Renormalization flow of bound states

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A renormalization group flow equation with a scale-dependent transformation of field variables gives a unified description of fundamental and composite degrees of freedom. In the context of the effective average action, we study the renormalization flow of scalar bound states which are formed out of fundamental fermions. We use the gauged Nambu–Jona-Lasinio model at weak gauge coupling as an example. Thereby, the notions of a bound state or fundamental particle become scale dependent, being classified by the fixed-point structure of the flow of effective couplings.

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I. INTRODUCTION

Bound states, as opposed to fundamental particles, are commonly thought of as derived quantities, in the sense that the properties of positronium or atoms can be computed from the known electromagnetic interactions of their constituents. The conceptual separation between bound states and fundamental particles is, however, not always so obvious. As an example, it has been proposed that the Higgs scalar can be viewed as a top-quark–top-antiquark bound state [1], with a compositeness scale much above the characteristic scale of electroweak symmetry breaking. The mass of the bound state (and therefore the scale of electroweak symmetry breaking) depends in this model on a free parameter characterizing the strength of a four-fermion interaction. For a bound-state mass or momentum near the compositeness scale Λ, all the usual properties of bound states are visible. If the Higgs boson mass is substantially smaller than Λ, however, the bound state behaves like a fundamental particle for all practical aspects relating to momentum scales sufficiently below the compositeness scale. Depending on the momentum scale, the particle can therefore appear either as a typical bound state or a fundamental particle. The scale dependence of the physical picture can be cast into the language of the renormalization group (RG) by considering a scale-dependent effective action. It should be possible to understand the issues related to bound states or composite fields in this context. In this paper we demonstrate how the effective behavior as bound state or ''fundamental particle'' in dependence on a parameter of the model can be understood within the exact renormalization group equation for the effective average action [2].

In strong interactions, bound states or composite fields play an essential role in the dynamics at low momenta. In particular, scalar quark-antiquark bound states are responsible for chiral symmetry breaking with the associated dynamics of the pions. Furthermore, it has been proposed that the condensation of a color octet composite field may lead to ''spontaneous breaking of color'' [3] with a successful phenomenology of the spectrum and interactions of the light pseudoscalars, vector mesons and baryons. In order to verify or falsify such a proposal and connect the parameters of an effective low-energy description to the fundamental parameters of QCD one needs a reliable connection between short and long distance within the RG approach. In such a formalism it is convenient to represent fundamental particles and bound states by fields on equal footing. For quark-antiquark bound states this can be achieved by partial bosonization. A first picture of the flow of bound states in the exact RG approach has been developed in [4]. A shortcoming of these initial proposals is the fact that the bosonization is typically performed at a fixed scale. In a RG picture it would seem more appropriate that the relation between the fields for composite and fundamental particles becomes scale dependent. Furthermore, the simple observation that a typical bound-state behavior should not lead to the same relevant (or marginal) parameters as in the case of fundamental particles has not been very apparent so far.

In this paper we propose a modified exact renormalization group equation which copes with these issues. The field variables themselves depend on the renormalization scale k. For this purpose we use k-dependent nonlinear field transformations [5,6]. As a consequence, partial bosonization can be performed continuously for all k. This yields a description where explicit four-quark interactions which have the same structure as those produced by the exchange of a bound state are absent for every scale k. These interactions are then completely accounted for by the exchange of composite fields. We will demonstrate this approach in a simple model, whereas the more formal aspects can be found in the Appendixes. As a result, we conclude that ''fundamental behavior'' is related to a flow governed by an infrared unstable fixed point with the appropriate relevant parameters. For the typical ''bound-state behavior'' such a fixed point does not govern the flow. The parameters characterizing the bound-state mass and interactions are rather determined by an infrared attractive (partial) fixed point and become therefore predictable as a function of the relevant or marginal parameters characterizing masses and interactions of other ''fundamental

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fields.'' As a consequence, the notions of bound state or fundamental particle become scale dependent, with a possible crossover from one behavior to another.

As a simplified model sharing many features of electroweak or strong interactions we consider the gauged Nambu–Jona-Lasinio (NJL) model [7] (with one flavor $N_F = 1$), with the action

$$S = \int d^4x \left[\bar{\psi} i \gamma^\mu (\partial_\mu + ieA_\mu)\psi + 2\lambda_{\rm NJL}(\bar{\psi}_R \psi_L)(\bar{\psi}_L \psi_R) + \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 \right]. \tag{1}$$

We consider here a small gauge coupling e. This model has two simple limits: For small $\lambda_{\rm NJL}$ we recover massless quantum electrodynamics (QED), whereas for large enough $\lambda_{\rm NJL}$ one expects spontaneous chiral symmetry breaking. The region of validity of perturbative electrodynamics can be established by comparing $\lambda_{\rm NJL}$ to the effective four-fermion interaction generated by box diagrams in the limit of vanishing external momenta:

$$\Delta \mathcal{L}_B = \frac{1}{4}\Delta\lambda(\bar{\psi}\gamma_\mu\gamma_5\psi)(\bar{\psi}\gamma_\mu\gamma_5\psi) = \Delta\lambda\left[2(\bar{\psi}_L\psi_R)(\bar{\psi}_R\psi_L) + \frac{1}{4}(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma_\mu\psi)\right]. \tag{2}$$

Since the box diagrams are infrared divergent in the chiral limit of vanishing electron mass, we have introduced a scale by implementing an infrared cutoff $\sim k$ in the propagators such that[1]

$$\Delta\lambda = 6e^4 \int \frac{d^4q}{(2\pi)^4}[q^2(1+r_B(q))]^{-2}[q^2(1+r_F(q))^2]^{-1} = \frac{9}{16\pi^2}\frac{e^4}{k^2}. \tag{3}$$

(The second equality holds for the particular cutoff functions r_B, r_F described in Appendix E.) As long as perturbation theory remains valid (small e), and $\lambda_{\rm NJL} \lesssim \Delta\lambda$, we do not expect the four-fermion interaction $\sim\lambda_{\rm NJL}$ to disturb substantially the physics of massless QED. [In this case $\lambda_{\rm NJL}$ is an irrelevant parameter in the renormalization group (RG) language.]

The spontaneous breaking of the chiral symmetry for $\lambda_{\rm NJL} > \lambda_{\rm c}$ has been studied by a variety of methods [7–10]. For strong four-fermion interactions the dominant physics can be described by a Yukawa interaction with an effective composite scalar field. The phase transition at $\lambda_{\rm NJL} = \lambda_{\rm c}$ is of second order. In the vicinity of this transition the composite scalar has all the properties usually attributed to a fundamental field. In particular, its mass is governed by a relevant parameter. In this paper we present a unified description of all these different features in terms of flow equations for the effective average action.

II. FLOW EQUATION FOR THE GAUGED NJL MODEL

Our starting point is the exact renormalization group equation for the scale-dependent effective action Γ_k in the form [2]

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{STr}\{\partial_t R_k(\Gamma_k^{(2)} + R_k)^{-1}\}. \tag{4}$$

The solution Γ_k to this equation interpolates between its boundary condition in the ultraviolet Γ_Λ, usually given by the classical action, and the effective action $\Gamma_{k=0}$, representing the generating functional of the one particle irreducible (1PI) Green's functions. This flow is controlled by the to some extent arbitrary positive function $R_k(q^2)$ that regulates the infrared fluctuations at a scale k and falls off quickly for $q^2 > k^2$. Indeed, the insertion $\partial_t R_k$ suppresses the contribution of modes with momenta $q^2 \gg k^2$. The operator ∂_t represents a logarithmic derivative $\partial_t = k(d/dk)$. The heart of the flow equation is the fluctuation matrix $\Gamma_k^{(2)}$ that comprises second functional derivatives of Γ_k with respect to all fields, and together with R_k it corresponds to the exact inverse propagator at a given scale k. The (super-)trace runs over momenta and all internal indices including momenta and provides appropriate minus signs for the fermionic sector.

For our study, we use the following simple truncation for the gauged NJL model including the scalars arising from bosonization [10] (Hubbard-Stratonovich transformation):

$$\Gamma_k = \int d^4x \Big\{ \bar{\psi} i \partial\!\!\!/ \psi + 2\bar{\lambda}_{\sigma,k}\bar{\psi}_{\rm R}\psi_{\rm L}\bar{\psi}_{\rm L}\psi_{\rm R} + Z_{\phi,k}\partial_\mu\phi^*\partial_\mu\phi + \bar{m}_k^2\phi^*\phi + \bar{h}_k(\bar{\psi}_{\rm R}\psi_{\rm L}\phi - \bar{\psi}_{\rm L}\psi_{\rm R}\phi^*) + \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 - e\bar{\psi}A\!\!\!/\psi \Big\}. \tag{5}$$

This truncation is sufficient for our purposes. For quantitative estimates some of the simplifications could be improved in future work. This concerns, in particular: setting the fermion and gauge-field wave function renormalization constants to 1, reducing an *a priori* arbitrary scalar potential to a pure mass term, skipping all vector, axialvector, etc. channels of the four-fermion interaction as well as all higher-order operators, neglecting the running of the gauge coupling e and dropping all higher-order derivative terms. Especially the gauge sector is treated insufficiently, although this is appropriate for small e; for simplicity, we use Feynman gauge, $\alpha = 1$. The running of the scalar wave function renormalization $Z_{\phi,k}$ will also not be studied explicitly; since $Z_{\phi,k}$ is zero for the bosonization of a point-like four-fermion interaction, we shall assume that it remains small in the region of interest.

Nevertheless, the essential points of how fermionic interactions may be translated into the scalar sector can be studied

[1] We note that $\Delta\lambda$ does not depend on the gauge-fixing parameter α.

in this simple truncation. Of course, the truncation is otherwise not supposed to reveal all properties of the system even qualitatively; in particular, the interesting aspects of the gauged NJL model at strong coupling [11–13] cannot be covered unless the scalar potential is generalized.

The truncation (5) is related to the bosonized gauged NJL model, if we impose the relation

$$\lambda_{\mathrm{NJL}} := \frac{1}{2} \frac{\overline{h}_\Lambda^2}{\overline{m}_\Lambda^2} \tag{6}$$

as a boundary condition at the bosonization scale Λ and $\overline{\lambda}_{\sigma,\Lambda}=0$, $Z_{\phi,\Lambda}=0$; it is this bosonization scale Λ that we consider as the ultraviolet starting point of the flow. In fact, the action (1) can be recovered by solving the field equation of ϕ as functional of $\psi,\overline{\psi}$ and reinserting the solution into Eq. (5).

