{(1)}$ and $N_y^{(2)}$, the univariate renewal functions can be obtained from $H(x,y)$, since from (3.19), (3.20) and (4.8)

$$H_1(x) = \sum_{i=1}^{\infty} F_i(x,\infty) = H(x,\infty) ,$$

$$H_2(y) = \sum_{j=1}^{\infty} F_j(\infty,y) = H(\infty,y) .$$

Furthermore, from (4.8), we observe that

$$H ** F(x,y) = \sum_{k=1}^{\infty} F_{k+1}(x,y) = H(x,y) - F(x,y),$$

from whence we obtain the "*integral equation of two dimensional renewal theory*" (cf. (3.23)),

(4.9) $$H(x,y) = F(x,y) + \int_0^x \int_0^y H(x-u,y-v)dF(u,v) .$$

From either (4.8) or (4.9) we can derive an expression for the L.S.T. of $H(x,y)$, viz.

(4.10) $$H^*(p,q) = \frac{F^*(p,q)}{1-F^*(p,q)} ,$$

analogous to (3.24) and (3.27) which give the univariate L.S.T.'s of $H_1(x)$ and $H_2(y)$.

It should be remarked that knowledge of the two dimensional renewal function $H(x,y)$ implies complete knowledge of all aspects of the two-dimensional renewal process. (cf. Smith [8, p. 254]). This is obvious from (4.10), and expressions for the higher order moments of $N_{x,y}$ can be found in terms of $H(x,y)$.

If we assume $F(x,y)$ to be an absolutely continuous d.f. with j.p.d.f. $f(x,y)$ then we define the *two dimensional renewal density function* $h(x,y) = \frac{\partial^2}{\partial x \partial y} H(x,y)$; i.e.,

(4.11) $$h(x,y) = \sum_{h=1}^{\infty} f_n(x,y) .$$

From (4.9) we obtain the "*two dimensional renewal density integral equation*"

(4.12) $$h(x,y) = f(x,y) + \int_0^x \int_0^y h(x-u,y-v)f(u,v)du\,dv \ .$$

Taking the bivariate L.T. of (4.12) we obtain

(4.13) $$h^{\circ}(p,q) = \frac{f^{\circ}(p,q)}{1-f^{\circ}(p,q)} \ .$$

Note that (2.15) and (3.26) imply

$$h^{\circ}(p,0) = \frac{f^{1\circ}(p)}{1-f^{1\circ}(p)} = h_1^{\circ}(p) \ .$$

Similarly,

$$h^{\circ}(0,q) = \frac{f^{2\circ}(q)}{1-f^{2\circ}(q)} = h_2^{\circ}(q) \ .$$

Thus $h^{\circ}(p,q)$ contains all the information concerning the (univariate) L.T.'s of both $h_1(x)$ and $h_2(y)$, the univariate renewal densities.

In fact, since

$$h^{\circ}(p,0) = \int_0^{\infty} e^{-px}\left(\int_0^{\infty} h(x,y)dy\right) dx$$

$$= h_1^{\circ}(p) = \int_0^{\infty} e^{-px}h_1(x)\, dx$$

we may deduce that

$$h_1(x) = \int_0^{\infty} h(x,y)\, dy \ ,$$

and similarly

$$h_2(y) = \int_0^{\infty} h(x,y)\, dx \ .$$

These results could have been deduced directly from the definitions of $h_1(x)$, $h_2(y)$ and $h(x,y)$.

