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Renormalization of the energy-momentum tensor and the validity of the equivalence principle at finite temperature

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Using the techniques of finite-temperature field theory we renormalize the electromagnetic and gravitational couplings of an electron which is immersed in a heat bath with $T \ll m_e$. By taking the nonrelativistic limit, we demonstrate that the inertial and gravitational masses are unequal. The implications for the equivalence principle are discussed.

I. INTRODUCTION

A cornerstone of the theory of general relativity is the principle of equivalence, which states, in its weak form, that the gravitational acceleration is identical for all bodies, or that the inertial and gravitational masses are equal.[1] This equality has been tested via the Eötvös experiment and its successors and is now verified to the level of one part in 10^{12}.[2] Since in a quantum theory a portion of a particle's mass (formally infinite) arises from radiative corrections, these must also obey the equivalence principle and this has been demonstrated via detailed theoretical calculation.[3] However, in a quantum field theory at temperature $T>0$, a fraction of the mass of a particle arises through the finite-temperature component of the radiative corrections.[4] We have pointed out previously that these terms do *not* satisfy the equivalence principle.[5] However, the format was as a brief note, and we present here a more substantive discussion of this result.

There are two limits to finite-temperature calculations which are natural to consider, $T \ll m$ and $T \gg m$. In the former the temperature-dependent effects arise due to interaction with a photon heat bath, with the effect of massive particles being suppressed by $O(\exp(-m/T))$. The interpretation of the theory is simplest in this situation, as one can sensibly consider the case of nonrelativistic motion, for which our intuition is well developed. At very high temperatures, $T \gg m$, the interpretation becomes considerably more difficult, as there then exists a background heat bath of particle-antiparticle pairs. Then not only are the calculations more complex, but in addition all energy states are filled up to a Fermi energy $E \sim T$, so that the Pauli effect would appear to prevent the particle being studied from being nonrelativistic. For these reasons, we shall confine our discussion to the interpretively clearer case of $T \ll m$.

Our program is then as follows. In Sec. II, we outline, for completeness and in order to define terms, the demonstration of the equivalence principle at zero temperature for the case of an elementary spin-$\frac{1}{2}$ system. In Sec. III, we point out the modifications required in order to extend this result to finite temperature and demonstrate that the gravitational and inertial masses indeed renormalize differently. Section IV outlines the changes which obtain for the case of a spin-zero system. Although the calculation is somewhat different, the same inequality of gravitational and inertial mass is found. Finally, in Sec. V we discuss the implications of our results with respect to the validity of the equivalence principle.

II. RENORMALIZATION AT $T=0$

Before extending our discussion to the relatively new area of finite-temperature field theory, it is important to understand how the equivalence principle obtains in the familiar zero-temperature case. The only problem here is in carefully defining what is meant by gravitational and inertial mass. We shall do this by placing our test particle, an electron, in an external electromagnetic and/or gravitational field and studying its consequent motion in the nonrelativistic limit, where our intuition is best established.

It is well known that conventional or Pauli-Villars renormalization does not automatically respect the gauge invariance possessed by a fundamental theory, so we shall instead use dimensional regularization throughout. We begin with a bare spin-$\frac{1}{2}$ Lagrangian

$$\mathscr{L}_0=\bar{\psi}_0(i\not{\nabla}-m_0)\psi_0 \,. \tag{1}$$

As is well known, radiative corrections modify the simple theory. Thus, calculating the so-called self-energy diagram in Fig. 1, we find

$$\Sigma(p)=-ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{\gamma_\mu(\not{p}-\not{k}+m)\gamma^\mu}{k^2-2p\cdot k+p^2-m^2}$$
$$=-\frac{\alpha}{4\pi}[m_0-(\not{p}-m_0)]\left[\frac{3}{\epsilon}-4\right], \tag{2}$$

where

$$\frac{1}{\epsilon}=\frac{2}{d-4}+\gamma-\ln 4\pi+\ln m^2 \tag{3}$$

with d representing the number of spacetime dimensions. The modified Dirac equation becomes

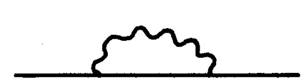

FIG. 1. The electron self-energy diagram.

$$[\not{p}-m_0-\Sigma(p)]u(p)=0\ . \tag{4}$$

From this, we can read off the usual mass and wave-function renormalization constants

$$\delta m=\Sigma(p)\,|_{\not{p}=m}=-\frac{\alpha}{4\pi}\left[\frac{3}{\epsilon}-4\right]m_0\ , \tag{5}$$

$$Z_2^{-1}-1=-\left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{\not{p}=m}=-\frac{\alpha}{4\pi}\left[\frac{3}{\epsilon}-4\right]. \tag{6}$$

The Lagrangian then becomes

$$\mathscr{L}=Z_2[\bar{\psi}_R(x)(i\not{\nabla}-m_R)\psi_R(x)+\bar{\psi}_R(x)\delta m\,\psi_R(x)]\ , \tag{7}$$

where

$$\psi_R(x)=Z_2^{-1/2}\psi_0(x) \tag{8}$$

is the renormalized field operator and

$$m_R=m_0+\delta m \tag{9}$$

is the renormalized mass, which represents the observed mass of the particle.

In order to find the inertial mass, we shall place the electron in an external electromagnetic field. Then the bare interaction Lagrangian

$$\mathscr{L}_0^{\rm int}=-e_0\bar{\psi}_0\gamma_\mu\psi_0A^\mu_{\rm ext} \tag{10}$$

must be modified by the usual radiative-correction diagrams shown in Fig. 2. Thus the vertex correction diagram, Fig. 2(d), yields

$$\Lambda_\mu(p',p)\equiv\Lambda_\mu(p,p)+\Lambda_\mu^{(c)}(p',p)$$
$$=-ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k^2-2p'\cdot k)(k^2-2p\cdot k)}$$
$$\times\gamma_\alpha(\not{p}'-\not{k}+m)\gamma_\mu(\not{p}-\not{k}+m)\gamma^\alpha\ . \tag{11}$$