Using the truncation (5), the flow equation (4) can be boiled down to first-order coupled differential equations for the couplings $\overline{m}_k^2$, $\overline{h}_k$ and $\overline{\lambda}_{\sigma,k}$. For this, we rewrite Eq. (4) in the form

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{STr}\, \tilde{\partial}_t \ln(\Gamma_k^{(2)} + R_k), \tag{7}$$

where the symbol $\tilde{\partial}_t$ specifies a formal derivative that acts only on the k dependence of the cutoff function R_k. Let us specify the elements of Eq. (4) more precisely:

$$(\Gamma_k^{(2)})_{ab} := \frac{\vec{\delta}}{\delta \Phi_a^T} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \Phi_b}, \quad \Phi = \begin{pmatrix} A \\ \phi \\ \phi^* \\ \psi \\ \overline{\psi}^T \end{pmatrix}, \tag{8}$$

$$\Phi^T = (A^T, \phi, \phi^*, \psi^T, \overline{\psi}).$$

Here $A \equiv A_\mu$ is understood as a column vector, and A^T denotes its Lorentz transposed row vector. For spinors the superscript T characterizes transposed quantities in Dirac space. The complex scalars ϕ and ϕ^* as well as the fermions $\overline{\psi}$ and ψ are considered as independent, but transposed quantities are not: e.g., Φ and Φ^T carry the same information.

Performing the functional differentiation, the fluctuation matrix can be decomposed as

$$\Gamma_k^{(2)} + R_k = \mathcal{P} + \mathcal{F}, \tag{9}$$

where $\mathcal{F}$ contains all the field dependence and $\mathcal{P}$ the propagators including the cutoff functions. Their explicit representations are given in Appendix B. Inserting Eq. (9) into Eq. (7), we can perform an expansion in the number of fields:

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{STr}\, \tilde{\partial}_t \ln(\mathcal{P}+\mathcal{F}) = \frac{1}{2} \mathrm{STr}\, \tilde{\partial}_t \left(\frac{1}{\mathcal{P}} \mathcal{F} \right) - \frac{1}{4} \mathrm{STr}\, \tilde{\partial}_t \left(\frac{1}{\mathcal{P}} \mathcal{F} \right)^2 + \frac{1}{6} \mathrm{STr}\, \tilde{\partial}_t \left(\frac{1}{\mathcal{P}} \mathcal{F} \right)^3 - \frac{1}{8} \mathrm{STr}\, \tilde{\partial}_t \left(\frac{1}{\mathcal{P}} \mathcal{F} \right)^4 + \cdots. \tag{10}$$

The ellipsis denotes field-independent terms and terms beyond our truncation. For our purposes it suffices to take the fields constant in space.

The corresponding powers of $(1/\mathcal{P})\mathcal{F}$ can be computed by simple matrix multiplication and the (super-)traces can be taken straightforwardly. This results in the following flow equations for the desired couplings:

$$\begin{aligned}
\partial_t \overline{m}_k^2 &\equiv \beta_m = 8 k^2 v_4 l_1^{(F)4}(0) \overline{h}_k^2, \\
\partial_t \overline{h}_k &\equiv \beta_h = -16 k^2 v_4 l_1^{(F)4}(0) \overline{\lambda}_{\sigma,k} \overline{h}_k \\
&\quad - 16 v_4 l_{1,1}^{(FB)4}(0,0) e^2 \overline{h}_k, \\
\partial_t \overline{\lambda}_{\sigma,k} &\equiv \beta_{\lambda_\sigma} = -24 k^{-2} v_4 l_{1,2}^{(FB)4}(0,0) e^4 \\
&\quad - 32 v_4 l_{1,1}^{(FB)4}(0,0) e^2 \overline{\lambda}_{\sigma,k} \\
&\quad + 8 v_4 \frac{1}{Z_{\phi,k}} l_{1,1}^{(FB)4}\left(0, \frac{\overline{m}_k^2}{Z_{\phi,k} k^2}\right) \overline{h}_k^2 \overline{\lambda}_{\sigma,k} \\
&\quad - 8 k^2 v_4 l_1^{(F)4}(0) \overline{\lambda}_{\sigma,k}^2 \\
&\quad + \frac{2 v_4}{Z_{\phi,k}^2 k^2} l_{1,2}^{(FB)4}\left(0, \frac{\overline{m}_k^2}{Z_{\phi,k} k^2}\right) \overline{h}_k^4,
\end{aligned} \tag{11}$$

where $v_4 = 1/(32\pi^2)$. The threshold functions l fall off for large arguments and describe the decoupling of particles with mass larger than k. They are defined in [10]; explicit examples are given in Appendix E. At this point, it is important to stress that all vector (V) and axial-vector (A) four-fermion couplings on the right-hand side of the flow equation have been brought into the form (V) and $(V+A)$, and then the $(V+A)$ terms have been Fierz transformed into the chirally invariant scalar four-fermion coupling $(S-P)$ used in our truncation [cf. Eq. (2)]. The pure vector coupling is omitted for the time being and will be discussed in Sec. VI. It should also be mentioned that no tensor four-fermion coupling is generated on the right-hand side of the flow equation.

Incidentally, the mass equation coincides with the results of [10]; we find agreement of the third equation with the results of [12] where the same model was investigated in a nonbosonized version.[2] We note that the last term in β_{λ_σ}, which is $\sim \overline{h}_k^4$, is suppressed by the threshold function as long as $\overline{m}^2/(Z_\phi k^2)$ remains large. For simplicity of the dis-

[2]Remaining numerical differences arise from wave function renormalization which was included in [12], but is neglected here for simplicity.

cussion we will first omit it and comment on its quantitative impact later on. The inclusion of this term does not change the qualitative behavior.

III. FERMION-BOSON TRANSLATION BY HAND

As mentioned above, the boundary conditions for the flow equation are such that the four-fermion interaction vanishes at the bosonization scale, $\overline{\lambda}_{\sigma,\Lambda}=0$. But lowering k a bit introduces the four-fermion interaction again according to Eq. (11):

$$\partial_t\overline{\lambda}_{\sigma,k}|_{k=\Lambda}=-24\Lambda^{-2}v_4 l_{1,2}^{(FB)4}(0,0)e^4\neq 0. \tag{12}$$

In Eq. (5), we may again solve the field equations for ϕ as a functional of $\overline{\psi},\psi$ and find in Fourier space

$$\phi(q)=\frac{\overline{h}_k(\overline{\psi}_{\rm L}\psi_{\rm R})(q)}{\overline{m}_k^2+Z_{\phi,k}q^2},\quad \phi^*(q)=-\frac{\overline{h}_k(\overline{\psi}_{\rm R}\psi_{\rm L})(-q)}{\overline{m}_k^2+Z_{\phi,k}q^2}. \tag{13}$$

Inserting this result into Γ_k yields the "total" four-fermion interaction $[\int_q\equiv\int(dq/2\pi)^4]$

$$\int_q\left(2\overline{\lambda}_{\sigma,k}+\frac{\overline{h}_k^2}{\overline{m}_k^2+Z_{\phi,k}q^2}\right)(\overline{\psi}_{\rm R}\psi_{\rm L})(-q)(\overline{\psi}_{\rm L}\psi_{\rm R})(q). \tag{14}$$

The local component (for $q^2=0$) contains a direct contribution $\sim 2\overline{\lambda}_{\sigma,k}$ (one-particle irreducible in the bosonized version) and a scalar exchange contribution (one-particle reducible in the bosonized version). From the point of view of the original fermionic theory, there is no distinction between the two contributions (both are 1PI in the purely fermionic language). This shows a redundancy in our parametrization, since we may change $\overline{\lambda}_{\sigma,k}$, $\overline{h}_k$ and $\overline{m}_k^2$ while keeping the effective coupling

$$2\lambda_\sigma^{\rm eff}(q)=2\overline{\lambda}_{\sigma,k}+\frac{\overline{h}_k^2}{\overline{m}_k^2+Z_{\phi,k}q^2} \tag{15}$$

fixed. Indeed, a choice $\overline{\lambda}'_{\sigma,k}$, $\overline{h}'_k$, $\overline{m}_k^2{}'$ leads to the same $\lambda_\sigma^{\rm eff}(0)$ if it obeys

$$\overline{\lambda}'_{\sigma,k}=\overline{\lambda}_{\sigma,k}+\frac{\overline{h}_k^2}{2\overline{m}_k^2}-\frac{\overline{h}_k^2{}'}{2\overline{m}_k^2{}'}. \tag{16}$$

In particular, we will choose a parametrization where $\overline{\lambda}'_{\sigma,k}$ vanishes for all k. In this parametrization, any increase $d\overline{\lambda}_{\sigma,k}$ according to Eq. (11) is compensated for by a change of $\frac{1}{2}d(\overline{h}_k^2/\overline{m}_k^2)$ of the same size. An increase in $\overline{\lambda}_{\sigma,k}$ is mapped into an increase in $\overline{h}_k^2/(2\overline{m}_k^2)$. In this parametrization, the four-fermion coupling remains zero, whereas the flow of $\overline{h}_k^2/\overline{m}_k^2$ receives an additional contribution

$$\partial_t\left(\frac{\overline{h}_k^2}{\overline{m}_k^2}\right)=\partial_t\left(\frac{\overline{h}_k^2}{\overline{m}_k^2}\right)\Bigg|_{\overline{\lambda}_{\sigma,k}}+2\partial_t\overline{\lambda}_{\sigma,k}|_{\overline{h}_k^2,\overline{m}_k^2}. \tag{17}$$

More explicitly, we can write

$$\partial_t\left(\frac{\overline{m}_k^2}{\overline{h}_k^2}\right)=\frac{1}{\overline{h}_k^2}\partial_t\overline{m}_k^2|_{\overline{\lambda}_{\sigma,k}}-2\frac{\overline{m}_k^2}{\overline{h}_k^3}\partial_t\overline{h}_k|_{\overline{\lambda}_{\sigma,k}}-2\frac{\overline{m}_k^4}{\overline{h}_k^4}\partial_t\overline{\lambda}_{\sigma,k}, \tag{18}$$

with $\partial_t\overline{m}_k^2|_{\overline{\lambda}_{\sigma,k}}$, $\partial_t\overline{h}_k|_{\overline{\lambda}_{\sigma,k}}$, $\partial_t\overline{\lambda}_{\sigma,k}$ given by Eq. (11) with the replacement $\overline{\lambda}_{\sigma,k}\to 0$ on the right-hand sides. One obtains

$$\partial_t\left(\frac{\overline{m}_k^2}{\overline{h}_k^2}\right)=v_4\Bigg[8l_1^{(F)4}(0)k^2+32l_{1,1}^{(FB)4}(0,0)e^2\frac{\overline{m}_k^2}{\overline{h}_k^2}+48l_{1,2}^{(FB)4}(0,0)\frac{e^4}{k^2}\left(\frac{\overline{m}_k^2}{\overline{h}_k^2}\right)^2\Bigg]. \tag{19}$$

