Ref.TH.2468-CERN

RENORMALIZATION OF YANG-MILLS THEORY DEVELOPED AROUND AN INSTANTON

D. Amati and A. Rouet

CERN — Geneva

ABSTRACT

The development of a quantum field theory around a classical solution implies the treatment of the corresponding zero frequency modes. For the Yang-Mills theory around the instanton there are the gauge, translation and dilatation zero modes. We treat them together by introducing the corresponding collective co-ordinates and re-expressing the Jacobian in terms of Faddeev-Popov fields. The propagators of the gauge and Faddeev-Popov fields are obtained together with the vertices that define the Feynman rules. Renormalization is performed in the zero mass version of the BPHZ formalism. Slavnov identities are thus proved; they allow to show the independence of the theory on the parameters introduced to deal with the zero modes.

Ref.TH.2468-CERN
7 March 1978

1. INTRODUCTION

We wish to present in this paper a careful study of the definitions of the perturbative expansion of gauge theories around a classical solution.

The interest for such an attempt is obvious. Gauge theories are known to have classical solutions with finite action in Euclidean space, which is the space in which quantum field theories are actually developed. These solutions represent the minima of the quantum actions[1)] and, therefore, contribute to the functional integral that defines the theory[2)]. The actual contribution of the classical solution is weighted by the vacuum fluctuations of the quantum modes which have been calculated by 't Hooft[3)]. This calculation implies the evaluation of the determinant of the quadratic form in the expansion of the Lagrangian around the classical solution and involves no major difficulties related to zero modes[4)]. These difficulties seem to us unavoidable if one wants to define the theory beyond this zero order contribution. Or, in other words, if one wants to have a well-defined expansion whose first term is the classical contribution evaluated by 't Hooft.

We would like, therefore, to reproduce for this case, and to be more definite for the Yang-Mills theory developed around the BPSF instanton[1)], a technique somewhat analogous to the one we have for the usual expansion around the zero field solutions. This implies the definition of a renormalization procedure that would allow the construction of a perturbative algorithm (Feynman diagrams) based on propagators, vertices and finite counterterms calculable in terms of a limited number of arbitrary parameters.

In order to do so we shall first recognize all zero modes. For the case under consideration these are those related to gauge invariance besides the translation and dilatation ones. A formalism which treats them together in terms of covariant quantities is introduced in Section 2.

Using functional integral techniques we then introduce collective co-ordinates[5),2)] to face the problems generated by the zero modes. The Jacobian is written in terms of generalized Faddeev-Popov ghosts. Finally, the Lagrangian so obtained is proved to be invariant under some transformations of the supersymmetric type[6)].

Starting from this Lagrangian we sketch in Section 3 the renormalization procedure and we derive the Feynman rules and the propagators. Section 4 is more technical; it strongly relies on the BPHZ formalism[7),8)] and the techniques developed in Ref. 6). It can be omitted without jeopardizing the understanding of the paper. Some aspects of the renormalization, related to the non-locality of the Lagrangian, are discussed. Slavnov's identity, transcribing the invariance found in Section 2, is then proved at any order in the perturbation theory. Section 5 summarizes the results. Using Slavnov's identity we finally prove that the connected Green functions which do not contain Faddeev-Popov ghosts related to translation and dilatation zero modes, are independent of the parameters related to these zero modes.

2. HEURISTIC DEFINITION OF THE THEORY: TREATMENT OF ZERO MODES

In this part we shall use functional integral methods to derive the form of a perturbative development of the Green functions around a classical solution with finite action in the Euclidean metric. The generating functional of the Green functions will be normalized to the sourceless generating functional perturbatively developed around the same classical solution. To be specific, we shall study the case of pure SU(2) Yang-Mills theory developed around the instanton[1]:

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + g\, A_\mu \times A_\nu \\
A^{cl}_{\mu,i} &= \frac{2}{g} \frac{\eta_{i\mu\nu} x_\nu}{x^2+1}
\end{aligned}
\tag{1}
$$

where the notations are the usual ones[3]. We want to compute the quantity $Z(j_\mu)/Z(0)$, where

$$
Z(j_\mu) = \int \mathcal{D} Q_\mu \, \exp \int \left(\mathcal{L}(A^{cl}_\mu + Q) + j_\mu Q_\mu \right) dx
\tag{2}
$$

A^{cl}_μ being a classical solution, $(A^{cl}_\mu + Q_\mu)$ has no linear term in Q_μ and the basic quantity to study to get a perturbation theory is the kernel of the quadratic form in Q_μ:

$$\mathcal{L}''_{\mu\nu} \equiv \frac{\delta^2 \mathcal{L}(A_\mu)}{\delta A_\mu \, \delta A_\nu}\Bigg|_{A_\mu = A_\mu^{cl}} \tag{3}$$

This kernel is not invertible because of zero modes appearing in its spectral decomposition. Occurrence of these zero modes is due to the fact that the chosen classical solution breaks some of the symmetries of the Lagrangian -- specifically gauge, translation and dilatation invariance in our example. The elimination of the zero modes due to gauge invariance is equivalent to a background gauge fixing term[9]. In fact, the equations of motion admit a family of solutions deduced from $A^{cl}_{\mu,i}(x)$:

$$\begin{aligned} A^{cl}_{\mu,i}(b,x) &= \Omega(\lambda x+a)\left[\lambda A^{cl}_{\mu,i}(\lambda x+a)\frac{\sigma_i}{2}\right]\Omega^{-1}(\lambda x+a) + \\ &\quad + \frac{i}{g}\,\Omega(\lambda x+a)\frac{\partial}{\partial(\lambda x+a)_\mu}\,\Omega^{-1}(\lambda x+a) \\ b &= \left(\omega^j(x),\, a_\nu,\, \lambda\right) \qquad ; \qquad \Omega = \exp\, i\,\omega^j(x)\,\frac{\sigma_j}{2} \\ \lambda A^{cl}_{\mu,i}(\lambda x+a) &= \frac{2}{g}\,\frac{\eta_{i\mu\nu}(x_\nu+a_\nu)}{(x+a)^2+1/\lambda^2} \end{aligned} \tag{4}$$

where σ_i are the Pauli matrices. Thus

$$\begin{aligned} \mathcal{L}'_\mu &\equiv \frac{\partial \mathcal{L}(A_\mu)}{\partial A_\mu}\Bigg|_{A_\mu = A^{cl}_{\mu,i}(b,x)} = 0 \qquad \forall\, b \\ \frac{\partial}{\partial b}\mathcal{L}'_\mu &= \mathcal{L}''_{\mu\nu}\,\frac{\partial A^{q}_\nu}{\partial b} = 0 \end{aligned} \tag{5}$$

the vectors

$$\frac{\partial A^{cl}_{\nu,i}(b,x)}{\partial b}$$

appear as zero modes for the kernel $\mathcal{L}''_{\mu\nu}$.