Here $\Lambda_\mu(p,p)$ can be determined via the Ward identities

$$\bar{u}(p')\Lambda_\mu(p,p)u(p)=\bar{u}(p')\frac{\partial}{\partial p^\mu}\Sigma(p)u(p)$$
$$=-(Z_2^{-1}-1)\bar{u}(p')\gamma_\mu u(p)\ , \tag{12}$$

while $\Lambda_\mu(p',p)$ can be calculated directly, yielding, correct to terms of $O(q/m)$,

$$\bar{u}(p')\Lambda_\mu^{(c)}(p',p)u(p)=\frac{\alpha}{2\pi}\bar{u}(p')(-i\sigma_{\mu\nu}q^\nu)\frac{1}{2m_R}u(p)\ , \tag{13}$$

where $q=p-p'$ is the four-momentum transferred by the external field. Then, adding the self-energy and mass-renormalization diagrams to the vertex correction and dividing by $\sqrt{Z_2}$ for each electron leg we find

$$\begin{aligned}
M_\mu^{(0)}&=-e_0\bar{u}(p')\gamma_\mu u(p)\ \ [\text{Fig. 2(a)}]\ ,\\
M_\mu^{\rm SE}&=-e_0\bar{u}(p')\left[\gamma_\mu\frac{1}{\not{p}-m}\delta m+\delta m\frac{1}{\not{p}-m}\gamma_\mu-2(Z_2^{-1}-1)\gamma_\mu\right]u(p)\ \ [\text{Fig. 2(b)}]\ ,\\
M_\mu^{\rm CT}&=e_0\bar{u}(p')\left[\gamma_\mu\frac{1}{\not{p}-m}\delta m+\delta m\frac{1}{\not{p}-m}\gamma_\mu\right]u(p)\ \ [\text{Fig. 2(c)}]\ ,\\
M_\mu^{V}&=-e_0\bar{u}(p')\left[\gamma_\mu(Z_2^{-1}-1)-i\frac{\alpha}{2\pi}\sigma_{\mu\nu}q^\nu\frac{1}{2m}\right]u(p)\ \ [\text{Fig. 2(d)}]\ .
\end{aligned} \tag{14}$$

Then

$$M_\mu=\frac{1}{Z_2}(M_\mu^{(0)}+M_\mu^{\rm SE}+M_\mu^{\rm CT}+M_\mu^{V})$$
$$=-e_0\bar{u}(p')\left[\gamma_\mu-i\frac{\alpha}{2\pi}\sigma_{\mu\nu}q^\nu\frac{1}{2m}\right]u(p)+O\left[\frac{q^2}{m^2}\right]. \tag{15}$$

Finally, we must append the vacuum polarization contribution, Fig. 3, which gives

$$M_\mu^{\rm VP}=-e_0\bar{u}(p')\gamma_\mu u(p)\frac{\alpha}{3\pi\epsilon}$$
$$\equiv e_0\bar{u}(p')\gamma_\mu u(p)(Z_3-1) \tag{16}$$

so that the full renormalized vertex becomes, including now a factor $Z_3^{-1/2}$ for the external photon,

$$M_\mu^{\rm total}=Z_2^{-1}Z_3^{-1/2}(M_\mu^{(0)}+M_\mu^{\rm SE}+M_\mu^{\rm CT}+M_\mu^{V}+M_\mu^{\rm VP})$$
$$=-e_R\bar{u}(p')\left[\gamma_\mu-i\frac{\alpha}{2\pi}\sigma_{\mu\nu}q^\nu\frac{1}{2m_R}\right]u(p)\ , \tag{17}$$

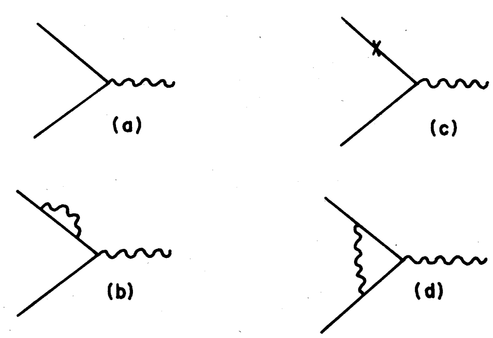

FIG. 2. The electromagnetic-vertex-renormalization diagrams in spinor electrodynamics.

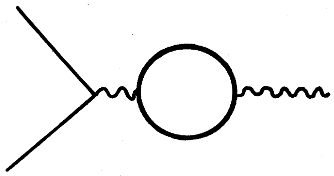

FIG. 3. The vertex correction due to vacuum polarization.

where

$$e_R=\sqrt{Z_3}e_0=e_0\left[1-\frac{\alpha}{6\pi\epsilon}\right] \tag{18}$$

is the renormalized charge and corresponds to the charge observed experimentally.

This discussion of renormalization of the electromagnetic vertex is well known, of course, and is described in many texts.[6] We have included it in detail, since it forms the outline of our later finite-temperature discussion as well.

One defines the inertial mass by placing the electron in an external electromagnetic field, which is described by a vector potential A^μ. The resultant Dirac equation is

$$\left[\not p-m_R-e_R\not A+ie_R\frac{1}{2m}\sigma_{\mu\nu}A^\mu q^\nu\frac{\alpha}{2\pi}\right]u(p)=0\ . \tag{19}$$

We can make a nonrelativistic reduction by writing the wave function in terms of two-component spinors

$$u(p)=\begin{bmatrix}u\\ l\end{bmatrix} \tag{20}$$

and solving for the leading (upper) component, yielding

$$i\frac{\partial}{\partial t}u=\left[m_R+\frac{p^2}{2m_R}-e_R\frac{\vec{A}\cdot\vec{p}}{m_R}+e_R\phi -e_R\vec{\sigma}\cdot\vec{B}\frac{1}{2m_R}\left[1+\frac{\alpha}{2\pi}\right]\right]u\ . \tag{21}$$

We can identify the inertial mass by measuring the acceleration of a particle acted upon by an external electric field $\vec{E}$. The corresponding electromagnetic potential may be chosen as

$$\vec{A}=0,\quad \vec{E}=-\vec{\nabla}\phi \tag{22}$$

and we find

$$\vec{a}=-[H,[H,\vec{r}]]=\frac{e_R\vec{E}}{m_R}\quad \text{or } m_I=m_R\ . \tag{23}$$

Thus the inertial mass is simply the renormalized mass, as is well known.

In order to define the gravitational mass, we must examine the corresponding renormalization of the energy-momentum tensor $T_{\mu\nu}$. In this case the bare vertex is given by[7]

$$T_{\mu\nu}=2\frac{\partial}{\partial g^{\mu\nu}}(\sqrt{-g}\,\mathscr{L}_0)=\bar\psi_0\left[\frac{i}{2}(\gamma_\mu\nabla_\mu+\gamma_\nu\nabla_\mu) -g_{\mu\nu}(i\not\nabla-m_0)\right]\psi_0\ . \tag{24}$$

The relevant radiative-correction diagrams are shown in Fig. 4. Comparing with Fig. 2, we note that there exist two additional terms. Figure 4(e) is a contact $\bar\psi\psi A_\mu$ term which arises because of the derivative couplings which are present in Eq. (24), and Fig. 4(f) accounts for the feature that the energy-momentum tensor can couple directly to the photon, unlike the corresponding electromagnetic vertex. The other diagrams, Figs. 4(a)—4(d), are just direct analogs of our previous calculation.