The characteristics of this flow can be understood best in terms of the dimensionless quantity

$$\tilde{\epsilon}_k=\frac{\overline{m}_k^2}{\overline{h}_k^2k^2}. \tag{20}$$

It obeys the flow equation

$$\begin{aligned}\partial_t\tilde{\epsilon}_k=\beta_{\tilde{\epsilon}}&=-2\tilde{\epsilon}_k+v_4[8l_1^{(F)4}(0)+32l_{1,1}^{(FB)4}(0,0)e^2\tilde{\epsilon}_k\\&\quad+48l_{1,2}^{(FB)4}(0,0)e^4\tilde{\epsilon}_k^2]\\&=-2\tilde{\epsilon}_k+\frac{1}{8\pi^2}+\frac{1}{\pi^2}e^2\tilde{\epsilon}_k+\frac{9}{4\pi^2}e^4\tilde{\epsilon}_k^{\,2},\end{aligned} \tag{21}$$

where in the last line we have inserted the values of the threshold functions for optimized cutoffs [14] discussed in Appendix E. Neglecting the running of the gauge coupling e, we note in Fig. 1 the appearance of two fixed points. For gauge couplings of order 1 or smaller and to leading order in e, these two fixed points are given by

$$\tilde{\epsilon}_1^*\simeq\frac{1}{16\pi^2}+\mathcal{O}(e^2/(16\pi^2)^2),\quad \tilde{\epsilon}_2^*\simeq\frac{8\pi^2}{9e^4}+\mathcal{O}(1/e^2). \tag{22}$$

The smaller fixed point $\tilde{\epsilon}_1^*$ is infrared unstable, whereas the larger fixed point $\tilde{\epsilon}_2^*$ is infrared stable. Therefore, starting with an initial value of $0<\tilde{\epsilon}_\Lambda<\tilde{\epsilon}_1^*$, the flow of the scalar mass-to-Yukawa-coupling ratio will be dominated by the first two terms in the modified flow equation (21) $\sim -2\tilde{\epsilon}_k+1/(8\pi^2)$. This is nothing but the flow of a theory involving a "fundamental" scalar with Yukawa coupling to a fermion sector. Moreover, we will end in a phase with (dynamical) chiral symmetry breaking, since $\tilde{\epsilon}$ is driven to negative values. (Higher order terms in the scalar potential need to be included once $\tilde{\epsilon}$ becomes zero or negative.) This all agrees

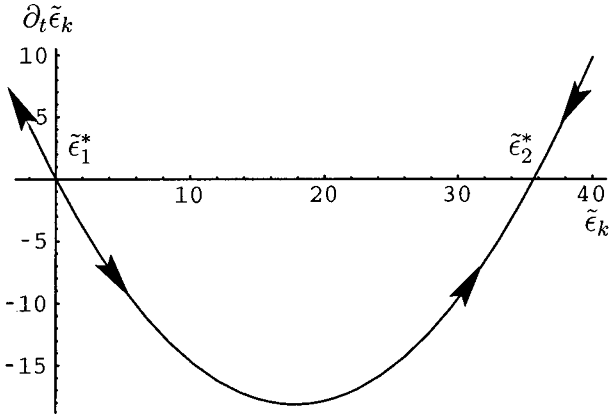

FIG. 1. Fixed-point structure of the $\tilde{\epsilon}_k$ flow equation after fermion-boson translation by hand. The graph is plotted for the threshold functions discussed in Appendix E with $e=1$. Note that $\tilde{\epsilon}_1^*$ is small but different from zero [cf. Eq. (22)]. Arrows indicate the flow towards the infrared, $k\to 0$.

with the common knowledge that the low-energy degrees of freedom of the strongly coupled NJL model are (composite) scalars which nevertheless behave as fundamental particles.

On the other hand, if we start with an initial $\tilde{\epsilon}_\Lambda$ value that is larger than the first (infrared unstable) fixed point, the flow will necessarily be attracted towards the second fixed point $\tilde{\epsilon}_2^*$; there, the flow will stop. This flow does not at all remind us of the flow of a fundamental scalar. Moreover, there will be no dynamical symmetry breaking, since the mass remains positive. The effective four-fermion interaction corresponding to the second fixed point reads

$$\lambda_\sigma^* = \frac{1}{2k^2\tilde{\epsilon}_2^*} \approx \frac{9}{16\pi^2}\frac{e^4}{k^2}. \tag{23}$$

It coincides with the perturbative value (3) of massless QED. We conclude that the second fixed point characterizes massless QED. The scalar field shows a typical bound-state behavior with mass and couplings expressed by e and k. [The question as to whether the bound state behaves like a propagating particle (i.e., ''positronium'') depends on the existence of an appropriate pole in the scalar propagator. At least for massive QED one would expect such a pole with renormalized mass corresponding to the ''rest mass'' of scalar positronium.]

From a different viewpoint, the fixed point $\tilde{\epsilon}_1^*$ corresponds directly to the critical coupling of the NJL model, which distinguishes between the symmetric and the broken phase. As long as the flow is governed by the vicinity of this fixed point, the scalar behaves like a fundamental particle, with mass given by the relevant parameter characterizing the flow away from this fixed point.

Our interpretation is that the ''range of relevance'' of these two fixed points tell us whether the scalar appears as a ''fundamental'' or a ''composite'' particle, corresponding to the state of the system being governed by $\tilde{\epsilon}_1^*$ or $\tilde{\epsilon}_2^*$, respectively.

The incorporation of the flow of the momentum-independent part of $\overline{\lambda}_{\sigma,k}$ into the flow of $\overline{h}_k$ and $\overline{m}_k^2$ affects only the ratio $\overline{m}_k^2/\overline{h}_k^2$. At this point, it does not differentiate which part of the correction appears in the separate flow equations for $\overline{m}_k^2$ and $\overline{h}_k$, respectively. This degeneracy can be lifted if we include information about the flow of $\overline{\lambda}_{\sigma,k}$ for two different values of the external momenta. Let us define $\overline{\lambda}_{\sigma,k}(s)$ as $\overline{\lambda}_{\sigma,k}(p_1,p_2,p_3,p_4)$ with $p_1=p_3=(1/2)(\sqrt{s},\sqrt{s},0,0)$, $p_2=p_4=(1/2)(\sqrt{s},-\sqrt{s},0,0)$, where $s=(p_1+p_2)^2=(p_3+p_4)^2$ is the square of the exchanged momentum in the s channel [5]. The coupling $\overline{\lambda}_{\sigma,k}$ appearing on the right-hand side of Eq. (17) corresponds in this notation to $\overline{\lambda}_{\sigma,k}(s=0)$. We can now achieve the simultaneous vanishing of $\overline{\lambda}_{\sigma,k}(s=0)$ and $\overline{\lambda}_{\sigma,k}(s=k^2)$ if we redefine $\overline{m}_k^2$ and $\overline{h}_k$ such that they obey in addition

$$\partial_t\left(\frac{\overline{h}_k^2}{\overline{m}_k^2+Z_{\phi,k}k^2}\right)=\partial_t\left(\frac{\overline{h}_k^2}{\overline{m}_k^2+Z_{\phi,k}k^2}\right)_{|\overline{\lambda}_{\sigma,k}}+2\,\partial_t\overline{\lambda}_{\sigma,k}(s=k^2). \tag{24}$$

Incorporation of this effect should improve a truncation where the 1PI four-fermion coupling is neglected subsequently, since we realize now a matching at two different momenta.

The combination of Eqs. (17) and (24) specifies the evolution of $\overline{m}_k^2$ and $\overline{h}_k$,

$$\partial_t\overline{m}_k^2=\beta_m|_{\overline{\lambda}_{\sigma,k}}+\frac{2\overline{m}_k^2(\overline{m}_k^2+Z_{\phi,k}k^2)}{\overline{h}_k^2}\left(\frac{\overline{m}_k^2+Z_{\phi,k}k^2}{Z_{\phi,k}k^2}\partial_t\Delta\overline{\lambda}_{\sigma,k}+\partial_t\overline{\lambda}_{\sigma,k}(s=0)\right), \tag{25}$$

$$\partial_t\overline{h}_k=\beta_h|_{\overline{\lambda}_{\sigma,k}}+\frac{2\overline{m}_k^2+Z_{\phi,k}k^2}{\overline{h}_k}\partial_t\overline{\lambda}_{\sigma,k}(s=0)+\frac{(\overline{m}_k^2+Z_{\phi,k}k^2)^2}{Z_{\phi,k}k^2\overline{h}_k}\partial_t\Delta\overline{\lambda}_{\sigma,k}, \tag{26}$$

where we have used

$$\Delta\overline{\lambda}_{\sigma,k}=\overline{\lambda}_{\sigma,k}(s=k^2)-\overline{\lambda}_{\sigma,k}(s=0). \tag{27}$$

Let us finally comment on the influence of the last term $\sim\overline{h}_k^4$ of Eq. (11), omitted up to now, on the flow equation (21) for $\tilde{\epsilon}_k$: the contribution of this term to Eq. (21) is $\sim(\overline{m}_k^2/Z_{\phi,k}k^2)^2 l_{1,2}^{(FB)4}(0,\overline{m}_k^2/Z_{\phi,k}k^2)$. For large $(\overline{m}_k^2/Z_{\phi,k}k^2)$, this term approaches a constant, so that a slight vertical shift of the parabola of Fig. 1 is induced. We observe that this shift leaves the position of the second fixed point $\tilde{\epsilon}_2^*$ unaffected to lowest order in e. This justifies the omission of the

$\sim \overline{h}_k^4$ term in the preceding discussion. The influence of the $\overline{h}_k^4$ term on the first fixed point is discussed at the end of the next section.

IV. FLOW WITH CONTINUOUS FERMION-BOSON TRANSLATION

The ideas of the preceding section shall now be made more rigorous by deriving the results directly from an appropriate exact flow equation. As a natural approach to this aim, we could search for a k-dependent field transformation of the scalars, $\phi \to \hat{\phi}_k[\phi]$. In terms of the new variables, the flow equation (7) should then provide for the vanishing of the four-fermion coupling in the transformed effective action. Indeed, we sketch this approach briefly in Appendix D. Instead, we propose here a somewhat different approach relying on a variant of the usual flow equation where the cutoff is adapted to k-dependent fields. The advantage is a very simple structure of the resulting flow equations in coincidence with those of the preceding section.