We can write explicitly these zero modes in our example

$$z^{k,y}_{\mu,i}(b,x) \equiv \frac{\partial A^{d}_{\mu,i}(b,x)}{\partial \omega_k(y)} = \left[\varepsilon_{kij}\, A^{d}_{\mu,j}(b,x) + \frac{1}{g}\,\delta_{ik}\,\partial_\mu\right]\delta(\lambda x+a-y)$$

$$\equiv D^{k,i}_{\mu}(b,x)\;\delta(\lambda x+a-y) \qquad (6)$$

$$\bar{z}^{\nu}_{\mu,i}(b,x) \equiv \frac{\partial A^{d}_{\mu,i}(b,x)}{\partial a_\nu} = \frac{\partial}{\partial(\lambda x+a)_\nu}\, A^{d}_{\mu,i}(b,x)$$

$$\bar{z}_{\mu,i}(b,x) \equiv \frac{\partial A^{d}_{\mu,i}(b,x)}{\partial \lambda} = \left[1+(\lambda x+a)_\nu\,\frac{\partial}{\partial(\lambda x+a)_\nu}\, A^{d}_{\mu,i}(b,x)\right]$$

However, these $\bar{z}$ are not naturally defined in terms of connections. Indeed, a space-time transformation should be accompanied by the corresponding gauge transformation. This leads naturally to the orthogonal basis

$$z^{B}_{\mu,i}(b,x) = \left\{ z^{k,y}_{\mu,i}(b,x)\,,\; z^{\nu}_{\mu,i}(b,x)\,,\; z_{\mu,i}(b,x) \right\} \qquad (7)$$

where $z^{\nu}_{\mu,i}(b,x)$ and $z_{\mu,i}(b,x)$ are linear combinations of the z defined in (6):

$$z^{\nu}_{\mu,i}(b,x) \equiv \bar{z}^{\nu}_{\mu,i}(b,x) - \int z^{k,y}_{\mu,i}(b,x)\, A^{d}_{\nu,k}(b,y)\, dy =$$

$$= \partial_\nu A_\mu - D_\mu A_\nu = -\, F^{d}_{\mu\nu,i}(b,x)$$

$$z_{\mu,i}(b,x) \equiv \bar{z}_{\mu,i}(b,x) - \int z^{k,y}_{\mu,i}(b,x)\,(\lambda y_\nu + a_\nu)\, A^{d}_{\nu,k}(b,y)\, dy = \qquad (8)$$

$$= -(\lambda x_\nu + a_\nu)\, F^{d}_{\mu\nu,i}(b,x) = -\left(x_\nu\, F^{d}_{\mu\nu,i}(x)\right)(b)$$

2.1 General treatment of the zero modes

To face the problem raised by the occurrence of these zero modes we shall use the now well-known collective co-ordinate method and the Faddeev-Popov trick.

Let us decompose the quantum field Q_μ on the basis of the eigenmodes of $\mathcal{L}''_{\mu\nu}$. Denoting the non-zero eigenmodes by $q^n_{\mu,i}(b,x)$ we write

$$A_{\mu,i}(x) \equiv A^{cl}_{\mu,i}(b,x) + Q_{\mu,i} = A^{cl}_{\mu,i}(b,x) + \xi^B z^B_{\mu,i}(b,x) + \xi^n q^n_{\mu,i}(b,x) \tag{9}$$

The functional integral on Q_μ (2) is an integration over the quantum variables ξ^B and ξ^n.

Let us change the variables and use b and ξ^n as integration variables, fixing the ξ^B at some given values c^B. The functional integration on Q_μ is now understood as

$$\int db\, d\xi^B d\xi^n\, \delta(\xi^B - c^B)\, \Delta(\xi^n, c^B)$$

where Δ is the Faddeev-Popov Jacobian

$$\Delta(\xi^n, c^B) = \det \frac{\partial \xi^B}{\partial b}\bigg|_{\xi^B = c^B} \tag{10}$$

Thanks to the orthogonality of the eigenmodes we get from (9) the equation:

$$\left[\int dx\, z^B_{\mu,i}(b,x)\, z^B_{\mu,i}(b,x)\right] \xi^B = \int A_{\mu,i}(x)\, z^B_{\mu,i}(b,x)\, dx - \int A^{cl}_{\mu,i}(b,x)\, z^B_{\mu,i}(b,x)\, dx \tag{11}$$

Using the expression on the left-hand side of Eq. (11)

as quantum variables in place of the ξ^B the functional integral (2) can be written:

$$Z(j_\mu) = \int db\, \mathcal{D}Q_\mu(x)\, \delta\left[\int Q_{\mu,i}(x)\, z^B_{\mu,i}(b,x)\, dx - c^B\right] \times$$
$$\times\, \Delta\, \exp \int dx \left[\mathcal{L}(A^{cl}_\mu + Q_\mu) + j_\mu Q_\mu\right] \tag{12}$$
$$\Delta = \mathrm{Det}\left\{ -\int \frac{\partial A^{cl}_{\mu,i}(b,x)}{\partial b}\, z^B_{\mu,i}(b,x)\, dx + \int Q_{\mu,i}(x)\, \frac{\partial z^B_{\mu,i}(b,x)}{\partial b}\, dx \right\}$$

Let us change the integration variable $Q_{\mu,i}(x)$ to $Q_{\mu,i}(b,x)$ defined so as to transform as the adjoint representation under the gauge transformations. Using the fact that $z^B_{\mu,i}(b,x)$ transform in the same way, the argument of the δ function in (12) is invariant under a b transformation. Then we can write (12) under the form

$$Z(j_\mu) = \int db\, \mathcal{D}Q_\mu(b,x)\, \delta\left[\int Q_{\mu,i}(b,x)\, z^B_{\mu,i}(b,x)\, dx - c^B\right] \times$$
$$\times\, \Delta\, \exp \int dx \left[\mathcal{L}(A^{cl}_\mu + Q_\mu) + j_\mu Q_\mu\right] \tag{13}$$
$$\Delta = \mathrm{Det}\left\{ -\int \frac{\partial\left[A^{cl}_{\mu,i}(b,x) + Q_{\mu,i}(b,x)\right]}{\partial b}\, z^B_{\mu,i}(b,x)\, dx \right\}$$

Using the classical trick due to 't Hooft we can multiply $Z(j_\mu)$ by $\exp - \lambda^B c^{B^2}$ without changing the ratio $Z(j_\mu)/Z(0)$ which does not depend on c^B and integrate over c^B using the δ function. Finally, the Jacobian Δ can be written making use of Faddeev-Popov anticommuting fields $c^B, \bar{c}^B$. This casts (13) into the form:

$$Z(j_\mu) = \int db\, \mathcal{D}Q_\mu\, \mathcal{D}c^B\, \mathcal{D}\bar{c}^B \exp \int \left[L(Q_\mu, c^B, \bar{c}^B) + j_\mu Q_\mu\right] dx \tag{14}$$

where

$$L(Q_{\mu}, c^B, \bar{c}^B) = \mathcal{L}(A^{cl}_{\mu,i}(b,x) + Q_{\mu,i}(b,x)) - \lambda^B\left[\int Q_{\mu,i}(b,x) z^B_{\mu,i}(b,x)dx\right]^2$$
$$- c^B\left[\int \frac{\partial[A^{cl}_{\mu,i}(b,x) + Q_{\mu,i}(b,x)]}{\partial B'} z^B_{\mu,i}(b,x)dx\right] \bar{c}^{B'} \tag{15}$$

$L(Q, c^B, \bar{c}^B)$ admits an exact symmetry which generalizes the one introduced in Ref. 16). In fact, (15) is invariant under the following set of transformations:

$$\begin{aligned}
\delta A^{cl}_{\mu,i}(b,x) &= 0 \\
\delta Q_{\mu,i}(b,x) &= \hat{\imath}\, \frac{\partial[A^{cl}_{\mu,i}(b,x) + Q_{\mu,i}(b,x)]}{\partial B'} \bar{c}^{B'} \\
\delta c^B &= -2\hat{\imath}\,\lambda^B \int Q_{\mu,i}(b,x) z^B_{\mu,i}(b,x)\,dx \\
\delta \bar{c}^B &= \frac{1}{2}\hat{\imath}\, f^{B'B''}_{B}\bar{c}^{B'}\bar{c}^{B''}
\end{aligned} \tag{16}$$

where $\hat{\imath}$ is an infinitesimal anticommuting parameter, and where $f^{B'B''}_{B}$ is defined by

$$\left[\frac{\partial}{\partial B'}, \frac{\partial}{\partial B''}\right](A^{cl}_{\mu} + Q_{\mu}) = f^{B'B''}_{B}\frac{\partial}{\partial B}(A^{cl}_{\mu} + Q_{\mu}) \tag{17}$$

This symmetry will be the basic tool for carrying out the renormalization program in Section 4.