For simplicity we perform our calculation at $q=0$ and find

$$\begin{aligned}
&\bar u(p)\tfrac{1}{2}(p_\mu\gamma_\nu+p_\nu\gamma_\mu)u(p)=M^{(0)}_{\mu\nu}\quad [\text{Fig. }4(\text{a})]\ ,\\
&-\frac{\alpha}{4\pi}\bar u(p)\tfrac{1}{2}(p_\mu\gamma_\nu+\gamma_\nu p_\mu)u(p)\left[-\frac{6}{\epsilon}+8\right]=M^{\text{SE}}_{\mu\nu}+M^{\text{CT}}_{\mu\nu}\quad [\text{Figs.}4(\text{b})\text{ and }4(\text{c})]\ ,\\
&-\frac{\alpha}{4\pi}\bar u(p)\left[\tfrac{1}{2}(p_\mu\gamma_\nu+\gamma_\nu p_\mu)\left[\frac{7}{3\epsilon}-\frac{56}{9}\right]-mg_{\mu\nu}\left[\frac{4}{3\epsilon}-\frac{29}{9}\right]\right]u(p)=M^{V}_{\mu\nu}\quad [\text{Fig. }4(\text{d})]\ ,\\
&-\frac{\alpha}{4\pi}\bar u(p)\left[\tfrac{1}{2}(p_\mu\gamma_\nu+\gamma_\mu p_\nu)\left[-\frac{2}{\epsilon}+6\right]+mg_{\mu\nu}\left[\frac{5}{\epsilon}-7\right]\right]u(p)=M^{C}_{\mu\nu}\quad [\text{Fig. }4(\text{e})]\ ,\\
&-\frac{\alpha}{4\pi}\bar u(p)\left[\tfrac{1}{2}(p_\mu\gamma_\nu+\gamma_\mu p_\nu)\left[\frac{8}{3\epsilon}-\frac{34}{9}\right]-mg_{\mu\nu}\left[\frac{2}{3\epsilon}+\frac{2}{9}\right]\right]u(p)=M^{P}_{\mu\nu}\quad [\text{Fig. }4(\text{f})]\ .
\end{aligned} \tag{25}$$

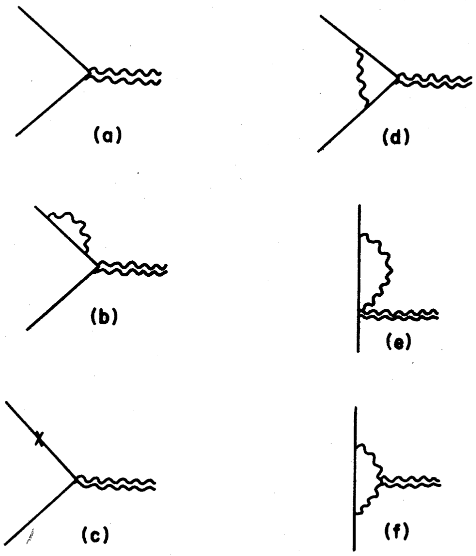

FIG. 4. The radiative corrections to the spinor energy-momentum tensor. The double wavy line represents the $T_{\mu\nu}$ coupling.

There is no "vacuum polarization" diagram in this case since we are treating gravity only to first order. If we now add the contributions and divide by Z_2 we find

$$M_{\mu\nu}^{\text{total}}=\frac{1}{Z_2}(M_{\mu\nu}^{(0)}+M_{\mu\nu}^{\text{SE}}+M_{\mu\nu}^{\text{CT}}+M_{\mu\nu}^{V}+M_{\mu\nu}^{C}+M_{\mu\nu}^{P})$$
$$=\bar{u}(p)\tfrac{1}{2}(\gamma_\mu p_\nu+\gamma_\nu p_\mu)u(p)\ . \tag{26}$$

Thus the energy-momentum tensor is unchanged by renormalization. However, here $E,\vec{p}$ are related using the renormalized mass

$$E=(\vec{p}^{\,2}+m_R{}^2)^{1/2}\ . \tag{27}$$

In order to define the gravitational mass we place the electron in an external gravitational field described by a potential $\phi_g(x)$. To lowest order then, we can write the resultant metric as

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\ , \tag{28}$$

where in the harmonic gauge[8]

$$h_{\mu\nu}=2\phi_g\delta_{\mu\nu}\ . \tag{29}$$

The corresponding Dirac equation

$$(\not p-m_R-\tfrac{1}{2}h_{\mu\nu}T^{\mu\nu})u(p)=0 \tag{30}$$

becomes, after nonrelativistic reduction,

$$i\frac{\partial}{\partial t}u=\left[m_R+\frac{p^2}{2m_R}+m_R\phi_g\right]u\ , \tag{31}$$

and the resultant acceleration

$$\vec{a}=-[H,[H,\vec{r}]]=-\frac{m_R\nabla\phi_g}{m_R} \tag{32}$$

yields the gravitational mass

$$m_g=m_R\ . \tag{33}$$

Thus we have demonstrated in the context of quantum field theory the well-known equivalence of inertial and gravitational mass. In Sec. III, we shall repeat this calculation for the situation where the electron finds itself immersed in a heat bath at temperature $T\ll m_R$.

III. RENORMALIZATION AT $0<T\ll m_R$

The modifications required for perturbation techniques when $T>0$ are well known. At $T=0$ the propagation of particles between spacetime points x_1,x_2 is described by the Feynman propagator

$$D_F(x_2,x_1)=\langle 0\,|\,T(\phi(x_2)\phi(x_1))\,|\,0\rangle\ . \tag{34}$$

Here if we expand $\phi(x)$ in terms of free-field creation and annihilation operators

$$\phi(x)=\sum_{\vec{p}}[a(\vec{p})e^{-ip\cdot x}+a^\dagger(\vec{p})e^{ip\cdot x}]\ , \tag{35}$$

and require that the vacuum is empty

$$a^\dagger(\vec{p})a(\vec{p})\,|\,0\rangle=0\ , \tag{36}$$

we obtain the usual propagator in momentum space

$$D_F(p)=\int d^4x\,e^{-ip\cdot(x_2-x_1)}D_F(x_2,x_1)$$
$$=\frac{i}{p^2-m^2}\ . \tag{37}$$

In the case of a theory at $T>0$, the only difference is that the $T=0$ vacuum $|0\rangle$ must be replaced by the finite-temperature vacuum $|\,0\rangle_\beta$, which is defined by[9]

$$a^\dagger(p)a(p)\,|\,0\rangle_\beta=n_B(E)\,|\,0\rangle_\beta\ , \tag{38}$$

where

$$n_B(E)=\frac{1}{\exp(\beta E)-1} \tag{39}$$

is the usual Bose-Einstein distribution function. Calculating the expectation value for this vacuum yields the finite-temperature propagator[10]

$${}^\beta D_F(p)=\frac{i}{p^2-m^2}+2\pi\delta(p^2-m^2)n_B(E)\ . \tag{40}$$

This is the only modification required at finite temperature. The vertices are unchanged by the heat bath.