The idea is to employ a flow equation for a scale-dependent effective action $\Gamma_k[\phi_k]$, where the field variable ϕ_k is allowed to vary during the flow; we derive this flow equation in Appendix C. To be precise within the present context, upon an infinitesimal renormalization group step from a scale k to $k-dk$, the scalar field variables also undergo an infinitesimal transformation of the type (in momentum space)

$$\begin{aligned}\phi_{k-dk}(q) &= \phi_k(q) + \delta\alpha_k(q)(\overline{\psi}_{\rm L}\psi_{\rm R})(q)\\ &\equiv \phi_k(q) + \delta\alpha_k(q)\int_p \overline{\psi}_{\rm L}(p)\,\psi_{\rm R}(p+q). \end{aligned} \tag{28}$$

Including the corresponding transformation of the complex conjugate variable, the flow of the scalar fields is given by

$$\begin{aligned}\partial_t\phi_k(q) &= -(\overline{\psi}_{\rm L}\psi_{\rm R})(q)\,\partial_t\alpha_k(q),\\ \partial_t\phi_k^*(q) &= (\overline{\psi}_{\rm R}\psi_{\rm L})(-q)\,\partial_t\alpha_k(q).\end{aligned} \tag{29}$$

The transformation parameter $\alpha_k(q)$ is an *a priori* arbitrary function, expressing a redundancy in the parametrization of the effective action. As shown in Eq. (C8), the effective action $\Gamma_k[\phi_k,\phi_k^*]$ obeys the modified flow equation

$$\begin{aligned}\partial_t\Gamma_k[\phi_k,\phi_k^*] &= \partial_t\Gamma_k[\phi_k,\phi_k^*]|_{\phi_k,\phi_k^*} + \int_q \left(\frac{\delta\Gamma_k}{\delta\phi_k(q)}\partial_t\phi_k(q)\right.\\ &\quad \left. + \frac{\delta\Gamma_k}{\delta\phi_k^*(q)}\partial_t\phi_k^*(q)\right),\end{aligned} \tag{30}$$

where the notation omits the remaining fermion and gauge fields for simplicity. The first term on the right-hand side is nothing but the flow equation for fixed variables evaluated at ϕ_k, ϕ_k^* instead of $\phi,\phi^* = \phi_\Lambda,\phi_\Lambda^*$. The second term reflects the flow of the variable. Projecting Eq. (30) onto our truncation (5), we arrive at modified flows for the couplings:

$$\begin{aligned}\partial_t\overline{m}_k^2 &= \partial_t\overline{m}_k^2|_{\phi_k,\phi_k^*},\\ \partial_t\overline{h}_k &= \partial_t\overline{h}_k|_{\phi_k,\phi_k^*} + (\overline{m}_k^2 + Z_{\phi,k}q^2)\,\partial_t\alpha_k(q),\\ \partial_t\overline{\lambda}_{\sigma,k} &= \partial_t\overline{\lambda}_{\sigma,k}|_{\phi_k,\phi_k^*} - \overline{h}_k\partial_t\alpha_k(q).\end{aligned} \tag{31}$$

Again, the first terms on the right-hand sides are nothing but the right-hand sides of Eq. (11), i.e., the corresponding beta functions $\beta_{m,h,\lambda_\sigma}$. The further terms represent the modifications owing to the flow of the field variables, as obtained from the two last terms in Eq. (30) by inserting Eq. (29). Obviously, we could have generalized the method easily to the case of momentum-dependent couplings (see below). In the following, however, it suffices to study the point-like limit, which we associate to $q=0$.

We exploit the freedom in the choice of variables in Eq. (29) by fixing $\alpha_k = \alpha_k(q=0)$ in such a way that the four-fermion coupling is not renormalized, $\partial_t\overline{\lambda}_{\sigma,k}=0$. This implies the flow equation for α_k,

$$\partial_t\alpha_k = \beta_{\lambda_\sigma}/\overline{h}_k. \tag{32}$$

Together with the boundary condition $\lambda_{\sigma,\Lambda}=0$, this guarantees a vanishing four-fermion coupling at all scales, $\overline{\lambda}_{\sigma,k}=0$. The (nonlinear) fields corresponding to this choice obtain for every k by integrating the flow (32) for α_k, with $\alpha_\Lambda = 0$.

Of course, imposing the condition (32) also influences the flow of the Yukawa coupling according to Eq. (31),

$$\partial_t\overline{h}_k = \beta_h + \frac{\overline{m}_k^2}{\overline{h}_k}\beta_{\lambda_\sigma}. \tag{33}$$

In consequence, the flow equation for the quantity of interest, $\overline{h}_k^2/\overline{m}_k^2$, then reads

$$\partial_t\left(\frac{\overline{h}_k^2}{\overline{m}_k^2}\right) = \partial_t\left(\frac{\overline{h}_k^2}{\overline{m}_k^2}\right)\Bigg|_{\phi_k,\phi_k^*} + 2\beta_{\lambda_\sigma}. \tag{34}$$

This coincides precisely with Eq. (17) where we have translated the fermionic interaction into the scalar sector by hand. The flow equation of the dimensionless combination $\tilde{\epsilon}_k = \overline{m}_k^2/k^2\overline{h}_k^2$ is therefore identical to the one derived in Eq. (21), so that the fixed-point structure described above is also recovered in the more rigorous approach. The underlying picture of this approach can be described as a permanent translation of four-fermion interactions, arising during each renormalization group step, into the scalar interactions. Thereby, bosonization takes place at any scale and not only at a fixed initial one.

One should note that the field transformation is not fixed uniquely by the vanishing of $\overline{\lambda}_{\sigma,k}$. For instance, an additional contribution in Eq. (28) $\sim \delta\beta_k(q)\phi_k(q)$ can absorb the momentum dependence of the Yukawa coupling by modifying the scalar propagator. Similarly to the discussion in Sec. III, this can be used in order to achieve simultaneously the vanishing of $\bar{\lambda}_{\sigma,k}(s)$ for all s and k and a momentum-independent $\bar{h}_k$. First, the variable change

$$\partial_t \phi_k(q) = -(\bar{\psi}_{\rm L}\psi_{\rm R})(q)\,\partial_t\alpha_k(q) + \phi_k(q)\,\partial_t\beta_k(q),$$
$$\partial_t \phi_k^*(q) = (\bar{\psi}_{\rm R}\psi_{\rm L})(-q)\,\partial_t\alpha_k(q) + \phi_k^*(q)\,\partial_t\beta_k(q) \tag{35}$$

indeed ensures the vanishing of $\bar{\lambda}_{\sigma,k}(s=q^2)$ if $\partial_t\alpha_k(q) = \bar{h}_k^{-1}\partial_t\bar{\lambda}_{\sigma,k}(q^2)$. This choice results in

$$\partial_t\bar{h}_k(q) = \partial_t\bar{h}_k(q)|_{\bar{\lambda}_{\sigma,k}} + \frac{Z_{\phi,k}q^2+\bar{m}_k^2}{\bar{h}_k}\,\partial_t\bar{\lambda}_{\sigma,k}(q^2) + \bar{h}_k\partial_t\beta_k(q),$$
$$\partial_t Z_{\phi,k}(q)q^2 + \partial_t\bar{m}_k^2 = \partial_t\bar{m}_k^2|_{\bar{\lambda}_{\sigma,k}} + 2\,\partial_t\beta_k(q)(Z_{\phi,k}q^2+\bar{m}_k^2), \tag{36}$$

where $\bar{h}_k(q)$ and $Z_{\phi,k}(q)$ depend now on q^2. Secondly, the momentum dependence of $\bar{h}_k(q)$ can be absorbed by the choice

$$\partial_t\beta_k(q) = -\frac{Z_{\phi,k}q^2+\bar{m}_k^2}{\bar{h}_k^2}\,\partial_t\bar{\lambda}_{\sigma,k}(q^2) + \frac{1}{Z_{\phi,k}k^2\bar{h}_k^2}\times[(Z_{\phi,k}k^2+\bar{m}_k^2)^2\partial_t\bar{\lambda}_{\sigma,k}(k^2) - \bar{m}_k^4\partial_t\bar{\lambda}_{\sigma,k}(0)]. \tag{37}$$

The particular form of the q-independent part of $\partial_t\beta_k$ was selected in order to obtain $\partial_t Z_{\phi,k}(q^2=k^2)=0$ such that our approximation of constant $Z_{\phi,k}$ is self-consistent. Inserting Eq. (37) into the evolution equation (36) for $\bar{h}_k$ and $\bar{m}_k^2$, we recover Eqs. (25) and (26). We also note that the evolution of $\tilde{\epsilon}=\bar{m}_k^2/(\bar{h}_k^2(0)k^2)$ is independent of the choice of $\beta_k(q)$.

It is interesting to observe that reinserting the classical equations of motion at a given scale in order to eliminate auxiliary variables is equivalent to the here-proposed variant of the flow equation with flowing variables. In contrast, the standard form of the flow equation in combination with a variable transformation, to be discussed in Appendix D, leads to a more complex structure, which is in general more difficult to solve.

V. BETWEEN MASSLESS QED AND SPONTANEOUS CHIRAL SYMMETRY BREAKING

In this section, we briefly present some quantitative results for the flow in the gauged NJL model. Despite our rough approximation, they represent the characteristic physics. We concentrate on the flow of the dimensionless renormalized couplings

$$\epsilon_k = \frac{\bar{m}_k^2}{Z_{\phi,k}k^2}, \quad h_k = \bar{h}_k Z_{\phi,k}^{-1/2}, \quad \tilde{\epsilon}_k = \frac{\epsilon_k}{h_k^2}, \quad \tilde{\alpha}_k = \alpha_k Z_{\phi,k}^{1/2}k^2 \tag{38}$$

in the symmetric phase. Inserting the specific threshold functions of Appendix E, we find the system of differential flow equations

$$\partial_t\epsilon_k = -2\,\epsilon_k + \frac{h_k^2}{8\pi^2} - \frac{\epsilon_k(\epsilon_k+1)}{h_k^2}\left(\frac{9e^4}{4\pi^2} - \frac{h_k^4}{16\pi^2}\frac{3+\epsilon_k}{(1+\epsilon_k)^3}\right)\times[1+(1+\epsilon_k)Q_\sigma],$$
$$\partial_t h_k = -\frac{e^2}{2\pi^2}h_k - \frac{2\,\epsilon_k+1+(1+\epsilon_k)^2Q_\sigma}{h_k}\left(\frac{9e^4}{8\pi^2} - \frac{h_k^4}{32\pi^2}\frac{3+\epsilon_k}{(1+\epsilon_k)^3}\right). \tag{39}$$

The resulting flow for $\tilde{\epsilon}_k$ is independent of $Q_\sigma \equiv \partial_t\Delta\bar{\lambda}_{\sigma,k}/\partial_t\bar{\lambda}_{\sigma,k}(0)$:

$$\partial_t\tilde{\epsilon}_k = \beta_{\tilde{\epsilon}} = -2\tilde{\epsilon}_k + \frac{1}{8\pi^2} + \frac{e^2}{\pi^2}\tilde{\epsilon}_k + \frac{9e^4}{4\pi^2}\tilde{\epsilon}_k^2 - \frac{1}{16\pi^2}\frac{\epsilon_k^2(3+\epsilon_k)}{(1+\epsilon_k)^3}. \tag{40}$$

Here the last term reflects the last contribution to β_{λ_σ} in Eq. (11), which has been neglected in the preceding section [cf. Eq. (21)]. We see that its influence is small for $\epsilon_k \ll 1$, whereas for $\epsilon_k \gg 1$ it reduces the constant term by a factor 1/2.

Note that, for a given Q_σ, Eqs. (39) form a closed set of equations. The same is true for the flows of $\tilde{\epsilon}_k$ and ϵ_k if we express h_k in terms of $\tilde{\epsilon}_k$ and ϵ_k in the first line of Eq. (39). In order to obtain Q_σ, the flow of $\bar{\lambda}_{\sigma,k}(s)$ has to be known; however, far less information is already sufficient for a qualitative analysis. First, it is natural to expect that $\bar{\lambda}_{\sigma,k}(s)$ is maximal for $s=0$, since $\bar{\lambda}_{\sigma,k}(s)$ will be suppressed for large s owing to the external momenta. This implies $\Delta\bar{\lambda}_{\sigma,k}/\bar{\lambda}_{\sigma,k}(s=0)<0$. With the simplifying assumption that $\Delta\bar{\lambda}_{\sigma,k}/\bar{\lambda}_{\sigma,k}(0)\simeq$const, we also conclude that

$$Q_\sigma < 0. \tag{41}$$

For a qualitative solution of the flow equations, we assume $|Q_\sigma|$ to be of order 1 or smaller.