2.2 Explicit formalism

Let us now fix the λ^B to be $1/2\alpha$ independent of x for the gauge modes, $1/2\beta$ independent of ν for the translations and $1/2\gamma$ for the dilatations. Let us also remark that if we change the quantum variables c^B and $\bar{c}^B$ to $c^B(b)$ and $\bar{c}^B(b)$ as we have done for Q_μ, (15) is invariant under b transformations, except for the terms in β and γ which vary under dilatations. This implies that we can perform a b transformation and forget the b variables except for λ in the β and γ terms. Using the notations $c^B = (c(y), \psi_\nu, \psi)$ and $\bar{c}^B = (\bar{c}(y), \bar{\psi}_\nu, \bar{\psi})$, (14) can be written:

$$Z(j_\mu) = \int \mathcal{D}\omega \, d\lambda \, da_\nu \int \mathcal{D}Q_\mu \mathcal{D}c \mathcal{D}\bar{c} \, d\psi_\nu \, d\bar{\psi}_\nu \, d\psi \, d\bar{\psi} \times \times \exp \int (L + j_\mu Q_\mu) dx \tag{18}$$

with

$$\begin{aligned} L = & -\frac{1}{4} F_{\mu\nu}(A^{cl}+Q) F^{\mu\nu}(A^{cl}+Q) - \frac{1}{2\alpha}(D_\mu Q_\mu)^2 \\ & - \frac{\lambda^2}{2\beta}\left(\int F^{cl}_{\mu\nu} Q_\mu\right)^2 - \frac{\lambda^4}{2\gamma}\left(\int x_\nu F^{cl}_{\mu\nu} Q_\mu\right)^2 \\ & + c D_\mu U_\mu + (\psi_\nu + x_\nu \psi) F^{cl}_{\mu\nu} U_\mu \end{aligned} \tag{19}$$

where we have used the notations $F^{cl}_{\mu\nu} = F_{\mu\nu}(A^{cl})$, so that $D_\mu F^{cl}_{\mu\nu} = 0$, and

$$U_\mu = D_\mu \bar{c} + Q_\mu \times c + (\bar{\psi}_\nu + x_\nu \bar{\psi})(D_\nu Q_\mu - F^{cl}_{\mu\nu}) + Q_\mu \bar{\psi} \tag{20}$$

The Lagrangian (19) is invariant under the transformations (16) which can be explicitly written:

$$\begin{aligned} \delta A^{cl}_{\mu,i}(x) &= 0 \\ \delta Q_{\mu,i}(x) &= \hat{\imath}\, U_{\mu,i}(x) \\ \delta c_i(x) &= \frac{\hat{\imath}}{\alpha} D^{ij}_\mu(x) Q_{\mu,j}(x) \\ \delta \psi_\nu &= \hat{\imath}\, \frac{\lambda^2}{\beta} \int F^{cl}_{\mu\nu,i}(x) Q_{\mu,i}(x)\, dx \\ \delta \psi &= \hat{\imath}\, \frac{\lambda^4}{\gamma} \int x_\nu F^{cl}_{\mu\nu,i}(x) Q_{\mu,i}(x)\, dx \end{aligned} \tag{21}$$

$$\delta \bar{c}_i(x) = \hat{i}\ \bar{U}_i(x)$$
$$\delta \bar{\Psi}_\nu = \hat{i}\ \bar{\Psi}_\nu \bar{\Psi} \qquad (21ctd)$$
$$\delta \bar{\Psi} = 0$$

with

$$\bar{U} = -\frac{1}{2}\bar{c} \times \bar{c} + (\bar{\Psi}_\nu + x_\nu \bar{\Psi}) D_\nu \bar{c} - \frac{1}{2}(\bar{\Psi}_\nu + x_\nu \bar{\Psi})(\bar{\Psi}_\rho + x_\rho \bar{\Psi}) F_{\nu\rho} \qquad (21')$$

Actually the four following terms composing L in (19) are separately invariant under (21)

$$L_1 = -\frac{1}{4} F_{\mu\nu}(A^\varphi + Q)\, F_{\mu\nu}(A^\varphi + Q)$$
$$L_2 = -\frac{1}{2\alpha}(D_\mu Q_\mu)^2 + c D_\mu U_\mu$$
$$L_3 = -\frac{\Lambda^2}{2\beta}\left(\int F^{cl}_{\mu\nu} Q_\mu\right)^2 + \Psi_\nu F^{cl}_{\mu\nu} U_\mu \qquad (22)$$
$$L_4 = -\frac{\Lambda^4}{2\gamma}\left(\int x_\nu F^{cl}_{\mu\nu} Q_\mu\right)^2 + x_\nu \Psi F^{cl}_{\mu\nu} U_\mu$$

Equation (18) is a proper definition of the generating functional which was ill defined in (2) due to the zero modes. The Lagrangian (19), which generates this functional integral, is invariant under the transformations (21).

Conversely the transformations (21) characterize the Lagrangian, up to the four normalization constants, corresponding to the four terms in (22), provided we demand the additional following property: apart from $(\int F_{\mu\nu} Q_\mu)^2$ and $(\int x_\nu F_{\mu\nu} Q_\mu)^2$, all the other terms in the Lagrangian should have canonical dimension 4, carry no Faddeev-Popov charge and be local in the quantum and external fields. Let us finally list in the Table the quantum and external fields of the theory. The first number in brackets represents the canonical dimension of the field, the second one its Faddeev-Popov charge.

External fields		x dependent quantum fields		x independent quantum fields	
$F^{\alpha}_{\mu\nu,i}(x)$	$[2,\alpha]$	$Q_{\mu,i}(x)$	$[1,0]$	$\bar{\Psi}_{\nu}$	$[-1,1]$
$x_\mu F^{\alpha}_{\mu\nu,i}(x)$	$[1,0]$	$\bar{c}_i(x)$	$[0,1]$	$\bar{\Psi}$	$[0,1]$
$D^{ij}_{\mu}(x)$	$[1,0]$	$c_i(x)$	$[2,-1]$	Ψ_{ν}	$[1,-1]$
$x_\mu D^{ij}_{\mu}(x)$	$[0,0]$			Ψ	$[2,-1]$

3. RENORMALIZATION PROCEDURE - FEYNMAN RULES - PROPAGATORS

Let us now forget the heuristic arguments which have led us to the Lagrangian (19) and the symmetry (21). We shall now start from this point, and try to construct a perturbation theory such that the Green functions satisfy the symmetry (21) at any order. Using this symmetry we shall show that the "physical" Green functions will not depend on the parameters α, β, γ, justifying a posteriori the heuristic approach. What is called "physical" will be discussed in Section 5.