It is now straightforward to apply these finite-temperature propagators to the previous diagrams in order to see what modifications are produced. We note that there is in the finite-temperature propagator a natural separation into a $T=0$ term, which we have already discussed, and a temperature-dependent modification. Also, because of the thermal distribution function all finite-temperature integrals are ultraviolet convergent. Since the infrared singular terms cancel in the usual way, all temperature-dependent modifications to the theory are finite. The theory can be renormalized at $T=0$, and then any additional changes in the parameters of the theory introduced by temperature must be calculable and finite.[11]

Our restriction to temperatures $T\ll m$ should be noted at this point. When one evaluates a specific perturbation-theory diagram, there are in general finite-temperature modifications which arise both from the fermion *and* from the photon propagators. However, at low temperatures any temperature-dependent modification due to a fermion line is suppressed by $O(e^{-m_e/T})$ and can be neglected. Only the $T\neq 0$ changes in the photon propagators will be required. Since $m_e \sim 10^{10}$ °K, while even the hottest star has $T \lesssim 10^8$ °K, this restriction to "low" temperatures is not particularly stringent.

Having made these introductory comments we can now proceed to calculate the temperature-dependent modifications to all diagrams discussed in Sec. II. We begin with the self-energy in order to identify the mass shift and wave-function renormalization constraints. We find[12]

$$^{\beta}\Sigma(p)=\frac{\alpha}{4\pi^2}[I_A(\not p-m)+\not I+\not L(p^2-m^2)+\cdots], \quad (41)$$

where

$$I_A=8\pi\int\frac{dk}{k}n_B(k),$$
$$I_\mu=2\int\frac{d^3k}{k_0}n_B(k)\frac{(k_0,\vec{k})}{Ek_0-\vec{p}\cdot\vec{k}}, \quad (42)$$
$$L_\mu=-\frac{1}{E}\int\frac{d^3k}{k_0}n_B(k)\frac{(k_0,\vec{k})}{(Ek_0-\vec{p}\cdot\vec{k})^2}.$$

Thus the standard decomposition into a Lorentz-invariant mass shift and a wave-function renormalization proportional to $\not p-m$ does not obtain. Instead, because of the preferred frame associated with the heat bath, noncovariant terms appear in the self-energy. One must then be very cautious in defining what is meant by "mass." One possible definition, which we shall call the "phase-space mass" is given by the location of the pole in the propagator. This occurs at

$$E^2=\vec{p}\,^2+m_p{}^2 \quad (43)$$

with

$$m_p{}^2=m_R{}^2+2\frac{\alpha}{4\pi^2}p\cdot I\cong\left[m_R+\frac{\alpha\pi}{3}\frac{T^2}{m_R}\right]^2. \quad (44)$$

Thus there exists a temperature-dependent mass shift

$$\delta m_\beta=\frac{\alpha\pi}{3}\frac{T^2}{m_R}. \quad (45)$$

This shift presumably represents the additional inertia generated by the interaction of the electron with the real photons which make up the heat bath.

Likewise the wave-function renormalization constant can be determined by requiring that the fields are properly normalized. The correct prescription has been given elsewhere, and we find[13]

$$^{\beta}Z_2{}^{-1}-1=-\frac{\alpha}{4\pi^2}(I_A+2p\cdot L)$$
$$=-\frac{\alpha}{4\pi^2}\left[I_A-\frac{I_0}{E}\right]. \quad (46)$$

Note that this phase-space mass is not usually included in a discussion of the different types of mass. Nevertheless, it can be given a clear operational definition in terms of threshold and phase-space behavior for particle reactions. Thus, for example, the decay of a neutral boson (H^0) into an e^+e^- pair cannot take place if the H^0 mass is below $2m_p$, even if m_{H^0} is greater than $2m_R$. One can thus imagine measuring this phase-space mass by looking for the threshold of various reactions. It could also be determined by careful study of phase-space distributions of a specific process. (Both techniques are presently being utilized in the search for a possible neutrino mass.) In principle, the phase-space mass can be distinct from either the inertial or gravitational masses. However, we shall show below that at least in the nonrelativistic limit the phase-space and inertial masses coincide.

In order to define the inertial mass operationally we need the finite-temperature modifications to the renormalized charge vertex, i.e., Figs. 2(a)—2(d). [Note that we need *not* consider $T\neq 0$ changes to the vacuum polarization diagram, as these are associated with fermion lines and hence are $O(e^{-m_eT})$.] Using the T-modified photon propagator, we then find

$$^{\beta}M_\mu^{(0)}=-e_R(\bar u_\beta\gamma_\mu u_\beta-\bar u\gamma_\mu u) \quad [\text{Fig. 2(a)}],$$
$$^{\beta}M_\mu^{\rm SE}=-e_R\bar u(p')\left[\gamma_\mu\left[\frac{1}{\not p-m}\not I(p)+I_A+(\not p+m)\not L(p)\right]+\left[\not I(p')\frac{1}{\not p'-m}+I_A+\not L(p')(\not p'+m)\right]\gamma_\mu\right]u(p)\frac{\alpha}{4\pi^2} \quad [\text{Fig. 2(b)}], \quad (47)$$
$$^{\beta}M_\mu^{\rm CT}=e_R\bar u(p')\left[\gamma_\mu\frac{1}{\not p-m}\frac{\alpha}{4\pi^2}\not I(p)+\frac{\alpha}{4\pi^2}\not I(p')\frac{1}{\not p'-m}\gamma_\mu\right]u(p) \quad [\text{Fig. 2(c)}],$$
$$^{\beta}M_\mu^{V}=-e_R\bar u(p')\left[-I_A\gamma_\mu+\frac{1}{2m}I_\mu(p)+\frac{1}{2m}I_\mu(p')+i\sigma_{\alpha\nu}q^\nu\frac{1}{2m}I_\mu^\alpha(p)\right]u(p)\frac{\alpha}{4\pi^2}+O\left[\frac{q^2}{m^2}\right] \quad [\text{Fig. 2(d)}],$$

where

$$\frac{\alpha}{4\pi^2}I_{\mu\nu}(p)=e^2\int\frac{d^4k}{(2\pi)^4}2\pi n_B(k_0)\delta(k^2)\frac{k_\mu k_\nu}{(k\cdot p)^2}. \quad (48)$$

Here the form of the counterterm is dictated by our temperature-dependent Dirac equation, which reads

$$\left[\not{p}-m_R-\frac{\alpha}{4\pi^2}\not{I}\right]u_\beta(p)=0\ . \tag{49}$$

Thus, one must use the finite-temperature counterterm

$$\delta\mathscr{L}=\frac{\alpha}{4\pi^2}\not{I}(p)\ , \tag{50}$$

and must employ finite-temperature spinors $u_\beta(p)$ which satisfy the modified Dirac equation and which we choose to normalize via

$$u_\beta^\dagger(p)u_\beta(p)=1\ . \tag{51}$$

The form which these spinors assume is easily found by defining

$$\tilde{p}_\mu=p_\mu-\frac{\alpha}{4\pi^2}I_\mu\ . \tag{52}$$

Then

$$u_\beta(p)=u(\tilde{p})\ . \tag{53}$$

Note that the leading-order amplitude must be evaluated *using finite-temperature spinors.* This is the origin of the temperature-dependent modification of the leading term of the form

$$-e(\bar{u}_\beta\gamma_\mu u_\beta-u\gamma_\mu u) \tag{54}$$

as given in Eq. (47).