5. EXAMPLES

5.1 Independent exponential distribution.

Let (X_n, Y_n), for each $n = 1,2,\ldots$, be independent exponential r.v.s. with means a and b. Thus the j.p.d.f. of (X_1, Y_1) is given by

$$f(x,y) = \frac{1}{ab} \exp[-(\frac{x}{a} + \frac{y}{b})]; \quad x \geq 0,\ y \geq 0 ,$$

with
$$f^{o}(p,q) = \frac{1}{(1+ap)(1+bq)} .$$

For $n = 1,2,\ldots$, $S_n^{(1)}$ and $S_n^{(2)}$ are independent, respectively gamma (n,a) and gamma (n,b), r.v.'s with densities

$$f_n^1(x) = \frac{1}{\Gamma(n)} \frac{x^{n-1}}{a^n} \exp[-\frac{x}{a}]$$

and

$$f_n^2(y) = \frac{1}{\Gamma(n)} \frac{y^{n-1}}{b^n} \exp[-\frac{y}{b}] .$$

Thus

$$(5.1) \qquad f_n(x,y) = f_n^1(x)\, f_n^2(y)$$

and $F_n(x,y)$, their joint d.f. is given by

$$(5.2) \qquad F_n(x,y) = P(n, \tfrac{x}{a})\, P(n, \tfrac{y}{b})$$

where $P(n,x) = \frac{1}{\Gamma(n)} \int_0^x u^{n-1} e^{-u} du$, the incomplete gamma function.

From univariate theory it is well known that $N_x^{(1)}$ is a Poisson $(\frac{x}{a})$ r.v. and $N_y^{(2)}$ is a Poisson $(\frac{y}{b})$ r.v. Hence, by Theorem 3.6, or otherwise we can conclude that $(N_x^{(1)}, N_y^{(2)})$ are independently distributed.

Note that $f^{o}(p,0) = \frac{1}{1+ap}$, and $f^{o}(0,q) = \frac{1}{1+bq}$ and that $f^{o}(p,q) = f^{o}(p,0)\, f^{o}(0,q)$. Thus the bivariate L.T. of $P(x,y;s_1,s_2)$, the j.p.g.f. of $N_x^{(1)}$ and $N_y^{(2)}$, is given, from Theorem 3.4, by

$$p^{o}(p,q;s_1,s_2) = \frac{ab}{(1-s_1+ap)(1-s_2+bq)} .$$

From this bivariate L.T., or otherwise, we can deduce that

$$p_{r,s}(x,y) = \frac{(\frac{x}{a})^r \ (\frac{y}{b})^s}{r! \ s!} \ \exp[-(\frac{x}{a}+\frac{y}{b})] .$$

Concerning the distribution of $N_{x,y}$ we have from Theorem 4.3 that the bivariate L.T. of the p.g.f. of $N_{x,y}$ is given by

$$\Pi^{o}(p,q;s) = \frac{ap + bq + abpq}{pq[1 - s + ap + bq + abpq]} .$$

Inversion of this bivariate L.T., or more directly, using Theorem 4.1 and (5.2), gives

$$P\{N_{x,y} = k\} = P(k, \frac{x}{a})P(k, \frac{y}{b}) - P(k+1,\frac{x}{a})P(k+1,\frac{y}{b}) .$$

The two dimensional renewal density $h(x,y)$ can be evaluated by inverting the bivariate L.T.

(5.3) $$h^{o}(p,q) = \frac{1}{abpq + ap + bq} .$$

The tables in Voekler and Doetsch [10, p. 209] give

(5.4) $$h(x,y) = \frac{1}{ab} \exp[-(\frac{x}{a}+\frac{y}{b})] \ I_0\left(2\sqrt{\frac{xy}{ab}}\right) ,$$

where $I_0(x) = J_0(ix)$ is the modified Bessel function of the first kind of zero order. Using the series expansion, for the k-th order function,

$$I_k(z) = \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{2r+k}}{r!\Gamma(k+r+1)!} ,$$

with $k = 0$; or directly from (4.11); we have the alternative expression

(5.5) $$h(x,y) = \frac{1}{ab} \exp[-(\frac{x}{a} + \frac{y}{b})] \sum_{r=0}^{\infty} (\frac{xy}{ab})^r \frac{1}{(r!)^2} .$$

The two dimensional renewal function $H(x,y)$ can be evaluated directly using Theorem 4.4, viz.