We start this section by drawing a crude scheme of the BPHZL [7),8)] renormalization frame. This procedure has been extensively exposed and used[6)] and we just intend to give some hint to the non-expert reader.

Then we write down the propagators which have to be used to compute Feynman graphs. Technical remarks on the specificities of the renormalization techniques in our case, due to the non-locality, are postponed to Section 4.

Let us first summarize the BPHZ procedure[7)] for the massive case.

1) Compute the unrenormalized integrand $I(\gamma)$ corresponding to each subgraph γ, i.e., the naïve product of the propagators and the vertices.

2) Assign to each subgraph γ a degree $\delta(\gamma)$ at best equal to its naïve divergence degree. Subtract to $I(\gamma)$ the $\delta(\gamma)$ first orders of its Taylor series with respect to the external momenta of γ. This defines the renormalized integrand.

3) Then you can carry out the integrations: they do converge.

4) The counterterms are formally like in the Bogoliubov-Parasiuk-Hepp usual procedure, but, because of the convergence of the integrals, just their finite part appears. This means that the counterterms are formal power series in $\hbar$ (loop expansion) with finite coefficients. These counterterms are to be defined by requiring some symmetry [the one given by (21) in our case] together with some normalization conditions.

Actually our theory is massless. This creates a supplementary problem which has been solved by Lowenstein and Zimmermann[8] in the following way:

1) Add to the Lagrangian a mass term of the form $(1 - s)^2 M^2$.
2) Put to zero - by hand - all the counterterms with canonical dimension less than four.
3) Compute $I(\gamma)$ as in the massive case.
4) Subtract to $I(\gamma)$ the $\delta(\gamma)$ first orders of its Taylor series with respect to the external momenta of γ and with respect to s.
5) Carry out the integrations. At non-exceptional momenta, they do converge for any $0 \leq s \leq 1$.
6) Put $s = 1$. Then you have really computed the massless theory.

3.1 Gauge field propagator

We shall now derive the propagator of the gauge field Q_μ. The part of the Lagrangian (19) quadratic in Q_μ is

$$\begin{aligned} L_Q^{\text{quadr.}} = & - F^d_{\mu\nu}\, Q_\mu \times Q_\nu - \frac{1}{2}\left(D_\mu Q_\nu\right)^2 + \frac{1}{2}\left(1-\frac{1}{\alpha}\right)\left(D_\mu Q_\mu\right)^2 \\ & - \frac{\Lambda^2}{2\beta}\left(\int F^d_{\mu\nu}\, Q_\mu\right)^2 - \frac{\Lambda^4}{2\gamma}\left(\int x_\nu F_{\mu\nu}\, Q_\mu\right)^2 + (1-s)^2 \Lambda^2 \frac{M^2}{2}\, Q_\mu^2 \end{aligned} \tag{22}$$

The last term has been added following the procedure mentioned above. Its dependence is to compensate its variance under dilatations. By defining the kernel P by $L_Q^{\text{quad}} = Q_\mu P_{\mu\nu} Q_\nu$ we find

$$P^{ij}_{\lambda\mu}(x,y) = -2\,\varepsilon_{ijk}\, F^{d}_{\lambda\mu,k}(x)\; \delta(x-y) + $$
$$+ g_{\lambda\mu}\left((D^2)^{ij}_{(x)} + (1-s)^2 \lambda^2 M^2 \delta_{ij} \right) \delta(x-y)$$
$$-\left(1-\frac{1}{\alpha}\right)\left(D_\lambda D_\mu\right)^{ij}_{(x)}\; \delta(x-y) - \frac{\lambda^2}{\beta} F^{d}_{\mu\nu,i}(x)\, F^{d}_{\mu\nu,j}(y) \tag{23}$$
$$-\frac{\lambda^4}{\gamma}\, x_\nu\, F^{d}_{\lambda\nu,i}(x)\; y_\rho\, F^{d}_{\mu\rho,j}(y)$$

The elimination of the zero modes which led to (22) ensures the invertibility of the kernel P. In other words, there exists a unique propagator $\Pi^{jk}_{\mu\nu}(y,z)$ defined by

$$\int dy\;\; P^{ij}_{\lambda\mu}(x,y)\;\; \Pi^{jk}_{\mu\nu}(y,z) \;=\; g_{\lambda\nu}\; \delta_{ik}\; \delta(x-z) \tag{24}$$

We have been able to compute this propagator which can be written under the form

$$\Pi^{jk}_{\mu\nu}(y,z) = \left[g_{\mu\rho}\, \delta_{j\ell} - \left(D_\mu (D^2)^{-1} D_\rho \right)^{j\ell}_{(y)} \right] \Delta^{\ell k}_{\rho\nu}(y)\; \delta(y-z)$$
$$-\frac{1}{N_1}\int dt\; F^{d}_{\mu\sigma,j}(y)\; F^{d}_{\rho\sigma,\ell}(t)\; \Delta^{\ell k}_{\rho\nu}(t)\; \delta(t-z)$$
$$-\frac{1}{N_2}\int dt\; y_\rho\; F^{d}_{\mu\sigma,j}(y)\; t_\tau\; F^{d}_{\rho\tau,\ell}(t)\; \Delta^{\ell k}_{\rho\nu}(t)\; \delta(t-z) \tag{25}$$
$$+\,\alpha\left[D_\mu (D^2)^{-1} \left(D^2 + \alpha (1-s)^2 \lambda^2 M^2 \right)^{-1} D_\nu \right]^{jk}_{(y)} \delta(y-z)$$

where

$$\Delta^{\ell k}_{\rho\nu}(t) = \left[\left(D^2 + (1-s)^2 \lambda^2 M^2 \right) g_{\rho\nu} + 2\left[D_\rho, D_\nu \right] \right]^{-1}_{\ell k}(t) \tag{26}$$

N_1 and N_2 are the normalization constants given by

$$\begin{aligned} &\int dx\, F^{d}_{\mu\nu,i}(x)\, F^{d}_{\mu\rho,i}(x) = N_1 \delta_{\nu\rho} \\ &\int dx\, x_\nu F^{d}_{\mu\nu,i}(x)\, x_\rho\, F^{d}_{\mu\rho,i}(x) = N_2 \end{aligned} \tag{27}$$

Equation (24) can be verified using the relations

$$\begin{aligned} &D^{ij}_\mu(x)\, \Delta^{jk}_{\mu\nu}(x) = \left[D^2 + (1-s)^2\lambda^2 M^2\right]^{-1}_{ij}(x)\, D^{jk}_\nu(x) \\ &(1-s)^2\lambda^2 M^2 F^{d}_{\mu\nu,i}(x) = \Delta^{ij}_{\mu\rho}(x)\, F^{d}_{\rho\nu,j}(x) \end{aligned} \tag{28}$$

These relations can be easily derived from the identities:

$$\begin{aligned} &D^2 D_\mu + D_\mu D^2 = 2 D_\nu D_\mu D_\nu \\ &D^2 F^{d}_{\mu\nu} = 2 D_\rho D_\mu F^{d}_{\rho\nu} \end{aligned} \tag{29}$$

Let us now comment on the structure of the propagator (25).