Adding all of these contributions, both for $T\neq0$ and $T=0$, we find then

$$M_\mu^{\text{total}}\cong-e_R\bar{u}_\beta(p')\left[\gamma_\mu\left[1-\frac{\alpha}{4\pi^2}\right]\left[\frac{1}{2E_p}I_0(p)+\frac{1}{2E_{p'}}I_0(p')\right]+\frac{\alpha}{4\pi^2}\left[\frac{1}{2m}I_\mu(p)+\frac{1}{2m}I_\mu(p')\right]\right.$$
$$\left.+i\sigma_{\alpha\nu}q^\nu\frac{1}{2m_p}\left[\frac{\alpha}{2\pi}g_\mu^\alpha+\frac{\alpha}{4\pi^2}I_\mu^\alpha(p)\right]\right]u_\beta(p)\ . \tag{55}$$

Note that if we set $q=p-p'=0$, the vertex becomes

$$M_\mu^{\text{total}}(p=p')=-e_R\bar{u}_\beta(p)\left[\gamma_\mu\left[1-\frac{\alpha}{4\pi^2}\frac{I_0(p)}{E}\right]+\frac{\alpha}{4\pi^2}\frac{1}{m}I_\mu(p)\right]u_\beta(p)$$
$$=-e_R\left[\frac{\tilde{p}_\mu}{\tilde{E}}\left[1-\frac{\alpha}{4\pi^2}\frac{I_0(p)}{E}\right]+\frac{1}{E}\frac{\alpha}{4\pi^2}I_\mu(p)\right]=-e_R\frac{p_\mu}{E_\beta}\ , \tag{56}$$

which is identical to the result obtained at $T=0$ except that now the energy E_β and momentum p are related via the phase-space mass m_p,

$$E_\beta=(p^2+m_p{}^2)^{1/2}\ . \tag{57}$$

The time component gives

$$M_0^{\text{total}}(p=p')=-e_R \tag{58}$$

so that the charge is unrenormalized by temperature effects when $T\ll m$.[14]

This analysis suggests then that the inertial mass is to be identified with the phase-space mass. That this is *required* can be seen by placing the electron in an external electromagnetic field A_μ. The Dirac equation becomes

$$\left[\not{p}-\frac{\alpha}{4\pi^2}\not{I}(p)-e_R\not{A}\left[1-\frac{\alpha}{4\pi^2}\frac{1}{E}I_0(p)\right]+\frac{\alpha}{4\pi^2}\frac{1}{m_R}A^\mu I_\mu(p)+\sigma_{\mu\nu}\frac{1}{2m_R}\left[\partial^\nu A^\mu\frac{\alpha}{2\pi}+\frac{\alpha}{4\pi^2}\partial^\nu A^\alpha I_\alpha^\mu(p)\right]-m_R\right]\psi_\beta(p)=0\ . \tag{59}$$

Using the nonrelativistic reduction $u\gg l$ and

$$\frac{\alpha}{4\pi^2}I_\mu \approx \alpha\frac{\pi}{3}\frac{T^2}{m}\left[1-\tfrac{1}{6}\frac{p^2}{m^2},\tfrac{1}{2}\frac{\vec{p}}{m}\right],$$

$$\frac{\alpha}{4\pi^2}I_{\mu\nu} \approx \begin{cases} \alpha\dfrac{\pi T^2}{9m^2}(-g_{\mu\nu}+4\delta_{\mu 0}\delta_{\nu 0}), & \mu=\nu \\ 2\alpha\dfrac{\pi T^2}{9m^2}\dfrac{p_i}{m}, & \mu=i,\ \nu=0 \\ & \mu=0,\ \nu=i \end{cases} \tag{60}$$

we find the effective Schrödinger equation

$$i\frac{\partial}{\partial t}u = \left[m_R+\delta m_\beta+\frac{p^2}{2m_R}\left[1-\frac{\delta m}{m_R}\right]-e_R\frac{\vec{A}\cdot\vec{p}}{m_R}\left[1-\frac{\delta m}{m_R}\right]+e_R\phi-e_R\frac{1}{2m_R}\vec{\sigma}\cdot\vec{B}\left[1-\tfrac{5}{3}\frac{\delta m_\beta}{m_R}+\frac{\alpha}{2\pi}\right]\right]u$$

$$\approx\left[m_p+\frac{p^2}{2m_p}-e_R\frac{\vec{A}\cdot\vec{p}}{m_p}+e_R\phi-e_R\frac{1}{2m_p}\vec{\sigma}\cdot\vec{B}\left[1-\tfrac{2}{3}\frac{\delta m_\beta}{m_R}+\frac{\alpha}{2\pi}\right]\right]u\ . \tag{61}$$

We see that there exist essentially two finite-temperature modifications.

(i) There is a new contribution to the anomalous moment[15]

$$\delta\mu_a = -\tfrac{2}{3}\frac{\delta m_\beta}{m_R}\ . \tag{62}$$

(ii) The inertial mass is identical to the phase-space mass since if $\vec{A}=0$, $\vec{E}=-\nabla\phi$

$$\vec{a}=-[H,[H,\vec{r}]]=\frac{e\vec{E}}{m_p}\quad\text{or } m_I=m_p\ . \tag{63}$$

Likewise we can define the temperature-dependent gravitational mass by repeating our calculation of the renormalization of $T_{\mu\nu}$, but now at finite temperature. The evaluation is straightforward, except for the case of the graviton-photon coupling diagram, where overlapping singularities require special attention. The evaluation of this diagram, Fig. 4(f), is indicated in the Appendix. We then find (again calculating at $q=0$)

$$
\begin{aligned}
{}^\beta M^{(0)}_{\mu\nu} &= \bar{u}_\beta(p)\tfrac{1}{2}(p_\mu\gamma_\nu+p_\nu\gamma_\mu)u_\beta(p)-\bar{u}(p)\tfrac{1}{2}(p_\mu\gamma_\nu+p_\nu\gamma_\mu)u(p)\quad \text{[Fig. 4(a)]}\ ,\\
{}^\beta M^{\rm Se}_{\mu\nu}+{}^\beta M^{\rm CT}_{\mu\nu} &= -\frac{\alpha}{4\pi^2}\bar{u}(p)\tfrac{1}{2}(\gamma_\mu p_\nu+\gamma_\nu p_\mu)u(p)\left[-2I_A+2\frac{I_0}{E}\right]\quad \text{[Figs. 4(b)+4(c)]}\ ,\\
{}^\beta M^{V}_{\mu\nu} &= -\frac{\alpha}{4\pi^2}\bar{u}(p)\left[\tfrac{1}{2}(\gamma_\mu p_\nu+\gamma_\nu p_\mu)I_A-\frac{1}{2m}[p_\mu I_\nu(p)+p_\nu I_\mu(p)]+4\frac{\pi^3T^2}{3m}g_{\mu\nu}+mI_{\mu\nu}(p)\right]u(p)\quad \text{[Fig. 4(d)]}\ ,\\
{}^\beta M^{C}_{\mu\nu} &= \frac{\alpha}{4\pi^2}\bar{u}(p)\left[\frac{1}{m}[p_\mu I_\nu(p)+p_\nu I_\mu(p)]+4\frac{\pi^3T^2}{3m}g_{\mu\nu}\right]u(p)\quad \text{[Fig. 4(e)]}\ ,\\
{}^\beta M^{P}_{\mu\nu} &= -\frac{\alpha}{4\pi^2}\bar{u}(p)\left[\frac{1}{m}[p_\mu I_\nu(p)+p_\nu I_\mu(p)]-mI_{\mu\nu}(p)+4\frac{\pi^3T^2}{3m}(g_{\mu\nu}-2g_{\mu 0}\delta_{\nu 0})\right]u(p)\quad \text{[Fig. 4(f)]}\ .
\end{aligned} \tag{64}
$$