(5.6) $$H(x,y) = \sum_{k=1}^{\infty} P(k, \frac{x}{a}) P(k, \frac{y}{b}) .$$

Using the result that

$$P(k,x) = \sum_{r=k}^{\infty} \frac{e^{-x} x^r}{r!} ,$$

(5.6) can be expressed as

(5.7) $$H(x,y) = \exp[-(\frac{x}{a} + \frac{y}{b})] \sum_{k=1}^{\infty} \{ \sum_{i=k}^{\infty} \frac{(\frac{x}{a})^i}{i!} \} \{ \sum_{j=k}^{\infty} \frac{(\frac{y}{b})^j}{j!} \} .$$

5.2 Bivariate exponential distribution.

Various bivariate exponential distributions have been proposed (see Johnson and Kotz [4, Chapter 41]) but not all have a j.p.d.f. and most involve severe restrictions on the correlation between the two r.v.'s. The bivariate exponential distribution used in this example suffers little from these restrictions and has other desirable characteristics. It is constructed using a slight modification of the distribution proposed by Moran [5].

Let $X_1 = \frac{a}{2}(U_1^2 + U_2^2)$ and $Y_1 = \frac{b}{2}(U_3^2 + U_4^2)$

where U_1, U_2, U_3, U_4 are all unit normal r.v.'s; (U_1, U_3) and (U_2, U_4) are mutually independent, but each pair has a bivariate normal distribution with correlation ω.

The joint characteristic function is given by

(5.8) $$E\, e^{it_1X_1+it_2Y_1} = \frac{1}{(1-iat_1)(1-ibt_2)+\omega^2 t_1 t_2} .$$

This can be inverted (cf. Moran [5]) to give the joint p.d.f. of (X_1,Y_1) as

$$f(x,y) = \sum_{n=0}^{\infty} \omega^{2n} g_n(x,y)$$

where

$$g_n(x,y) = \frac{1}{ab} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{j!} \left(\frac{x}{a}\right)^j e^{-x/a} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} \left(\frac{y}{b}\right)^k e^{-y/b} .$$

We can also obtain alternative expressions for $f(x,y)$. From (5.8) we obtain the bivariate L.T. of $f(x,y)$ as

$$f^{o}(p,q) = E\, e^{-pX_1 - qY_1} = \frac{1}{1+ap+bq+(1-\omega^2)abpq} .$$

This may be inverted using the tables in Voelker and Doetsch [10, p. 209] to yield the j.p.d.f.

$$f(x,y) = \frac{1}{ab(1-\omega^2)} \exp\left[-\frac{1}{(1-\omega^2)}\left(\frac{x}{a} + \frac{y}{b}\right)\right] I_0 \left(\frac{2\omega}{1-\omega^2}\sqrt{\frac{xy}{ab}}\right) . \tag{5.9}$$

This distribution has also been discussed by Nagao and Kadoga [6] who also give details as to the estimation of the parameters.

Another expression for $f(x,y)$ has been given by Vere-Jones [9] using Laguerre polynomials. (See (5.11) to follow with $n = 1$).

Since $f^{1o}(p) = (1+ap)^{-1}$ and $f^{2o}(q) = (1+bq)^{-1}$ we can deduce that the marginal densities $f^1(x)$ and $f^2(y)$ are both exponential with means a and b respectively.

The correlation between (X_1,Y_1), $\rho = \omega^2$.

This distribution generalizes the distribution given in Section 5.1 since when $\rho = 0$ the bivariate exponential distribution reduces to two independent exponential distributions. Furthermore, we can find explicit expressions for the j.p.d.f. of $(S_n^{(1)}, S_n^{(2)})$. Since

$$f_n^{o}(p,q) = \frac{1}{[1+ap+bq+(1-\rho)abpq]^n}$$

inversion using the tables of Voekler and Doetsch [10, p. 235] gives

$$(5.10)\qquad f_n(x,y) = \frac{1}{ab(1-\rho)(n-1)!}\left[\frac{xy}{\rho ab}\right]^{\frac{n-1}{2}} \exp\left[-\frac{1}{1-\rho}\left(\frac{x}{a}+\frac{y}{b}\right)\right] I_{n-1}\left(\frac{2\sqrt{\rho}}{1-\rho}\sqrt{\frac{xy}{ab}}\right)$$