1) If in place of the classical solution (1) we should consider $A^{cl}_\mu = 0$, (25) reduces to the ordinary propagator

$$\Pi_{\mu\nu} = \frac{g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}}{\partial^2 + (1-s)^2\lambda^2 M^2} + \alpha\, \frac{\frac{\partial_\mu \partial_\nu}{\partial^2}}{\partial^2 + (1-s)^2\lambda^2 \alpha M^2} \tag{30}$$

2) Let us denote by $\hat{\Pi}^{jk}_{\mu\nu}(y,z)$ the three first terms of (25). $\hat{\Pi}^{jk}_{\mu\nu}(y,z)$ generalizes the transverse propagator, the first term of (30). Indeed it is transverse to the gauge, to the translation and to the dilatation zero modes:

$$
\begin{aligned}
&D_\mu^{ij}(y)\, \hat{\Pi}^{jk}_{\mu\nu}(y,z) = \hat{\Pi}^{jk}_{\mu\nu}(y,z)\, \overleftarrow{D}^{ki}_\nu(z) = 0 \\
&\int dy\, F^{cl}_{\rho\mu,j}(y)\, \hat{\Pi}^{jk}_{\mu\nu}(y,z) = \int dz\, \hat{\Pi}^{jk}_{\mu\nu}(y,z)\, F^{cl}_{\nu\rho,k}(z) = 0 \qquad (31)\\
&\int dy\, y_\rho\, F^{cl}_{\rho\mu,j}(y)\, \hat{\Pi}^{jk}_{\mu\nu}(y,z) = \int dz\, \hat{\Pi}^{jk}_{\mu\nu}(y,z)\, z_\rho\, F^{cl}_{\nu\rho,k}(z) = 0
\end{aligned}
$$

As expected, $\hat{\Pi}^{jk}_{\mu\nu}(y,z)$ does not depend on the "gauge parameters" α, β, γ.

3) The last term in (25) generalizes the longitudinal propagator -- the second term of (30). It propagates the gauge zero mode and depends explicitly on α. Let us remark that it is, however, transverse to the translation and dilatation zero modes.

4) We could have naïvely expected in $\Pi_{\mu\nu}$ terms propagating the translation and dilatation zero modes and depending on β and γ, i.e., terms of the form

$$
\begin{aligned}
&\frac{\beta}{N_1}\int dt\, F^{cl}_{\mu\sigma,j}(y)\, F^{cl}_{\rho\sigma,\ell}(t)\, \chi^{\ell k}_{\rho\nu}(t,z) \\
&\frac{\gamma}{N_2}\int dt\, y_\sigma\, F^{cl}_{\mu\sigma,j}(y)\, t_\tau\, F^{cl}_{\rho\tau,\ell}(t)\, \hat{\chi}^{\ell k}_{\rho\nu}(t,z) \qquad (32)
\end{aligned}
$$

Besides, we might be puzzled by the role of the β and γ dependent terms in P in order to derive Π which finally does not depend on them. Actually, if the terms depending on β and γ had been omitted, i.e., if we had not taken into account the translation and dilatation zero modes, the propagator would have been ill defined[10] because any term of the form (32) could have been added to the propagator (25) without spoiling equation (24). The effect of the terms depending on β and γ in the kernel (23) is just to forbid terms of the form (32) providing a unique definition of the propagator. We emphasize that our prescription is to treat the translation and dilatation zero modes exactly as the gauge ones[9]. This differs -- at least in principle -- from other approaches in which the propagator is the inverse of the quadratic form in the subspace orthogonal to the dilatation and translation zero modes. A propagator defined in this way has recently been obtained[12]. Due to the fact that it seems to propagate at least the dilatation zero mode, the relation with ours is not immediate.

3.2 Propagators of the Faddeev-Popov ghosts

Let us now derive the other propagators. The part of the Lagrangian (19) quadratic in c and $\bar{c}$ can be written

$$\mathcal{L}^{\text{quadr.}}_{c\bar{c}} = c\, D^2 \bar{c} + (1-s)^2 \lambda^2 m^2\, c\bar{c} \tag{33}$$

which defines the $c\bar{c}$ propagator

$$\Pi^{ij}_{c\bar{c}}(x,y) = \left(D^2 + (1-s)^2 \lambda^2 m^2\right)^{-1}_{ij} \delta(x-y) \tag{34}$$

Finally the part of the Lagrangian (19) quadratic in the x independent fields ψ and ψ_ν is

$$\begin{aligned} \mathcal{L}^{\text{quadr}}_{\psi\psi_\nu} &= - F^{cl}_{\mu\nu} F^{cl}_{\mu\nu} \left(\psi_\nu + x_\nu \psi\right)\left(\bar{\psi}_\rho + x_\rho \bar{\psi}\right) \\ &= - N_1\, \psi_\nu \bar{\psi}_\nu - N_2\, \psi\, \bar{\psi} \end{aligned} \tag{35}$$

N_1 and N_2 are defined in (27). This leads to associate to the ψ fields propagators which are in p space δ distributions. Introducing such propagators allows us to define a perturbation theory in the number of loops. Computing a Green function at some given order requires the calculation of a finite number of Feynman graphs, each graph being computed using perfectly well-defined rules. Ultra-violet and infra-red convergence is ensured by the renormalization procedure indicated at the beginning of the section.

4. RENORMALIZATION: PROOF OF THE SLAVNOV IDENTITIES

We shall begin this part by transcribing invariance (21) into a Slavnov-type identity. Invariance (21) being satisfied at zeroth order, the same is true for the Slavnov identity. Then we shall study the structure of the right-hand side of Slavnov's identity for an arbitrary choice of the counterterms. Finally, we shall prove that it is possible to choose these counterterms in such a way that this right-hand side of the Slavnov identity vanishes at any order. To a large extent this proof will follow closely Ref. 6), so that we shall just sketch these aspects, giving a larger emphasis to the points which are specific to our case.

To transcribe invariance (21) into a Slavnov identity, let us add to the Lagrangian (19) external fields coupled to the variations U_μ and $\bar{U}$ defined in (21) and (21')

$$\hat{L} = L + \eta^i \bar{U}_i + \eta^i_\mu U_{\mu,i} \tag{36}$$

In order that the terms added have dimension four and zero Faddeev-Popov charge, η^i and η^i_μ will be respectively assigned dimension 4 and 3 and Faddeev-Popov charges -2 and -1. We shall define the action of the Slavnov operator S onto the generating functional Γ of the renormalized one irreducible particle Green functions computed at $s = 1$ by:

$$\begin{aligned} S(\Gamma) \equiv \int dx \Big[\frac{\delta \Gamma}{\delta Q_{\mu,i}(x)} \frac{\delta \Gamma}{\delta \eta_{\mu,i}(x)} - \frac{\delta \Gamma}{\delta \bar{c}_i(x)} \frac{\delta \Gamma}{\delta \eta_i(x)} + \frac{1}{\alpha} \frac{\delta \Gamma}{\delta c_i(x)} D_\mu^{ij} Q_{\mu,j}(x) \Big] \\ - \frac{\delta \Gamma}{\delta \bar{\psi}_\nu} \bar{\psi}_\nu \bar{\psi} + \frac{\lambda^2}{\beta} \frac{\delta \Gamma}{\delta \psi_\nu} \int F^{\psi}_{\mu\nu,i}(x)\, Q_{\mu,i}(x)\, dx \; + \\ + \frac{\lambda^4}{\gamma} \frac{\delta \Gamma}{\delta \psi} \int x_\nu F^{\psi}_{\mu\nu,i}(x)\, Q_{\mu,i}(x)\, dx \end{aligned} \tag{37}$$

At zeroth order Γ coincides with the Lagrangian $\hat{L}(s = 1)$ and, as a result of invariance (21),

$$S(\Gamma)\Big|_{\text{zeroth order}} = 0 \tag{38}$$

We shall now state that, for an arbitrary choice of the counterterms allowed by our renormalization procedure, $S(\Gamma)$ satisfies to any order

$$S(\Gamma) = \Delta\Gamma \tag{39}$$

where $\Delta\Gamma$ denotes the generating functional of the renormalized irreducible Green functions with an insertion Δ satisfying the following properties

1) Δ is of order $\hbar$ as implied by (38).
2) Δ is the integral over x of a polynomial of η, η_μ, and the fields described in the Table -- including the external ones -- which are local in these fields.
3) The products of fields composing Δ have a canonical dimension which is strictly equal to 4 and a Faddeev-Popov charge +1.