Including Z_2^{-1} for wave-function renormalization we can now add all terms to find the renormalized energy-momentum tensor at finite temperature

$$
\begin{aligned}
M^{\rm total}_{\mu\nu} &= \bar{u}_\beta(p)\left[\tfrac{1}{2}(\gamma_\mu p_\nu+\gamma_\nu p_\mu)\left[1-\frac{\alpha}{4\pi^2}\frac{1}{E}I_0(p)\right]-2\alpha\frac{\pi T^2}{3m}\delta_{\mu 0}\delta_{\nu 0}+\frac{\alpha}{4\pi^2}\frac{1}{2m}[p_\mu I_\nu(p)+p_\nu I_\mu(p)]\right.\\
&\qquad\left.-g_{\mu\nu}\left[\not{p}-m_R-\frac{\alpha\pi T^2}{3m}\right]\right]u_\beta(p)\\
&= \frac{p_\mu p_\nu-2(\alpha\pi/3)T^2\delta_{\mu 0}\delta_{\nu 0}}{E_\beta}\ .
\end{aligned} \tag{65}
$$

Thus a noncovariant temperature-dependent component appears. We can now evaluate the gravitational mass by performing a nonrelativistic reduction in the presence of an external gravitational potential, in which case the Dirac equation

$$\left[\not p - m_R - \frac{\alpha}{4\pi^2}\not I(p) - \tfrac{1}{2}h_{\mu\nu}T^{\mu\nu}\right]\psi=0 \tag{66}$$

becomes

$$i\frac{\partial}{\partial t}u = \left[m_p + \frac{p^2}{2m_p} + (m_R - \delta m_\beta)\phi_g\right]u \;. \tag{67}$$

Computing the acceleration then yields the gravitational mass

$$\vec{a} = -\frac{m_R - \delta m_\beta}{m_p}\vec{\nabla}\phi_g \quad \text{or } m_g = m_R - \delta m_\beta \tag{68}$$

which is clearly different from the inertial mass.

That this inequality is more general than our simple spinor electrodynamic calculation can be seen in Sec. IV where we show that similar results obtain for the case of a scalar field.

IV. INERTIAL AND GRAVITATIONAL MASS IN SCALAR ELECTRODYNAMICS

In this section we calculate the radiative corrections to the inertial and gravitational masses of a charged spin-zero particle both at zero and at finite T. Just as in the case of spinor electrodynamics we find that at $T=0$ the energy-momentum tensor is unchanged by renormalization while at finite $T \ll m$ (m the scalar mass) the inertial and gravitational masses are shifted in opposite directions by finite-temperature effects.

We begin with the bare spin-zero Lagrangian

$$\mathscr{L}_0 = (\partial_\mu - eA_\mu)\phi^\dagger(\partial_\mu - eA_\mu)\phi - m^2\phi^\dagger\phi \;, \tag{69}$$

and first calculate the self-energy diagrams of Fig. 5 at $T=0$, and find

$$\Sigma(p) = \frac{-\alpha m^2}{4\pi}\left[\frac{3}{\epsilon} - 7\right] . \tag{70}$$

FIG. 5. The scalar self-energy diagrams.

(The bubble diagram vanishes when dimensional regularization is used.) This yields the usual mass and wave-function renormalization constants

$$\delta m^2 = \frac{-\alpha m^2}{4\pi}\left[\frac{3}{\epsilon} - 7\right] ,$$
$$Z_2^{-1} - 1 = 0 \tag{71}$$

and we see that there is no wave-function renormalization to order α.

To calculate the gravitational mass we need the energy-momentum tensor of a charged scalar particle. Because we will only consider gravitons with $q=0$ interacting with the scalar field we can use the canonical energy-momentum tensor[16]

$$T_{\mu\nu} = \frac{2\partial}{\partial g^{\mu\nu}}(\sqrt{-g}\,\mathscr{L}_0)$$
$$= 2(\partial_\mu - eA_\mu)\phi^\dagger(\partial_\nu - eA_\nu)\phi - g_{\mu\nu}\mathscr{L}_0 \;, \tag{72}$$

since the improved tensor of Callan, Coleman, and Jackiw[17] gives an identical result in this limit.[16] [The tensor of Eq. (72) is such that the trace of $T_{\mu\nu}$ gives $2m^2$ for a free scalar particle.] The relevant radiative-correction diagrams are shown in Fig. 6. The effects of the graph in Fig. 6(h) were taken into account for the fermion case by the use of finite-temperature spinors, but here must be included explicitly. We then find

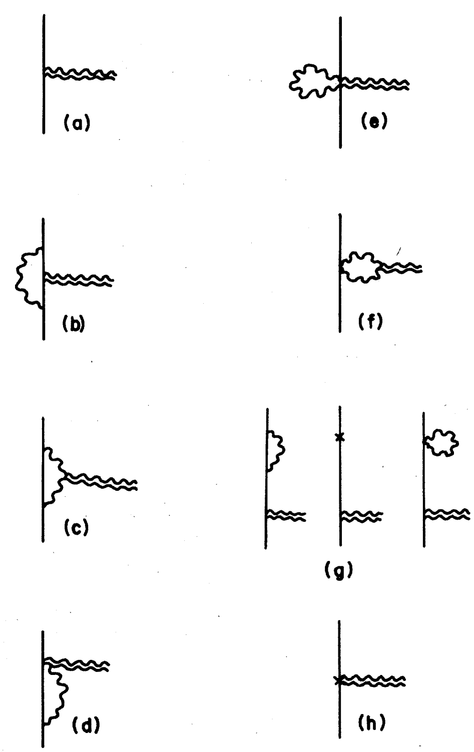

FIG. 6. The radiative corrections to the scalar energy-momentum tensor. The double wavy line represents the $T_{\mu\nu}$ coupling.