$$= f_n^1(x) f_n^2(y) \frac{(n-1)!}{(1-\rho)}\left[\frac{\rho xy}{ab}\right]^{\frac{-(n-1)}{2}} \exp\left[\frac{-\rho}{1-\rho}\left(\frac{x}{a}+\frac{y}{b}\right)\right] I_{n-1}\left(\frac{2\sqrt{\rho}}{1-\rho}\sqrt{\frac{xy}{ab}}\right),$$

where $f_n^1(x)$ and $f_n^2(y)$ are gamma (n,a) and gamma (n,b) p.d.f.'s respectively, the marginal densities of $S_n^{(1)}$ and $S_n^{(2)}$.

Another expression for $f_n(x,y)$ can be derived from (5.10) by using generalized Laguerre polynomials $L_k^{(\alpha)}(z)$ and the Hille-Hardy formula (Szego [7, p. 101], cf. Vere-Jones [9]) viz.

$$(5.11)\qquad f_n(x,y) = f_n^1(x) f_n^2(y) \sum_{k=0}^{\infty} \frac{(n-1)!k!}{(k+n-1)!} L_k^{(n-1)}\left(\frac{x}{a}\right) L_k^{(n-1)}\left(\frac{y}{b}\right) \rho^k .$$

An explicit expression for $F_n(x,y)$ can be found from (5.11) using the fact (Szego [7, p. 100]) that

$$L_k^{(n-1)}(t) = \sum_{r=0}^{k} \binom{k+n-1}{k-r} \frac{(-t)^r}{r!} ,$$

to give

$$(5.12)\qquad F_n(x,y) = \sum_{k=0}^{\infty} \binom{k+n-1}{k} \rho^k \left\{\sum_{i=0}^{k} \binom{k}{i} (-1)^i P\left(n+i, \frac{x}{a}\right)\right\}\left\{\sum_{j=0}^{k} \binom{k}{j} (-1)^j P\left(n+j, \frac{y}{b}\right)\right\} .$$

Note that (5.12) reduces to (5.2) when $\rho = 0$, as expected.

Explicit expressions for the joint distribution of $N_x^{(1)}$ and $N_y^{(2)}$ are difficult to obtain but the bivariate L.T. of the j.p.g.f. of $(N_x^{(1)}, N_y^{(2)})$ is easily derived, viz.

$$P^O(p,q;s_1,s_2) = \frac{\rho ab(1-s_1)(1-s_2)}{[1-s_1 s_2+ap+bq+(1-\rho)abpq][1-s_1+ap][1-s_2+bq]}$$

$$+ \frac{ab}{[1-s_1+ap][1-s_2+bq]} .$$

Theoretically this can be inverted and the coefficients of $s_1^r s_2^s$ extracted to find $p_{rs}(x,y)$. The marginal distributions are both Poisson, as in Example 5.1.

The joint moments of $N_x^{(1)}$ and $N_y^{(2)}$ are of interest and we obtain $K^o(p,q) = L^2\{cov(N_x^{(1)}, N_y^{(2)}\}$ from (3.29).

(5.13) $$K^o(p,q) = \frac{\rho}{pq[ap+bq+(1-\rho)abpq]} .$$

We shall see later that, for this example, $cov(N_x^{(1)}, N_y^{(2)})$ is related to the two dimensional renewal function $EN_{x,y}$.

The bivariate L.T. of the p.g.f. of $N_{x,y}$ is given by

$$\Pi^o(p,q;s) = \frac{ap+bq+(1-\rho)abpq}{pq[1-s+ap+bq+(1-\rho)abpq]} .$$

This can be inverted, with some difficulty, but useful expressions for $P\{N_{xy} = k\}$ can be obtained by using (4.1) and (5.12).