Let us now prove these assertions. To compute $S(\Gamma)$ we have to use the quantum equations of motion[11] of Q multiplied by $\delta\Gamma/\delta\eta_\mu$, that of $\bar{c}$ multiplied by $\delta\Gamma/\delta\eta$, that of c multiplied by $D_\mu Q_\mu$, that of ψ_ν multiplied by $\bar{\psi}_\nu\bar{\psi}$, that of ψ_ν multiplied by $\int F^{cl}_{\mu\nu} Q_\mu$, and the one of ψ multiplied by $\int x_\nu F^{cl}_{\mu\nu} Q_\mu$. The contributions to the equation of motion of terms -- or counterterms -- of the Lagrangian which have dimension 4 give rise to contributions to Δ, which clearly have the stated structure. The two mass terms -- proportional to $\lambda^2(1-s)^2$ -- and the non-local terms proportional to β and γ give rise to contributions more difficult to analyze. Actually they generate in the equations of motion anisotropic terms[12] which must be reduced using the Zimmermann identities. This is due to the fact that these terms - do not forget that there is no counterterm of this type - which belong to the free Lagrangian are over subtracted and appear in the Zimmermann terminology as $N_4 Q_\mu^2$, $N_4 \bar{c}c$, $N_4(\int F_{\mu\nu} Q_\mu)^2$, $N_4(\int x_\nu F_{\mu\nu} Q_\mu)^2$. In the Zimmermann reduction, extra terms are due to the contributions of the graphs in which a subgraph is or is not subtracted following whether the prescription is the N_4 one or the minimal one. Hence the extra terms have the form of a product of a subgraph at zero s and external momenta, times the contribution of the total graph in which the subgraph has been contracted to a point. These contributions are visualized in Figs. 1 and 2. We notice that all the non-locality is factorized out in the subtraction terms which appears as a c number coefficient. These contributions to Δ have also the structure stated above. In other words, despite the presence of the non-local terms proportional to β and γ in the Lagrangian (19), Δ remains local in terms of quantum and external fields.

4.1 Consistency condition

Using (39) together with what we have learnt on the structure of Δ we shall now derive consistency conditions with a method identical to that of Ref. 6) which we summarize. Due to the anticommuting character of the Slavnov operator S we can associate to the operator S^2 defined by

$$S^2(\Gamma) = \int dx \frac{1}{\alpha} \frac{\delta \Gamma}{\delta c_i(x)} D_\mu^{ij}(x) \frac{\delta \Gamma}{\delta \eta_{\mu,j}(x)} \tag{40}$$

$$+ \frac{\lambda^2}{\beta} \frac{\delta \Gamma}{\delta \bar{\Psi}_\nu} \int dx\, F^d_{\mu\nu,i}(x) \frac{\delta \Gamma}{\delta \eta_{\mu,i}(x)} + \frac{\lambda^4}{\gamma} \frac{\delta \Gamma}{\delta \Psi} \int dx\, x_\nu F^d_{\mu\nu,i}(x) \frac{\delta \Gamma}{\delta \eta_{\mu,i}(x)}$$

symmetry transformations deduced from (21):

$$\begin{aligned}
\delta^2 Q_{\mu,i}(x) &= \delta^2 \bar{c}_i = \delta^2 \bar{\Psi}_\nu = \delta^2 \bar{\Psi} = 0 \\
\delta^2 c_i(x) &= \hat{j} \frac{1}{\alpha} D_\mu^{ij}(x)\, U_{\mu,j}(x) \\
\delta^2 \Psi_\nu &= \hat{j} \frac{\lambda^2}{\beta} \int dx\, F^d_{\mu\nu,i}(x)\, U_{\mu,i}(x) \\
\delta^2 \Psi &= \hat{j} \frac{\lambda^4}{\gamma} \int dx\, x_\nu F^d_{\mu\nu,i}(x)\, U_{\mu,i}(x)
\end{aligned} \tag{41}$$

where $\hat{j}$ is a commuting infinitesimal parameter. To be able to apply the operator S to (39) it is necessary to introduce a source σ coupled to Δ in the Lagrangian. This obviously modifies (39) in two ways: firstly Δ may vary under S, which will introduce in the right-hand side of (39) a term of the form $\beta S(\Delta)$. Secondly Δ may depend on the sources η_μ and η introduced in $\hat{L}$ (36). Then the Slavnov operator defined by (37) is no more equivalent to the transformations (21). A new variation is then introduced and we shall denote it by S^Δ so that (39) becomes

$$S(\Gamma) = \frac{\delta}{\delta\sigma}\Gamma + \sigma\left[S(\Delta) + S^\Delta \hat{L}\right] + \text{terms of order } \hbar\Delta \tag{42}$$

Applying S to (42) we get the consistency condition

$$S^2(\Gamma)\Big|_{\sigma=0} = S(\Delta) + S^{\Delta}\hat{L} + \text{terms of order } \hbar\Delta \tag{43}$$

Following Ref. 6) we shall now use (43) to prove that Δ satisfies

$$\Delta = S(\ell) + \text{terms of order } \hbar\Delta \tag{44}$$

where ℓ is a polynomial in the fields of dimension 4. Equation (44) allows us to prove recursively that it is possible to choose the counterterms at any order in $\hbar$ such that the generating functional of the one irreducible Green functions computed at $s = 1$ satisfy the Slavnov identity

$$S(\Gamma) = 0 \tag{45}$$

to any order in $\hbar$.

Indeed, suppose we have been able to choose the counterterms of the Lagrangian $L^{(n-1)}$ up to the order $\hbar^{n-1}$ so that

$$S(\Gamma^{n-1}) = \Delta^n + O(\hbar^{n+1}) \tag{46}$$

where Δ^n begins at order $\hbar^n$. If (44) holds, $\Delta^n = S(\ell^n)$ up to the order $\hbar^{n+1}$. Let us fix the counterterms of order $\hbar^n$ by defining $L^n = L^{n-1} - \ell^n$. The generating functional Γ^n computed from L^n then satisfies

$$S(\Gamma^n) = O(\hbar^{n+1}) \tag{47}$$

This ends the recursive proof of (45) provided we are able to prove (44).