$$
\begin{aligned}
&\phi^\dagger(p)(2p_\mu p_\nu)\phi(p) \quad [\text{Fig. 6(a)}]\,,\\
&\left[-\frac{\alpha}{4\pi}\right]\phi^\dagger(p)\left[p_\mu p_\nu\left[\frac{16}{3\epsilon}-\frac{152}{9}\right]+m^2 g_{\mu\nu}\left[-\frac{4}{3\epsilon}+\frac{44}{9}\right]\right]\phi(p) \quad [\text{Fig. 6(b)}]\,,\\
&\left[\frac{-\alpha}{4\pi}\right]\phi^\dagger(p)\left[p_\mu p_\nu\left[\frac{4}{\epsilon}-8\right]+m^2 g_{\mu\nu}\left[-\frac{1}{\epsilon}+1\right]\right]\phi(p) \quad [\text{Fig. 6(c)}]\,,\\
&\left[\frac{\alpha}{4\pi}\right]\phi^\dagger(p)\left[p_\mu p_\nu\left[\frac{28}{3\epsilon}-\frac{224}{9}\right]+m^2 g_{\mu\nu}\left[-\frac{16}{3\epsilon}+\frac{116}{9}\right]\right]\phi(p) \quad [\text{Fig. 6(d)}]\,,\\
&0 \ \ (\text{in dimensional regularization}) \quad [\text{Fig. 6(e)}]\,,\\
&0 \ \ (\text{in dimensional regularization}) \quad [\text{Fig. 6(f)}]\,,\\
&0 \ \ (Z_2=1 \text{ to order } \alpha) \quad [\text{Fig. 6(g)}]\,,\\
&\left[\frac{\alpha}{4\pi}\right]\phi^\dagger(p)\left[m^2 g_{\mu\nu}\left[\frac{3}{\epsilon}-7\right]\right]\phi(p) \quad [\text{Fig. 6(h)}]\,.
\end{aligned} \tag{73}
$$

Adding the contributions 6(a)—6(h) and dividing by $Z_2(=1)$ we find

$$M_{\mu\nu}^{\text{total}}=\phi^\dagger(p)(2p_\mu p_\nu)\phi(p) \tag{74}$$

and the energy-momentum tensor is unchanged by renormalization just as in the spin-$\frac{1}{2}$ case. Thus, radiative correction at $T=0$ respects the principle of equivalence for scalars.[16]

For the finite-temperature case we restrict ourselves to $T \ll m$ and, as before, consider only finite-temperature modifications to the photon propagator. We then calculate the temperature-dependent corrections to the diagrams already discussed in this section. We begin with the self-energy and find

$$^{\beta}\Sigma(p)=\frac{\alpha}{\pi^2}\left[K+\frac{(p^2-m^2)I_A}{4}\right], \tag{75}$$

where

$$K=\int \frac{d^3k}{k_0}n_B(k)=\tfrac{1}{2}p\cdot I=\frac{2\pi^3T^2}{3}\,. \tag{76}$$

This gives the temperature-dependent mass shift and wave-function renormalization

$$
\begin{aligned}
&\delta m^2=\alpha K/\pi^2=2\alpha\pi T^2/3\,,\\
&Z_2=1+\frac{\alpha I_A}{4\pi^2}
\end{aligned} \tag{77}
$$

and we note that this is the same inertial mass shift as in the spinor-electrodynamics case.

We then repeat the calculation of the renormalization of $T_{\mu\nu}$, but now at finite T, to obtain

$$
\begin{aligned}
&\phi^\dagger(p)(2p_\mu p_\nu)\phi(p) \quad [\text{Fig. 6(a)}]\,,\\
&\left[-\frac{\alpha}{2\pi^2}\right]\phi^\dagger(p)[p_\mu p_\nu I_A+(p_\mu I_\nu+p_\nu I_\mu)+2m^2 I_{\mu\nu}-2g_{\mu\nu}K]\phi(p) \quad [\text{Fig. 6(b)}]\,,\\
&\left[\frac{\alpha}{4\pi^2}\right]\phi^\dagger(p)[4m^2 I_{\mu\nu}-2(p_\mu I_\nu+p_\nu I_\mu)+2g_{\mu\nu}K]\phi(p) \quad [\text{Fig. 6(c)}]\,,\\
&\left[-\frac{\alpha}{\pi^2}\right]\phi^\dagger(p)[2g_{\mu\nu}K-(p_\mu I_\nu+p_\nu I_\mu)]\phi(p) \quad [\text{Fig. 6(d)}]\,,\\
&\left[\frac{\alpha}{\pi^2}\right]\phi^\dagger(p)(g_{\mu\nu}K)\phi(p) \quad [\text{Fig. 6(e)}]\,,\\
&\left[\frac{\alpha}{\pi^2}\right]\phi^\dagger(p)(4J^{\mu\nu}-g^{\mu\nu}I)\phi(p) \quad [\text{Fig. 6(f)}]\,,
\end{aligned} \tag{78}
$$

$$\left[\frac{\alpha}{\pi^2}\right]\phi^\dagger(p)(p_\mu p_\nu I_A)\phi(p) \quad \text{[Fig. 6(g)]},$$

$$\left[-\frac{\alpha}{\pi^2}\right]\phi^\dagger(p)(g_{\mu\nu}K)\phi(p) \quad \text{[Fig. 6(h)]},$$

where

$$I_{\mu\nu}=\int \frac{d^4k\,\delta(k^2)n_B(k)k_\mu k_\nu}{(p\cdot k)^2},$$
$$J_{\mu\nu}=\frac{\partial}{\partial\mu^2}\int d^4k\,\delta(k^2-\mu^2)n_B((k_0{}^2+\mu^2)^{1/2})k_\mu k_\nu\,|_{\mu^2=0}=\frac{K}{2}(g_{\mu\nu}-2\delta_{\mu 0}\delta_{\nu 0}). \tag{79}$$

Adding the contributions 6(a)—6(h) and multiplying by $Z_2{}^{-1}=1-\alpha I_A/4\pi^2$ we find the energy-momentum tensor

$$2p_\mu p_\nu-\frac{2\alpha K}{\pi^2}\delta_{\mu 0}\delta_{\nu 0}=2\left[p_\mu p_\nu-\frac{2\alpha\pi T^2}{3}\delta_{\mu 0}\delta_{\nu 0}\right], \tag{80}$$

which yields the same shift in the gravitational mass as in the fermion case.

V. CONCLUSION

We have thus demonstrated that at finite temperature the gravitational and inertial masses are not the same. Thus the acceleration in a gravitational field will be *different* for particles of different mass. This would in principle yield a violation of the equivalence principle in an Eötvös-type experiment, although at accessible temperatures the effect is small. Thus for an electron at 300 °K

$$\frac{\delta m_\beta}{m_e}=\alpha\frac{\pi T^2}{3m_e{}^2}=2\times 10^{-17} \tag{81}$$

yielding a totally undetectable effect.