A simple expression for $h(x,y)$, the two dimensional renewal density is obtained by inverting

(5.14) $$h^o(p,q) = \frac{1}{ap+bq+(1-\rho)abpq}$$

using the tables of Voekler and Doetsch [10, p. 209] to give

(5.15) $$h(x,y) = \frac{1}{ab(1-\rho)} \exp[-\frac{1}{1-\rho}(\frac{x}{a} + \frac{y}{b})] I_0 \left(\frac{2}{1-\rho}\sqrt{\frac{xy}{ab}}\right) .$$

Using the series expansion for $I_0(z)$ we may write

(5.16) $$h(x,y) = \frac{1}{ab(1-\rho)} \exp[-\frac{1}{1-\rho}(\frac{x}{a} + \frac{y}{b})] \sum_{r=0}^{\infty} [\frac{xy}{ab(1-\rho)^2}]^r \frac{1}{(r!)^2} .$$

Expressions (5.14), (5.15), (5.16) are the correlated analogs of (5.2), (5.3), (5.4) which follow when $\rho = 0$.

There are various ways in which we can find expressions for the two dimensional renewal function. We can invert the bivariate L.T.

(5.17) $$H^{o}(p,q) = \frac{1}{pq}\,\frac{1}{[ap+bq+(1-\rho)abpq]} ,$$

or we can use (4.8) in conjunction with (5.12). However, the simplest expression is obtained by using (5.16) and the fact that

$$H(x,y) = \int_0^x \int_0^y h(u,v)\,du\,dv$$

to obtain

$$H(x,y) = (1-\rho)\sum_{n=1}^{\infty} P(n, \frac{x}{(1-\rho)a})\, P(n, \frac{y}{(1-\rho)b}) .$$

We remarked earlier that we can, for this example, relate $\mathrm{cov}(N_x^{(1)}, N_y^{(2)})$ to $EN_{x,y}$. We note that (5.13) and (5.17) lead to the result that

$$K^{o}(p,q) = \rho H^{o}(p,q)$$

and thus $$\mathrm{cov}(N_x^{(1)}, N_y^{(2)}) = \rho EN_{x,y} .$$

Since $N_x^{(1)}$ and $N_y^{(2)}$ are each marginally distributed as Poisson $(\frac{x}{a})$ and Poisson $(\frac{y}{b})$ r.v.'s respectively, var $N_x^{(1)} = \frac{x}{a}$ and var $N_y^{(2)} = \frac{y}{b}$. Thus the correlation between $N_x^{(1)}$ and $N_y^{(2)}$ is given by

$$\mathrm{corr}(N_x^{(1)}, N_y^{(2)}) = \rho\sqrt{\frac{ab}{xy}}\; H(x,y) .$$

REFERENCES

[1] Cox, D,R, (1962). *Renewal theory.* London: Methuen.

[2] Ditkin, V.A. and Prudnikov, A.P. (1962). *Operational calculus in two variables and its applications.* Translated by Wishart, D.M.G. London: Pergamon.

[3] Feller, W. (1968). *An introduction to probability theory and its applications*, Volume 1, Third edition. New York: Wiley.

[4] Johnson, N.L. and Kötz, S. (1972). *Distributions in Statistics: Continuous multivariate distributions.* New York: Wiley.

[5] Moran, P.A.P. (1967). "Testing for correlation between non-negative variates." *Biometrika*, 54, 385-394.

[6] Nagao, M. and Kadoya, M. (1971). "Two-variate exponential distribution and its numerical table for engineering application." *Bull. Disas. Prev. Res. Inst.*, Kyoto Univ., 20, 183-215.

[7] Szego, G. (1939). *Orthogonal polynomials.* Amer. Math. Soc. Coll. Publ. XXIII.

[8] Smith, W.L. (1958). "Renewal theory and its applications." *J.R. Statist. Soc.*, B, 20, 243-302.

[9] Vere-Jones, D. (1967). "The infinite divisibility of a bivariate gamma distribution." *Sankhyā*, A, 27, 421-422.

[10] Voekler, D. and Doetsch, G. (1950). *Die Zweidimensionale Laplace-transformation.* Basel: Verlag Birkhäuser.