The proof of (44) paraphrases the analogous proof in Ref. 6) and will only be sketched here. Firstly, one can verify that $S^2(\Gamma)|_{\sigma=0}$ is equal to $S^2(R^{\Delta})|_{\sigma=0}$ modulo terms of order $\hbar$, where R^{Δ} is a polynomial of fields of dimension 4 and of the same order in $\hbar$ as Δ. In a first step consider only in (43) the terms linear in η_μ or η. Such terms cannot appear in $S^\Delta\hat{L}$, since Δ cannot contain terms more than linear in η_μ and η, because of the power counting. Thus (43), restricted, for instance, to terms linear in η, yields:

$$S^2(\eta R_\eta) = S(\eta \Delta_\eta) \tag{48}$$

where ηR_η and $\eta\Delta_\eta$ are the terms linear in η, and in R^Δ and Δ, respectively. Let us divide Δ_η in its invariant part $\Delta_\eta^\natural$ and its non-invariant part $\Delta_\eta^\flat$. η itself being invariant, (48) implies

$$\eta \Delta_\eta^\flat = S(\eta R_\eta) \tag{49}$$

which is of the form (44). Hence $\eta\Delta_\eta^\flat$ can be absorbed by choosing the correspond-ing counterterm properly. The invariant terms $\Delta_\eta^\natural$ can easily be listed and seen to be also of the form (44). The same arguments work for the terms depending on η_μ. Having then absorbed these terms, S^Δ reduces to zero, by definition, and the consistency condition is reduced to:

$$S^2(R) = S(\Delta) \tag{50}$$

Again Δ is divided in its invariant part $\Delta^\natural$ and its non invariant part $\Delta^\flat$. From (50), $\Delta^\flat$ clearly has the form (44) and thus can be absorbed. At this point we stay only with terms $\Delta^\natural$ satisfying

$$S(\Delta^\natural) = 0 \tag{51}$$

These terms cannot contain ψ or ψ_ν fields, because of the form of the varia-tions of ψ and ψ_ν.

If we consider in $\Delta^\natural$ the terms independent of $\bar{\Psi}$, the analysis is iden-tical to the one done in Ref. 6): all these terms can be absorbed. We are left with the terms depending on $\bar{\Psi}$, $\bar{\psi}\Delta_{\bar{\psi}}$. $\bar{\Psi}$ being invariant, having dimension 0 and Faddeev-Popov charge +1, $\Delta_{\bar{\psi}}$ is invariant, has dimension 4 and carries no Faddeev-Popov charge. Such terms are well known: they are the basic terms of the tree approximation Lagrangian (22), that is, since the non-local terms $(\int F^{cl}_{\mu\nu} Q_\mu)^2$ are excluded in Δ,

$$L_1 = \int F_{\mu\nu}(A^{cl}+Q)\, F^{\mu\nu}(A^{cl}+Q)$$
$$L_2 = \int \left[\frac{1}{2\alpha}(D_\mu Q_\mu)^2 - c\, D_\mu U_\mu \right] \tag{52}$$

These terms are not variations and cannot be absorbed by fixing some counterterm. In fact, we have fixed all the counterterms at first order and proved

$$S(\Gamma) = a_1 L_1 \bar{\Psi} + a_2 L_2 \bar{\Psi} + O(\hbar^2) \tag{53}$$

where a_1 and a_2 are unknown coefficients.

In fact, we shall verify that a_1 and a_2 vanish. Let us differentiate (53) with respect to $\bar{\psi}$ and put to zero the sources Q_μ, c, $\bar{c}$, ψ_ν, $\bar{\psi}_\nu$, ψ, $\bar{\psi}$. The left-hand side vanishes, while the right-hand side is $4a_1N_1$, where N_1 is the normalization constant given in (27). This proves that a_1 is actually zero. To prove that a_2 is zero, let us differentiate with respect to $cD_\mu Q_\mu$ the equality by now proved:

$$S^2(\Gamma) = O(\hbar^2) \tag{54}$$

and then set the sources to zero. All the terms vanish except the commutator of S with the derivatives with respect to $cD_\mu Q_\mu$ applied on S. But the action of this commutator on S amounts to derive S with respect to $(D_\mu Q_\mu)^2 + cD_\mu U_\mu$. From (54) we learn that this action is zero at order $\hbar$ while from (53) we learn that it is proportional to a_2. Thus a_2 has also to vanish and the Slavnov identity is proved up to order $\hbar^2$. These arguments can be recursively repeated, proving that it is possible to fix the counterterms at any order in such a way that the Slavnov identity is satisfied at any order in the perturbation theory.

5. SUMMARY AND INTERPRETATION OF THE RESULTS

We have now completely defined a perturbation theory around a classical solution: from the classical Lagrangian (19), and taking into account the mass terms due to the BPHZL renormalization procedure, we have been able to calculate the propagators of the different quantum fields (25), (34) and (35). The counterterms are local in the quantum and external fields, have a canonical dimension 4 and carry no Faddeev-Popov charge. They are uniquely defined by requiring that the generating functional of the one irreducible particle Green functions calculated at s = 1 satisfy the Slavnov identity (37)

$$S(\Gamma) = 0 \tag{54}$$

at any order in the perturbation theory, together with four normalization conditions. At a given order, calculation of a one irreducible particle Green function requires therefore the computation of a finite number of Feynman graphs. The rules to compute each graph are well defined. They have been stated at the beginning of Section 3, the section in which the propagators have been also explicitly given. Convergence is granted by the subtractions involved in the renormalization procedure.

Physically we are interested in Z, the generating functional of the connected Green functions (computed at s = 1), more than in its Legendre transform Γ. To define Z we must add source terms to the Lagrangian which becomes

$$\begin{aligned} \mathcal{L} = L &+ \gamma_i(x)\,\bar{U}_i(x) + \eta_{\mu,i}(x)\,U_{\mu,i}(x) + \\ &+ j_{\mu,i}(x)\,\varphi_{\mu,i}(x) + \bar{\xi}_i(x)\,c_i(x) + \xi_i(x)\,\bar{c}_i(x) + \\ &+ \bar{\zeta}_\nu\psi_\nu + \zeta_\nu\bar{\psi}_\nu + \bar{\zeta}\psi + \zeta\bar{\psi} \end{aligned} \tag{55}$$

The action of the S operator on Z is defined by

$$\begin{aligned} SZ = \Big\{ \int dx \Big[j_{\mu,i}(x)\frac{\delta}{\delta\eta_{\mu,i}(x)} - \xi_i(x)\frac{\delta}{\delta\gamma_i(x)} + \frac{1}{\alpha}\bar{\xi}_i(x) D_\mu^{ij}(x)\frac{\delta}{\delta j_{\mu,j}(x)}\Big] \\ - \zeta_\nu\frac{\delta}{\delta\bar{\zeta}_\nu}\frac{\delta}{\delta\zeta} + \frac{\Lambda^2}{\beta}\bar{\zeta}_\nu\int dx\, F^{d}_{\mu\nu,i}(x)\frac{\delta}{\delta j_{\mu,i}(x)} + \\ + \frac{\Lambda^4}{\gamma}\bar{\zeta}\int dx\, x_\nu F^{d}_{\mu\nu,i}(x)\frac{\delta}{\delta j_{\mu,i}(x)}\Big\} Z = 0 \end{aligned} \tag{56}$$

From the squared Slavnov identity (40) we can derive, as in Ref. 6), the equations of motion of the Faddeev-Popov ghosts which, written in terms of Z, are:

$$
\begin{aligned}
&D_\mu^{ij}(x) \frac{\delta}{\delta \eta_{\mu,j}(x)} Z = \bar{\xi}_i(x) \\
&\int dx\, F^{\alpha}_{\mu\nu,i}(x) \frac{\delta}{\delta \eta_{\mu,i}(x)} Z = \bar{\xi}_\nu \\
&\int dx\, x_\nu\, F^{\alpha}_{\mu\nu,i}(x) \frac{\delta}{\delta \eta_{\mu,i}(x)} Z = \bar{\xi}
\end{aligned}
\tag{57}
$$