We next need initial values ϵ_Λ, $\tilde{\epsilon}_\Lambda$ for solving the system of differential equations. We note that the initial value ϵ_Λ diverges for the pure NJL model, since $Z_{\phi,\Lambda}=0$. For large ϵ_k, one has

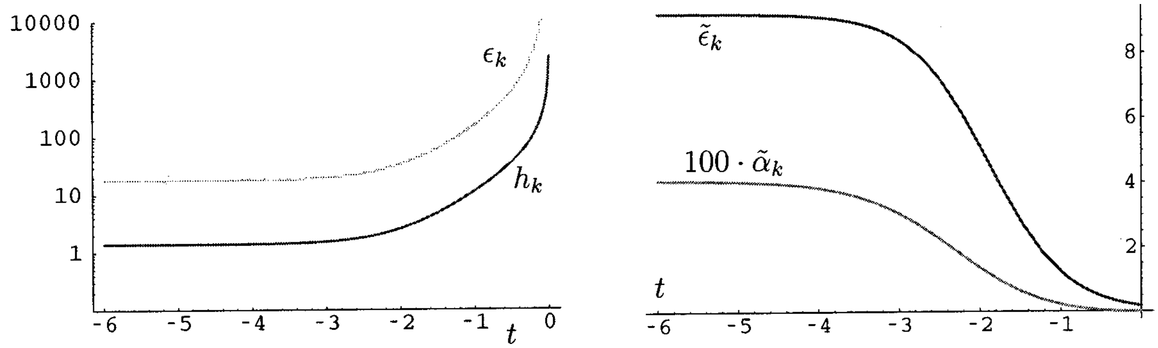

FIG. 2. Flows of ϵ_k, h_k, $\tilde{\epsilon}_k$ and $\tilde{\alpha}_k$ in the symmetric phase according to Eqs. (39), (40) and (46) for the initial values $\epsilon_\Lambda = 10^6$, $\tilde{\epsilon}_\Lambda = 0.17 \gtrsim 1/(6e^2)$, $e=1$, $Q_\sigma = -0.1$. For a better visualization, $\tilde{\alpha}_k$ has been multiplied by a factor of 100. The plot of $\tilde{\epsilon}_k$ on the right panel exhibits the crossover behavior between the fixed points $\tilde{\epsilon}_1^*$ at small t to $\tilde{\epsilon}_2^*$ for $t \to -\infty$.

$$\partial_t \epsilon_k = \left[-2 + \frac{1}{8\pi^2 \tilde{\epsilon}_k} + (|Q_\sigma| \epsilon_k - 1) \left(\frac{9 e^4 \tilde{\epsilon}_k}{4\pi^2} - \frac{1}{16\pi^2 \tilde{\epsilon}_k} \right) \right] \epsilon_k , \tag{42}$$

and we find that ϵ_k decreases rapidly for

$$\tilde{\epsilon}_\Lambda > \frac{1}{6e^2}. \tag{43}$$

(In a more complete treatment, it decreases rapidly for arbitrary $\tilde{\epsilon}_k$ owing to the generation of a nonvanishing $Z_{\phi,k}$ by the fluctuations. In the present truncation, the qualitative behavior of the flow will depend on the details of Q_σ if $\tilde{\epsilon}_\Lambda$ does not satisfy this bound.) We confine our discussion to initial values satisfying Eq. (43), which can always be accomplished without fine-tuning.

In Fig. 2 we present a numerical solution for large ϵ_Λ (small nonzero $Z_{\phi,k}$), $Q_\sigma = -0.1$, and $\tilde{\epsilon}_\Lambda$ slightly above the bound given by Eq. (43) for $e=1$; these initial conditions correspond to the symmetric phase. We observe that both h_k and ϵ_k approach constant values in the infrared. This corresponds to the "bound-state fixed point" for $\tilde{\epsilon}_k$: $\tilde{\epsilon}_2^* \simeq 8\pi^2/(9e^4)$. A constant ϵ_k implies that the renormalized mass term $m_k^2 = \epsilon_k k^2$ decreases $\sim k^2$ in the symmetric phase. The precise value of the Yukawa coupling at the fixed point depends on e and $|Q_\sigma|$:

$$(h^*)^2 = 16\pi^2 \epsilon^* - \frac{8\epsilon^*(\epsilon^*+1)[1-|Q_\sigma|(\epsilon^*+1)]}{[2\epsilon^*+1-|Q_\sigma|(\epsilon^*+1)^2]} e^2. \tag{44}$$

If $\epsilon^* \gg 1$ still holds, the fixed-point values can be given more explicitly:

$$\epsilon^* \simeq \frac{2}{|Q_\sigma|}, \quad h^* \simeq \frac{3e^2}{2\pi\sqrt{|Q_\sigma|}}. \tag{45}$$

Note that $\epsilon^* \gg 1$ is equivalent to $|Q_\sigma| \ll 1$; numerically, we find that Eqs. (45) describe the fixed-point values reasonably well already for $|Q_\sigma| \lesssim 0.1$. We observe that the fixed-point values are independent of the initial values ϵ_Λ and $\tilde{\epsilon}_\Lambda$, so that the system has "lost its memory."

Finally, the parameter $\tilde{\alpha}_k$ governing the field redefinition obeys the flow equation

$$\partial_t \tilde{\alpha}_k = 2\tilde{\alpha}_k - \frac{9e^4}{8\pi^2 h_k} + \frac{h_k^3}{32\pi^2} \frac{3+\epsilon_k}{(1+\epsilon_k)^3}. \tag{46}$$

A numerical solution is plotted in Fig. 2, right panel. Also $\tilde{\alpha}_k$ approaches a constant for small k. Therefore, the transformation parameter $\alpha_k \sim \tilde{\alpha}_k / k^2$ increases for small k.

The physical picture of the fixed point (45) is quite simple. We may first translate back to an effective four-fermion interaction by solving the scalar field equations:

$$\bar{\lambda}_{\sigma,k}(q^2) = \frac{1}{2} \frac{(h^*)^2}{(q^2 + \epsilon^* k^2)} = \frac{9e^4}{8\pi^2} \frac{1}{(|Q_\sigma| q^2 + 2k^2)}. \tag{47}$$

In the limit $k \to 0$, this mimics the exchange of a massless positronium-like state with effective coupling $h^* = 3e^2/(2\pi\sqrt{|Q_\sigma|})$. Indeed, if we switch on the electron mass m_e, we expect that the running of the positronium mass term stops at $k^2 \simeq m_e^2$. In consequence, the positronium state will acquire a mass $\sim m_e$, which is, in principle, calculable by an improved truncation within our framework.

On the other hand, starting with small enough $\tilde{\epsilon}_\Lambda$, one will observe chiral symmetry breaking as we have already argued in Sec. III. Quantitative accuracy should include at least the flow of the scalar wave function renormalization in this case. Near the boundary between the two phases, the infrared physics is described by a renormalizable theory for QED with a neutral scalar coupled to the fermion.

VI. MODIFIED GAUGE FIELDS

The possibility of k-dependent field redefinitions is not restricted to composite fields. We demonstrate this here by a transformation of the gauge field, which becomes a k-dependent nonlinear combination according to

$$\partial_t A_\mu(q) = -\partial_t \gamma_k (\bar{\psi}\gamma_\mu\psi)(q) - \partial_t \delta_k (\partial_\nu F_{\mu\nu})(q) - \partial_t \zeta_k (\partial_\mu \partial_\nu A_\nu)(q). \tag{48}$$

This transformation can absorb the vector channel in the four-fermion interaction. Indeed, we may enlarge our truncation (2) by a term

$$\Gamma_k^{(V)} = \int d^4x \bar{\lambda}_{v,k} (\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma_\mu\psi) \tag{49}$$

(or a corresponding generalization with momentum-dependent coupling $\bar{\lambda}_{v,k}$). The flow equation for $\bar{\lambda}_v$ reads

$$\partial_t \bar{\lambda}_{v,k} = -6k^{-2}v_4 l_{1,2}^{(FB)4}(0,0)e^4 + \frac{1}{2}k^{-2}v_4 \frac{1}{Z_{\phi,k}^2} l_{1,2}^{(FB)4}\left(0, \frac{\bar{m}_k^2}{Z_{\phi,k}k^2}\right)\bar{h}_k^4 + e\partial_t\gamma_k + \mathcal{O}(\bar{\lambda}_{\sigma,k}, \bar{\lambda}_{v,k}). \tag{50}$$

In the following, we again omit the term $\sim \bar{h}_k^4$, whose contributions are subdominant once the scalars have decoupled from the flow. Choosing γ_k according to

$$\partial_t \gamma_k = 6k^{-2}v_4 l_{1,2}^{(FB)4}(0,0)e^3, \tag{51}$$

we can obtain a vanishing of $\bar{\lambda}_v$ for all k. This procedure introduces additional terms $\sim \bar{\sigma}_k (\partial_\nu F_{\mu\nu})\bar{\psi}\gamma_\mu\psi$ with $\bar{\sigma}_k$ obeying

$$\partial_t \bar{\sigma}_k = -\partial_t \gamma_k + e\partial_t \delta_k + \cdots, \tag{52}$$

where the dots correspond to contributions from $\partial_t \Gamma$ at fixed fields. Adjusting δ_k permits us to enforce $\bar{\sigma}_k = 0$. As a result, only the gauge field propagator gets modified by higher derivative terms. We note that the modified gauge field has the same gauge transformation properties as the original field only for $\zeta_k = 0$. In fact, the gauge fixing becomes dependent on the fermions by a term $\bar{\sigma}_k^{(gf)} \bar{\psi}\gamma_\mu\psi\partial_\mu\partial_\nu A_\nu$ according to

$$\partial_t \bar{\sigma}_k^{(gf)} = \frac{1}{\alpha}\partial_t \gamma_k + e\partial_t \zeta_k + \cdots. \tag{53}$$

Again, we can enforce a vanishing $\bar{\sigma}_k^{(gf)}$ for all k by an appropriate choice of ζ_k. The contribution to the evolution of the gauge field propagator resulting from the field redefinition (48) is

$$\partial_t \Gamma^{(A2)} = -\partial_t \delta_k (\partial_\nu F_{\mu\nu})(\partial_\rho F_{\mu\rho}) + \frac{1}{\alpha}\partial_t \zeta_k (\partial_\mu \partial_\nu A_\nu)(\partial_\mu \partial_\rho A_\rho) + \cdots. \tag{54}$$

With $\partial_t \gamma_k \sim e^3$, $\partial_t \delta_k \sim e^2$, $\partial_t \zeta_k \sim e^2/\alpha$ we see that the field redefinitions lead to a modification of the kinetic term (or a momentum-dependent wave function renormalization of the gauge field) already in leading order $\sim e^2$. Depending on the precise definition of the renormalized gauge coupling this can modify the β-function for the "composite gauge field" as compared to the original one. This modification is the counterpart of the elimination of the effective vertices $\sim \bar{\sigma}_k, \bar{\sigma}_k^{(gf)}$. (We note that no corrections arise if e is defined by the effective electromagnetic vertex at very small momentum.)