Though surprising on the surface, this inequality does not conflict with any of the fundamental tenets of general relativity. The key here is that the existence of a heat bath introduces a preferred frame—the one in which the blackbody radiation is isotropic. This introduces noncovariant contributions into the renormalization scheme. That such noncovariant terms can cause trouble is well known from work on the energy-momentum trace anomaly where use of a Pauli-Villars or some other noncovariant coupling scheme also leads to an apparent inequality of gravitational and inertial masses at $T=0$. However, this result is only illusory. The photon-graviton coupling, Fig. 4(f), has a trace proportional to $d-4$, where d is the number of space-time dimensions. If $d=4$ such a diagram appears to vanish, but in dimensional regularization the $d-4$ from the trace cancels a similar term in the denominator, yielding a finite contribution which will be missed in a noncovariant renormalization scheme. This additional term is the anomaly and restores the equality between gravitational and inertial masses at $T=0$. Thus it is perhaps not surprising that the existence of a finite-temperature heat bath—which also breaks the explicit covariance of the calculation—gives rise to an asymmetry between gravitational and inertial masses.

As far as the validity of the equivalence principle is concerned, we note that the fundamental ideas which led to its postulation in the first place include the impossibility of detection of absolute motion through the vacuum and the indistinguishability of acceleration and gravity. However, one *can* measure absolute velocity and/or acceleration with respect to a heat bath (indeed this has been done for the motion of the earth through the 3°K microwave radiation which remains from the early universe). Thus the conditions under which the equivalence principle were formulated are not met at $T\neq 0$. In principle, such effects are detectable in an Eötvös-type experiment. However, the observable consequences at attainable temperatures are negligible.

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APPENDIX

The calculation of finite-temperature modifications is generally straightforward. Thus, the self-energy diagram in Fig. 1,

$$\Sigma(p)=-ie^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{\gamma_\mu(\not p-\not k+m)\gamma^\mu}{k^2-2p\cdot k+p^2-m^2}, \tag{A1}$$

picks up a $T\neq 0$ component

$${}^\beta\Sigma(p)=-e^2\int\frac{d^4k}{(2\pi)^4}2\pi\delta(k^2)n_B(k)\frac{\gamma_\mu(\not p-\not k+m)\gamma^\mu}{k^2-2p\cdot k+p^2-m^2} \tag{A2}$$

which can be evaluated via standard techniques, yielding the result given in Eq. (41). Likewise the other diagrams in Figs. 1—4 can be evaluated by the same substitution

$$\frac{1}{k^2+i\epsilon}\to-2\pi i\delta(k^2)n_B(k_0) \tag{A3}$$

with the exception of the $T_{\mu\nu}$ coupling to the photon, Fig. 4(f). The problem here is that at $q=0$ the photon propagators both carry identical four-momenta k_μ. Thus, the finite-temperature integral will contain a piece

$$\int d^4k \frac{1}{k^2} 2\pi i \delta(k^2) \tag{A4}$$

with a delta function $\delta(k^2)$ divided by a factor of k^2. This is a singular integral to say the least, and special care must be exercised in its evaluation. That such spurious singularities can arise in this way was already noted by Dolan and Jackiw,[18] who also suggested the solution. The point is that our calculations are performed in the so-called "real time" formalism, wherein one writes the propagator in coordinate space in terms of the usual integral over a three-momentum d^3k and an energy dk_0 of a momentum-space propagator

$$iD_F(k) = \frac{1}{k^2+i\epsilon} - i\,2\pi\delta(k^2) n_F(k) \tag{A5}$$

which has the very convenient feature of naturally separating into a $T \neq 0$ and $T=0$ component. Unfortunately, correct analyticity properties are not automatically obtained via this technique and when problems arise it is necessary to write the corresponding integral in the so-called imaginary-time representation wherein the momentum-space propagator retains its familiar form

$$iD_F(k) = \frac{1}{k^2+i\epsilon}, \tag{A6}$$

but where now the "energy" k_0 takes on only discrete values

$$k_0 = 2\pi i n T, \quad -\infty < n < \infty \tag{A7}$$

and where the Fourier transform to coordinate space involves the usual integral over d^3k, but now a *sum* over the integers n from $-\infty$ to $+\infty$.[19] Thus the propagator becomes

$$D_F(x) = 2T \sum_n \int \frac{d^3k}{(2\pi)^3} e^{-ik_0 t + i\vec{k}\cdot\vec{x}} \frac{-i}{k^2+i\epsilon} \tag{A8}$$

which may be evaluated via standard techniques and can be shown to be identical to the real-time expression

$$D_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \left[\frac{-i}{k^2+i\epsilon} + 2\pi\delta(k^2) n_F(k) \right]. \tag{A9}$$

Of course, the separation into $T=0$ and $T\neq 0$ components is lost in the imaginary-time expression. Nevertheless it is often very useful. Thus the integral corresponding to Fig. 4(f) can be written in the imaginary-time form

$$I_{\mu\nu}(p) = e^2 2T \sum_n \int \frac{d^3k}{(2\pi)^3} \bar{u}(p)\gamma_\alpha \frac{i}{\not p - \not k - m} \times \gamma_\beta u(p) \frac{1}{k^2+i\epsilon} T^{\alpha\beta}_{\mu\nu}(k) \frac{1}{k^2+i\epsilon}, \tag{A10}$$

where

$$T^{\alpha\beta}_{\mu\nu}(k) = -(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu) k^2 - 2k_\mu k_\nu g^{\alpha\beta} + (k_\mu \delta^\beta_\nu + k_\nu \delta^\beta_\mu) k^\alpha + (k_\mu \delta^\alpha_\nu + k_\nu \delta^\alpha_\mu) k^\beta + g_{\mu\nu}(k^2 g^{\alpha\beta} - k^\alpha k^\beta) \tag{A11}$$

represents the coupling of the energy-momentum tensor to the photon. The integral can now be reexpressed in terms of a single photon propagator by use of a derivative

$$I_{\mu\nu}(p) = \frac{d}{d\lambda^2}\bigg|_{\lambda^2=0} e^2 2T \times \sum_n \int \frac{d^3k}{(2\pi)^3} \bar{u}(p)\gamma_\alpha \frac{i}{\not p - \not k - m} \gamma_\beta u(p) \times T^{\alpha\beta}_{\mu\nu}(k) \frac{1}{k^2-\lambda^2+i\epsilon}, \tag{A12}$$

which can in turn be rewritten in the real-time form

$$I_{\mu\nu}(p) = \frac{d}{d\lambda^2}\bigg|_{\lambda^2=0} e^2 \times \int \frac{d^4k}{(2\pi)^4} \bar{u}(p)\gamma_\alpha \frac{i}{\not p - \not k - m} \gamma_\beta u(p) \times T^{\alpha\beta}_{\mu\nu}(k) \left[\frac{1}{k^2-\lambda^2+i\epsilon} - i\,2\pi\delta(k^2-\lambda^2) \times n_B((k^2+\lambda^2)^{1/2}) \right]. \tag{A13}$$

The latter integral can now be evaluated in the standard fashion. Note that the separation into $T=0$ and $T\neq 0$ components again results and upon performing the requisite differentiation one finds the results quoted in Eqs. (25) and (64), respectively.

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