VII. CONCLUSIONS

It is an inherent feature of quantum field theory that a system with certain fundamental degrees of freedom at a "microscopic" scale can exhibit completely different degrees of freedom at a "macroscopic" scale, which appear to be equivalently "fundamental" in an operational sense. A prominent example are the pions in a low-momentum effective theory for strong interactions. These different faces of one and the same system are related by the action of the renormalization group. In the present work, we realize this formal concept with the aid of a renormalization group flow equation for the effective average action whose field variables are allowed to change continuously under the flow from one scale to another. In particular, this generally nonlinear transformation of variables is suitable for studying the renormalization flow of bound states.

We illustrate these ideas by way of example by considering the gauged NJL model at weak gauge coupling. Our flow equations can clearly identify the phase transition to spontaneous chiral symmetry breaking. In our picture, the interaction between the fermions, representing the fundamental degrees of freedom at high momentum scales, gives rise to a pairing into scalar degrees of freedom. These so-formed bound states may still appear effectively as composite objects at lower scales or rather as fundamental degrees of freedom, depending on the strength of the initial interaction. As the criterion that distinguishes between these two cases, we classify the renormalization flow of the scalar bound states: "fundamental behavior" is governed by a typical infrared unstable fixed point with the relevant parameter corresponding to the mass of the scalar. Contrary to this, "bound-state behavior" is related to an infrared attractive (partial) fixed point that is governed by the relevant and marginal parameters of the "fundamental" fermion and photon—massless QED in our case. The flow may show a crossover from one to the other characteristic behavior. This physical picture is obtained from the continuous transformation of the field variables under the flow that translates the fermion interactions into the parameters of the scalar sector. In the case of spontaneous chiral symmetry breaking, the scalars always appear as "fundamental" on scales characteristic for the phase transition and the order parameter.

From a different perspective, we propose a technique for performing a bosonization of self-interactions of fundamental fermion fields permanently at all scales during the renormalization flow. Provided that appropriate low-energy degrees of freedom of a quantum system are known, our modified flow equation for the average effective action is capable of describing the crossover from one set of variables to another during the flow in a well-controlled manner.

Thereby, the notions of fundamental particle or bound state become scale dependent.

For the translation from fermion bilinears to scalars, the gauge field acts rather as a spectator, permanently catalyzing the generation of four-fermion interactions under the flow. In the vector channel, however, the gauge field can also participate in the field transformation. Hereby, a four-fermion interaction $\sim(\bar{\psi}\gamma^\mu\psi)^2$ is absorbed at the expense of a modified photon kinetic term, which can lead to a change in the beta function β_e of an appropriately defined effective gauge coupling. We expect this type of transformation to be particularly useful in the strong-gauge-coupling sector of the gauged NJL model. Here it is known that the four-fermion interaction can acquire an anomalous scaling dimension of 4 (instead of 6) [11], so that it mixes with the gauge interaction (in a renormalization-group sense) anyway. It should be worthwhile to employ this transformation for a search for the existence of ultraviolet stable fixed points in the β_e function, to be expected for a large number of fermion species $N_{\rm F}$ [9].

In view of the motivating cases of top quark condensation in the Higgs sector and color octet condensation in low-energy QCD, we now have an important tool at our disposal which allows for a nonperturbative study of the transition from the underlying theory to the condensing degrees of freedom. Particularly in the case of ''spontaneous breaking of color,'' a quantitatively reliable calculation of the potential for the quark-antiquark degrees of freedom seems possible. Analogously to the gauged NJL model, the effective quark self-interactions, being induced by the exchange of gluons and instantons, have to be translated into the scalar bound-state sector. The renormalization flow of the latter and the symmetry properties of their corresponding potential shall finally adjudicate on ''spontaneous breaking of color.''

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APPENDIX A: DIRAC ALGEBRA AND FIERZ TRANSFORMATIONS

We work in a chiral basis, $\psi=\binom{\psi_{\rm L}}{\psi_{\rm R}}$, $\bar{\psi}=(\bar{\psi}_{\rm R},\bar{\psi}_{\rm L})$, where ψ and $\bar{\psi}$ are anticommuting Grassmann variables and should be considered as independent, $\psi_L=\frac{1}{2}(1+\gamma_5)\psi$. The Dirac algebra for 4-dimensional Euclidean spacetime is given by

$$\begin{aligned}
&\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}, \quad \gamma_\mu=(\gamma_\mu)^\dagger,\\
&\gamma_\mu\gamma_\nu=\delta_{\mu\nu}-\mathrm{i}\sigma_{\mu\nu}, \quad \sigma_{\mu\nu}=\frac{\mathrm{i}}{2}[\gamma_\mu,\gamma_\nu],\\
&\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_0.
\end{aligned} \tag{A1}$$

Defining $O_{\rm S}=\mathbb{1}$, $O_{\rm V}=\gamma_\mu$, $O_{\rm T}=(1/\sqrt{2})\sigma_{\mu\nu}$, $O_{\rm A}=\mathrm{i}\gamma_\mu\gamma_5$, $O_{\rm P}=\gamma_5$, we obtain the Fierz identities in the form

$$(\bar{\psi}^a O_X\psi_b)(\bar{\psi}_c O_X\psi_d)=\sum_Y C_{XY}(\bar{\psi}_a O_Y\psi_d)(\bar{\psi}_c O_Y\psi_b), \tag{A2}$$

where $X,Y=$ S,V,T,A,P and

$$C_{XY}=\begin{pmatrix} -\frac14 & -\frac14 & -\frac14 & -\frac14 & -\frac14\\ -1 & \frac12 & 0 & -\frac12 & 1\\ -\frac32 & 0 & \frac12 & 0 & -\frac32\\ -1 & -\frac12 & 0 & \frac12 & 1\\ -\frac14 & \frac14 & -\frac14 & \frac14 & -\frac14 \end{pmatrix}. \tag{A3}$$

The structure $(\bar{\psi}O_{\rm V}\psi)^2-(\bar{\psi}O_{\rm A}\psi)^2$ is invariant under Fierz transformations, and $(\bar{\psi}O_{\rm V}\psi)^2+(\bar{\psi}O_{\rm A}\psi)^2$ can be completely transformed into (pseudo-)scalar channels:

$$(\bar{\psi}O_{\rm V}\psi)^2+(\bar{\psi}O_{\rm A}\psi)^2=-2[(\bar{\psi}O_{\rm S}\psi)^2-(\bar{\psi}O_{\rm P}\psi)^2]. \tag{A4}$$

Further useful identities are

$$\begin{aligned}
&(\bar{\psi}O_{\rm T}\gamma_5\psi)^2=(\bar{\psi}O_{\rm T}\psi)^2,\\
&(\bar{\psi}\gamma_\alpha\gamma_\beta\gamma_\delta\psi)(\bar{\psi}\gamma_\alpha\gamma_\beta\gamma_\delta\psi)=10(\bar{\psi}\gamma_\mu\psi)^2+6(\bar{\psi}\gamma_\mu\gamma_5\psi)^2,\\
&(\bar{\psi}\gamma_\alpha\gamma_\beta\gamma_\delta\psi)(\bar{\psi}\gamma_\delta\gamma_\beta\gamma_\alpha\psi)=10(\bar{\psi}\gamma_\mu\psi)^2-6(\bar{\psi}\gamma_\mu\gamma_5\psi)^2.
\end{aligned} \tag{A5}$$

APPENDIX B: DETAILS OF THE FLUCTUATION MATRIX

In Eq. (9), we decompose the fluctuation matrix into $\mathcal{P}$ and $\mathcal{F}$, the latter containing the field dependence. The inverse propagator is diagonal in momentum space,

$$\mathcal{P}=\begin{pmatrix} q^2(1+r_B) & & & & \\ & 0 & Z_{\phi,k}q^2(1+r_B)+\overline{m}_k^2 & & \\ & Z_{\phi,k}q^2(1+r_B)+\overline{m}_k^2 & 0 & & \\ & & & 0 & -\not{q}^T(1+r_{\rm F})\\ & & & -\not{q}(1+r_{\rm F}) & 0 \end{pmatrix}. \tag{B1}$$

It involves the dimensionless cutoff functions r_B and r_{F}, being related to the components of R_k by

$$R_k^A = q^2 r_B\,, \quad R_k^\phi = Z_{\phi,k} q^2 r_B\,, \quad R_k^\psi = -\not{q}\, r_{\mathrm{F}}\,. \tag{B2}$$

Of course, these cutoff functions are supposed to satisfy the usual requirements of cutting of the infrared and suppressing the ultraviolet sufficiently strongly. The conventions for the Fourier transformation employed here can be characterized by

$$\psi(x) = \int \frac{d^4 q}{(2\pi)^4} \mathrm{e}^{\mathrm{i}qx} \psi(q), \quad \bar{\psi}(x) = \int \frac{d^4 q}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}qx} \bar{\psi}(q) \tag{B3}$$

for the fermions. As a consequence, the Fourier modes of the field Φ and Φ^T in Eq. (8) are then given by $\Phi(q) = [A(q), \phi(q), \phi^*(-q), \psi(q), \bar{\psi}^T(-q)]$ (column vector) and $\Phi^T(-q) = [A^T(-q), \phi(-q), \phi^*(q), \psi^T(-q), \bar{\psi}(q)]$ (row vector). Owing to the sign difference in the arguments of ψ and $\bar{\psi}$, the inverse propagator $\mathcal{P}$ in Eq. (B1) is symmetric under transposition T. Concerning the field-dependent part, the matrix $\mathcal{F}$ is also diagonal in momentum space for constant "background" fields and antisymmetric under transposition in all fermion-related components:

$$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 & -e\bar{\psi}\gamma_\mu & e\psi^T\gamma_\mu^T \\ 0 & 0 & 0 & \bar{h}_k \bar{\psi}_{\mathrm{R}} & -\bar{h}_k \psi_{\mathrm{L}}^T \\ 0 & 0 & 0 & -\bar{h}_k \bar{\psi}_{\mathrm{L}} & \bar{h}_k \psi_{\mathrm{R}}^T \\ e\gamma_\mu^T \bar{\psi}^T & -\bar{h}_k \bar{\psi}_{\mathrm{R}}^T & \bar{h}_k \bar{\psi}_{\mathrm{L}}^T & \bar{H} & -F^T \\ -e\gamma_\mu \psi & \bar{h}_k \psi_{\mathrm{L}} & -\bar{h}_k \psi_{\mathrm{R}} & F & H \end{pmatrix}, \tag{B4}$$

where

$$H = -\bar{\lambda}_{\sigma,k}[\psi\psi^T - \gamma_5 \psi\psi^T \gamma_5], \quad H^T = -H,$$
$$\bar{H} = -\bar{\lambda}_{\sigma,k}[\bar{\psi}^T\bar{\psi} - \gamma_5 \bar{\psi}^T \bar{\psi}\gamma_5], \quad \bar{H}^T = -\bar{H}, \tag{B5}$$
$$F = \bar{h}_k (P_{\mathrm{L}}\phi - P_{\mathrm{R}}\phi^*) + \bar{\lambda}_{\sigma,k}[(\bar{\psi}\psi) - \gamma_5(\bar{\psi}\gamma_5\psi) + \psi\bar{\psi} - \gamma_5 \psi\bar{\psi}\gamma_5],$$

and γ_μ^T is understood as transposition in Lorentz and/or Dirac space. The projectors P_{L} and P_{R} are defined as $P_{\mathrm{L,R}} = (1/2)(1 \pm \gamma_5)$. In Eq. (B5), we have dropped the A_μ dependence of the quantity F which is not needed for our computation.

APPENDIX C: EXACT FLOW EQUATION FOR FLOWING FIELD VARIABLES

In the standard formulation of the flow equation [2], the field variables of the k-dependent effective action $\Gamma_k[\phi]$ correspond to the so-called classical field defined via

$$\phi = \frac{\delta W_k[j]}{\delta j} \equiv \phi_\Lambda\,, \tag{C1}$$

where all explicit k dependence is contained in the cutoff dependence of W_k, the generating functional for connected Green's functions. The last identity in Eq. (C1) symbolizes that no explicit k dependence occurs for this classical field, and thereby the field ϕ at any scale is identical to the one at the ultraviolet cutoff Λ. (The functional dependence of ϕ on j contains, of course, an implicit k dependence.)

In the present work, we would like to study the flow of the effective action, now depending on a field variable that is allowed to vary during the flow. For an infinitesimal change of k, ϕ_k also varies infinitesimally:

$$\phi_{k-dk}(q) = \phi_k(q) + \delta\alpha_k(q) F[\phi_k, \ldots](q),$$
$$\partial_k \phi_k = -\partial_k \alpha_k F[\phi_k, \ldots], \tag{C2}$$

where $\delta\alpha_k$ is infinitesimal and F denotes some functional of possibly all fields of the system. The desired effective action $\Gamma_k[\phi_k]$ is derived from a modified functional W_k:

$$\mathrm{e}^{W_k[j, \ldots]} = \int \mathcal{D}\chi \mathcal{D}(\ldots) \mathrm{e}^{-S[\chi] - \Delta S_k[\chi_k] + \int j \chi_k + \cdots}. \tag{C3}$$

The dots again indicate the contributions of further fields, suppressed in the following, and we assume the quantum field χ to be a real scalar for simplicity. In contrast to the common formulation [2] the source j multiplies a k-dependent nonlinear field combination χ_k which obeys

$$\partial_k \chi_k = -\partial_k \alpha_k G[\chi_k, \ldots]. \tag{C4}$$

We also modify the infrared cutoff

$$\Delta S_k[\chi_k] = \frac{1}{2} \int \chi_k R_k \chi_k\,, \tag{C5}$$

which ensures that the momentum modes $\sim k$ of the actual field χ_k contribute to the flow at the scale k, regardless of its different form at other scales. Furthermore, the cutoff form of Eq. (C5) shall lead us to a simple form of the flow equation. The k-dependent classical field is given by

$$\phi_k := \langle \chi_k \rangle = \frac{\delta W_k}{\delta j}, \tag{C6}$$

and, as a consequence, the higher derivatives of $W_k[j]$ are now related to correlation functions of χ_k and no longer of[3] χ_Λ. The desired effective action is finally defined in the usual way via a Legendre transformation including a subtraction of the cutoff:

[3]Equation (C6) implies the relation $F[\phi_k, \ldots] = \langle G[\chi_k, \ldots] \rangle$. However, the definition of χ_k is often not needed explicitly. For our purposes it suffices to define $F[\phi_k, \ldots]$.

$$\Gamma_k[\phi_k] = -W_k[j[\phi_k]] + \int j[\phi_k]\phi_k - \Delta S_k[\phi_k]. \quad \text{(C7)}$$

Its flow equation is obtained by taking a derivative with respect to the RG scale k,

$$\begin{aligned}\partial_k\Gamma_k[\phi_k] &= \partial_k\Gamma_k[\phi_k]\big|_{\phi_k} + \int \frac{\delta\Gamma_k[\phi_k]}{\delta\phi_k}\partial_k\phi_k \\ &= \frac{1}{2}\mathrm{Tr}\frac{\partial_k R_k}{\Gamma_k^{(2)}[\phi_k]+R_k} \\ &\quad - \int \frac{\delta\Gamma_k[\phi_k]}{\delta\phi_k}F[\phi_k,\ldots]\partial_k\alpha_k. \end{aligned} \quad \text{(C8)}$$

The first term of this flow equation is evaluated for fixed ϕ_k and hence leads to the form of the standard flow equation with ϕ_Λ replaced by ϕ_k; the second term describes the contribution arising from the variation of the field variable under the flow. Some comments should be made:

(1) The variation (C2) of the field during the flow is *a priori* arbitrary; therefore, Eq. (C8) (together with some boundary conditions) determines $\Gamma_k[\phi_k]$ completely only if α_k is fixed.

(2) This redundancy can be used to arrive at a simple form for $\Gamma_k[\phi_k]$ adapted to the problem under consideration. For example, one may determine α_k (and $F[\phi_k,\ldots]$) in such a way that some unwanted coupling vanishes.

(3) This program can be generalized straightforwardly to a whole set of transformations α_k^i for different fields i. Furthermore, the whole functional dependence may be k dependent by replacing $\partial_k\phi_k^i = -\partial_k\alpha_k^i F^i \to -\mathcal{F}_k^i$.

(4) The generating functional of ϕ_Λ 1PI Green's functions $\Gamma_{k=0}[\phi_\Lambda]$ can be obtained from $\Gamma_{k=0}[\phi_{k=0}]$ by choosing $\alpha_{k=0}=0$. In practice, however, it is often more convenient to use "macroscopic degrees of freedom" $\phi_{k=0}$ different from the "microscopic" ones ϕ_Λ. Their respective relation then needs the computation of the flow of α_k.

(5) The present definition of the average action $\Gamma_k[\phi_k]$ is different from the effective action $\Gamma_k[\hat\phi_k]$ that is obtained by a field transformation of the flow equation with fixed fields as described in Appendix D. More precisely, consider the flow of the effective action $\Gamma_k[\phi_\Lambda]$ for fixed ϕ_Λ and perform a finite k-dependent field transformation $\hat\phi_k = \hat\phi_k[\phi,\alpha_k]$; then, even if the transformation was chosen in such a way that $\hat\phi_k$ were identical with ϕ_k of the present method, these effective actions would not coincide. The cutoff term acts differently in the two cases. In the case of a field transformation, the cutoff involves ϕ_Λ, which is subsequently expressed in terms of the new variables, whereas, in the present case, the cutoff is readjusted at each scale and involves χ_k. Although this does not affect physical results for exact solutions of the flow, this might lead to differences in approximate solutions of the flow, even if the approximation is implemented in the same way in either case.

APPENDIX D: FERMION-BOSON TRANSLATION BY FIELD TRANSFORMATIONS WITH FIXED CUTOFF

Here, we shall present a third approach to fermion-boson translation relying on the standard formulation of the flow equation in addition to a finite field transformation. We intend to identify a field transformation of the type

$$\begin{aligned}\hat\phi &= \phi + \hat\alpha_k\bar\psi_{\mathrm{L}}\psi_{\mathrm{R}} \Leftrightarrow \phi = \hat\phi - \hat\alpha_k\bar\psi_{\mathrm{L}}\psi_{\mathrm{R}}, \\ \hat\phi^* &= \phi^* - \hat\alpha_k\bar\psi_{\mathrm{R}}\psi_{\mathrm{L}} \Leftrightarrow \phi^* = \hat\phi^* + \hat\alpha_k\bar\psi_{\mathrm{R}}\psi_{\mathrm{L}},\end{aligned} \quad \text{(D1)}$$

so that an appropriate choice of a finite $\hat\alpha_k$ can transform the four-fermion coupling to zero. For simplicity, we work in the limit of a point-like interaction and dispense with an additional transformation of the type $\sim\beta_k\phi_k$. Within these restrictions, we shall not find the physical infrared behavior described in Secs. IV and V. The present study is intended only for a quantitative comparison of the different approaches, which can be done by restricting the field redefinitions in Sec. IV to Eq. (28) with q-independent α_k.

In contrast to the modified flow equation of Sec. IV and Appendix C, the source term and the infrared cutoff considered here involve the original fields. This approach therefore corresponds simply to a variable transformation in a given differential equation (exact flow equation). The transformed effective action for the hatted fields is obtained by simple insertion, $\Gamma_k[\hat\phi,\psi,A] \coloneqq \Gamma_k[\phi[\hat\phi],\psi,A]$. Except for additional derivative terms arising from the scalar kinetic term, the two actions are formally equivalent, where the new "hatted" couplings read in terms of the original ones

$$\begin{aligned}\hat m_k^2 &= \overline{m}_k^2, \\ \hat h_k &= \overline{h}_k + \overline{m}_k^2\hat\alpha_k, \\ \hat\lambda_{\sigma,k} &= \overline{\lambda}_{\sigma,k} - \overline{h}_k\hat\alpha_k - \tfrac{1}{2}\overline{m}_k^2\hat\alpha_k^2.\end{aligned} \quad \text{(D2)}$$

Again, the transformation function $\hat\alpha_k$ is finally fixed by demanding that the beta function $\hat\beta_{\lambda_\sigma}$ for the hatted four-fermion coupling $\hat\lambda_{\sigma,k}$ vanishes,

$$\hat\beta_{\lambda_\sigma}(\hat m_k^2, \hat h_k, \hat\lambda_{\sigma,k}, \hat\alpha_k, \partial_t\hat\alpha_k) = 0, \quad \text{(D3)}$$

with the boundary conditions $\overline{\lambda}_{\sigma,k=\Lambda}=0$ and $\overline{\alpha}_{k=\Lambda}=0$, which express complete bosonization at Λ (this also implies $\hat\lambda_{\sigma,k=\Lambda}=0$). The new beta functions can now be determined from the standard flow equation, being subject to the field transformation. Following Appendix A of [5], the basic equation is

$$\partial_t\Gamma_k[\hat\Phi]=\partial_t\Gamma_k|_\Phi-\partial_t\hat\phi^*\bigg|_\Phi\frac{\delta}{\delta\hat\phi^*}\Gamma_k[\hat\Phi]-\partial_t\hat\phi\bigg|_\Phi\frac{\delta}{\delta\hat\phi}\Gamma_k[\hat\Phi]$$
$$=\frac{1}{2}\mathrm{STr}\,\tilde\partial_t\ln(\Gamma_k^{(2)}+R_k)$$
$$-\hat\alpha_k\left[\bar\psi_{\rm L}\psi_{\rm R}\frac{\delta\Gamma_k}{\delta\hat\phi}-\bar\psi_{\rm R}\psi_{\rm L}\frac{\delta\Gamma_k}{\delta\hat\phi^*}\right].\tag{D4}$$

Although there seems to be a formal resemblance to Eq. (30), there is an important difference: Eq. (D4) is equivalent to the standard flow equation, whereas Eq. (30) is not; the latter is derived with a different cutoff term. Without resorting to the calculation of Sec. II, we can evaluate this equation completely from the transformed truncation $\Gamma_k[\hat\phi,\psi,A]$ and the field transformations (D1) according to

$$(\Gamma_k^{(2)})^T_{ab}\equiv\frac{\vec\delta}{\delta\Phi_a^T}\Gamma_k\frac{\overleftarrow\delta}{\delta\Phi_b}=\left(\frac{\vec\delta}{\delta\Phi_a^T}\hat\Phi_i^T\right)\frac{\vec\delta}{\delta\hat\Phi_i^T}\Gamma_k\frac{\overleftarrow\delta}{\delta\hat\Phi_j}\left(\hat\Phi_j\frac{\overleftarrow\delta}{\delta\Phi_b}\right)$$
$$+(-1)^{(\hat\Phi,\Phi^T)}\left(\Gamma_k\frac{\overleftarrow\delta}{\delta\hat\Phi_i}\right)\left(\frac{\vec\delta}{\delta\Phi_a^T}\hat\Phi_i\frac{\overleftarrow\delta}{\delta\Phi_b}\right),\tag{D5}$$

where $(\Phi_l,\Phi_m)=1$ if fermionic components in Φ_l as well as Φ_m are considered, and $(\Phi_l,\Phi_m)=0$ otherwise; the indices a,b,i,j label the different field components of $\Phi,\hat\Phi$.

From Eq. (D4), or equivalently Eq. (D2), we deduce that the desired hatted beta functions are related to the original ones by

$$\partial_t\hat m_k^2\equiv\hat\beta_m=\beta_m,$$
$$\partial_t\hat h_k\equiv\hat\beta_h=\beta_h+\hat\alpha_k\beta_m+\hat m_k^2\partial_t\hat\alpha_k,\tag{D6}$$
$$\partial_t\hat\lambda_{\sigma,k}\equiv\hat\beta_{\lambda_\sigma}=\beta_{\lambda_\sigma}-\hat\alpha_k\beta_h-\tfrac{1}{2}\hat\alpha_k^2\beta_m-\hat h_k\partial_t\hat\alpha_k,$$

where the right-hand sides of Eq. (D6) have to be expressed in terms of the hatted couplings by means of the relations (D2). Now we determine $\hat\alpha_k$ by demanding that $\hat\beta_{\lambda_\sigma}$ vanishes for vanishing $\hat\lambda_{\sigma,k}$, so that no four-fermion coupling arises during the flow. Introducing dimensionless quantities for the hatted couplings, $\tilde\alpha_k=k^2Z_{\phi,k}^{1/2}\hat\alpha_k$, $\epsilon_k=k^{-2}Z_{\phi,k}^{-1}\hat m_k^2$, $h_k=Z_{\phi,k}^{-1/2}\hat h_k$, we end up with the flow equations

$$\partial_t\epsilon_k=-2\epsilon_k+\frac{1}{8\pi^2}(h_k-\epsilon_k\tilde\alpha_k)^2,$$

$$\partial_t(h_k-\epsilon_k\tilde\alpha_k)=\left[-\frac{e^2}{2\pi^2}-\frac{1}{4\pi^2}\tilde\alpha_k(h_k-\tfrac{1}{2}\epsilon_k\tilde\alpha_k)\right]$$
$$\times(h_k-\epsilon_k\tilde\alpha_k),\tag{D7}$$

$$\partial_t\tilde\alpha_k=2\tilde\alpha_k-\frac{9}{8\pi^2}\frac{e^4}{h_k}-\frac{1}{2\pi^2}e^2\tilde\alpha_k$$
$$+\frac{1}{16\pi^2}\left(h_k-2\epsilon_k\tilde\alpha_k+\frac{\epsilon_k^2}{2}\frac{\tilde\alpha_k^2}{h_k}\right)\tilde\alpha_k$$
$$+\frac{1}{8\pi^2}\frac{2+\epsilon_k}{(1+\epsilon_k)^2}\frac{1}{h_k}(h_k-\tfrac{1}{2}\epsilon_k\tilde\alpha_k)$$
$$\times(h_k-\epsilon_k\tilde\alpha_k)^2\tilde\alpha_k$$
$$+\frac{1}{32\pi^2}\frac{3+\epsilon_k}{(1+\epsilon_k)^3}\frac{1}{h_k}(h_k-\epsilon_k\tilde\alpha_k)^4,$$

where we have inserted the threshold-function values as given in Appendix E for illustrative purposes. These equations have to be read side by side with Eqs. (39) and (46). Contrary to the latter, the present flow equations are completely coupled; in particular, the flow for $\tilde\alpha_k$ is not disentangled as it is in the case of Eqs. (39) and (46). In the flow equation for the mass, we again observe a critical mass-to-Yukawa-coupling ratio at the bosonization scale, corresponding to the infrared unstable fixed point $\tilde\epsilon_1^*$ mentioned in Eq. (22): from a numerical solution, we find that $\tilde\epsilon_\Lambda|_{\rm crit}=\epsilon_\Lambda/h_\Lambda^2|_{\rm crit}\simeq\tilde\epsilon_1^*$ is hardly influenced by the $\overline{h_k^4}$ term. The actual initial value of this ratio at the bosonization scale with respect to $\tilde\epsilon_\Lambda|_{\rm crit}$ hence determines whether the system flows towards the phase with dynamical symmetry breaking or not.

In order to compare the present method with the one employed in Sec. IV, we plot a numerical solution of Eqs. (D7) in Fig. 3 (solid lines) and compare it to a solution of the corresponding equations (39) and (46) (dashed lines) without those terms arising from the additional transformation $\sim\partial_t\beta_k$, which is not considered in Eqs. (D7). In this figure, it becomes apparent that both methods do not only agree qualitatively, but also quantitatively to a high degree—as they should. The minor differences in these approaches can be attributed to the different formulation of the cutoff, and thereby reflect the inherent cutoff dependence for approximative solutions to the otherwise exact flow equation.

The same conclusion can be drawn from the flow equation for the dimensionless combination $\tilde\epsilon_k$ as defined in Eq. (20). Although the $\tilde\epsilon_k$ flow equation derived from Eqs. (D7) is comparably extensive (we shall not write it down here)

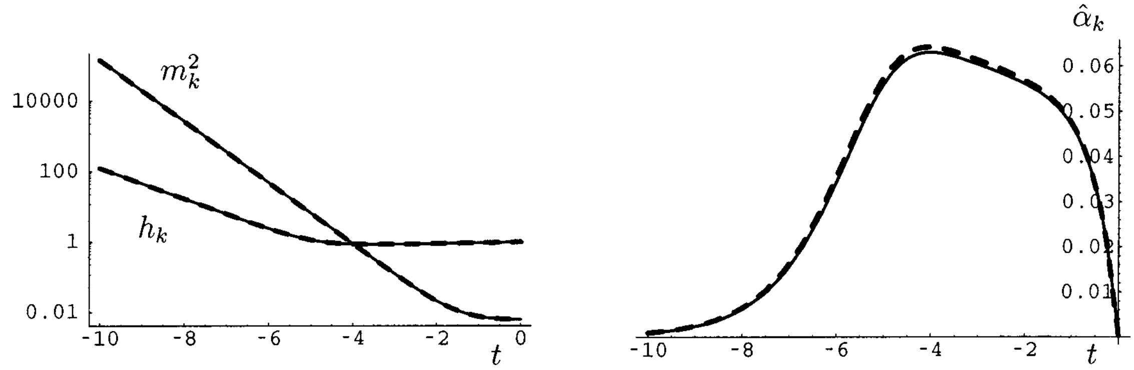

FIG. 3. Flows of ϵ_k, h_k and $\tilde{\alpha}_k$ in the symmetric phase ($h_\Lambda=1$, $e=1$, $\epsilon_\Lambda=1.16[1/(16\pi^2)]$). The solid lines represent a solution to Eqs. (D7); the dashed lines correspond to the analogous flow employing the method of Sec. IV and Appendix C (without the $\sim\partial_t\beta_k$ transformation). The plots are representative of a wide range of initial conditions.

and not identical to Eq. (21), the fixed-point structure remains nevertheless the same, and the $\tilde{\epsilon}_k$ flow reduces exactly to Eq. (21) for $k\to\Lambda$, where all our approaches agree. Moreover, the position of the infrared stable fixed point $\tilde{\epsilon}_2^*$ also remains the same in the infrared to leading order in e, so that the different approaches describe the same physics.

To summarize, employing the method of field transformation in the flow equation for fixed cutoff, the same properties of the system can be derived with a similar numerical accuracy in comparison to the flow equation proposed in Sec. IV and Appendix C. However, the structure of the resulting flow equations derived in this appendix appears to be more involved, and we expect this to be a generic feature of field transformation in the flow equation for fixed cutoff—at least within the usual approximation schemes.

APPENDIX E: CUTOFF FUNCTIONS

For concrete computations, we have to specify the cutoff functions. Here we shall use optimized cutoff functions as proposed in [14], which furnish a fast convergence behavior and provide for simple analytical expressions. Employing the nomenclature of [10], we use the dimensionless cutoff functions ($y=q^2/k^2$)

$$\begin{aligned} r_B(y)&=\left(\frac{1}{y}-1\right)\theta(1-y),\\ p(y)&=y(1+r_B(y))=y+(1-y)\theta(1-y),\\ r_F(y)&=\left(\frac{1}{\sqrt{y}}-1\right)\theta(1-y),\\ p_F(y)&=y(1+r_F(y))^2\to p(y). \end{aligned} \tag{E1}$$

Here we have set the normalization constants c_B and c_F mentioned in [14] to the values $c_B=1/2$ and $c_F=1/4$, so that fermionic and bosonic fluctuations are cut off at the same momentum scale $q^2=k^2$. This is natural in our case in order to avoid the situation in which fermionic modes which are already integrated out are transformed into bosonic modes which still have to be integrated out or vice versa.

For these cutoff functions, the required threshold functions evaluate to

$$l_n^{(F)d}(\omega)=(\delta_{n,0}+n)\frac{2}{d}\frac{1}{(1+\omega)^{n+1}}, \tag{E2}$$

$$l_{n_1,n_2}^{(FB)d}(\omega_1,\omega_2)=\frac{2}{d}\frac{1}{(1+\omega_1)^{n_1}(1+\omega_2)^{n_2}}\left[\frac{n_1}{1+\omega_1}+\frac{n_2}{1+\omega_2}\right]. \tag{E3}$$

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