Finally, we can study the dependence on β and γ of the connected Green functions. Actually we have already found that the propagator of the Q_μ does not depend on these parameters. We shall discuss the dependence on β, the arguments being identical for γ. From Lowenstein's action principle[12)]

$$
\frac{\partial}{\partial \beta}\ \Delta^\beta Z \tag{58}
$$

where $\Delta^\beta Z$ is the generating functional of the connected Green functions with the insertion Δ^β which is $(\int F_{\mu\nu} Q_\mu)^2$ in the tree approximation. Let the Slavnov operator act on (58):

$$
[S, \Delta^\beta] Z = -\left[\frac{\partial}{\partial \beta}, S\right] Z = \frac{\lambda^2}{\beta^2}\, \bar{\xi}_\nu \int dx\, F^{\alpha}_{\mu\nu,i}(x) \frac{\delta}{\delta j_{\mu,i}(x)} Z \tag{59}
$$

On the other hand, let us consider the insertion

$$
\hat{\Delta} = \bar{\xi}_\nu \frac{\partial}{\partial \bar{\xi}_\nu} \tag{60}
$$

It verifies

$$[S,\hat{\Delta}]Z = -\frac{\lambda^2}{\beta}\bar{\xi}_\nu \int dx\, F^{a}_{\mu\nu,i}(x)\, \frac{\delta}{\delta j_{\mu,i}(x)} Z \tag{61}$$

which allows the following decomposition for Δ^β:

$$\begin{aligned} \Delta^\beta &= -\frac{1}{\beta}\hat{\Delta} + \Delta_{inv} \\ [S, \Delta_{inv}]Z &= 0 \end{aligned} \tag{62}$$

We know that there are six independent invariants which, in the tree approximation, are the four invariants L_i of (22) and two invariants related to the normalization of the external fields η_μ, η:

$$\begin{aligned} \Delta_1 &= \int F_{\mu\nu}(A^{a}+Q)F^{\mu\nu}(A^{a}+Q) + \text{terms of order } \hbar \\ \Delta_2 &= \int\left[-\frac{1}{2\alpha}(D_\mu Q_\mu)^2 + c\, D_\mu U_\mu\right] + \text{terms of order } \hbar \\ \Delta_3 &= -\frac{\lambda^2}{2\beta}\left(\int dx\, F^{a}_{\mu\nu,i}(x)\frac{\delta}{\delta j_{\mu,i}(x)}\right)^2 + \bar{\xi}_\nu \frac{\delta}{\delta \bar{\xi}_\nu} \\ \Delta_4 &= -\frac{\lambda^4}{2\gamma}\left(\int dx\, x_\nu F^{a}_{\mu\nu,i}(x)\frac{\delta}{\delta j_{\mu,i}(x)}\right)^2 + \bar{\xi}\frac{\delta}{\delta \bar{\xi}} \\ \Delta_5 &= j_\mu \frac{\delta}{\delta j_\mu} + \eta_\mu \frac{\delta}{\delta \eta_\mu} + \bar{\xi}\frac{\delta}{\delta \bar{\xi}} + \bar{\xi}_\nu \frac{\delta}{\delta \bar{\xi}_\nu} + \xi\frac{\delta}{\delta \bar{\xi}} \\ \Delta_6 &= \xi\, \delta/\delta\xi + \eta\, \delta/\delta\eta \end{aligned} \tag{63}$$

where use has been made of the equations of motion (57) for $\bar{\xi}_\nu, \bar{\xi}$. Let us now in (58) put to zero the sources ζ_ν, $\bar{\zeta}_\nu$, ζ, $\bar{\zeta}$. Taking into account the fact that the propagator of the Q_μ is orthogonal to $F^{cl}_{\mu\nu}$ and to $x_\nu F^{cl}_{\mu\nu}$, the first term of Δ_3 and Δ_4 does not contribute. On the other hand, $\hat{\Delta}$ and the second term of Δ_3 and Δ_4 do not contribute because of vanishing sources ζ_ν and $\bar{\xi}$. Then (58) can be written

$$\frac{\delta}{\delta\beta} Z \Big|_{\zeta=\bar{\zeta}=\zeta_\nu=\bar{\zeta}_\nu=0} = b_1\Delta_1 Z + b_2\Delta_2 Z + b_5\Delta_5 Z + b_6\Delta_6 Z \tag{64}$$

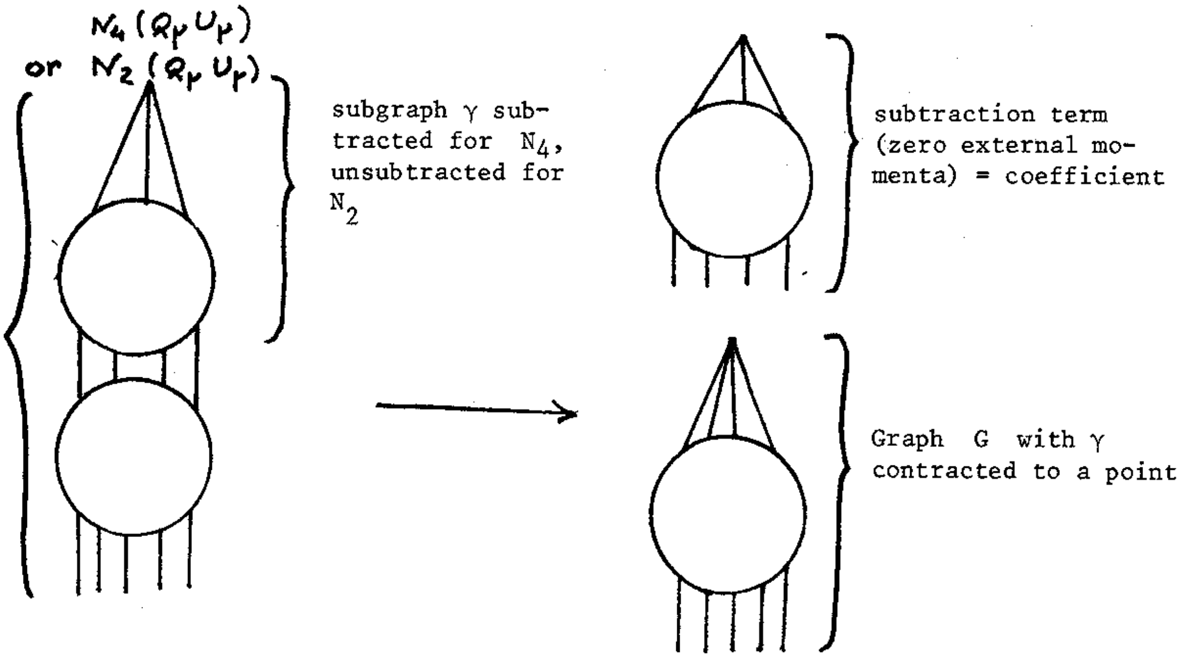
where b_1, b_2, b_5, b_6 are some unknown coefficients. It is the point now to recall that we must require normalization conditions to define the theory, precisely to fix in the Lagrangian the coefficients of L_1, L_2, L_3, L_4 and the normalization of the external fields. For L_1, L_2, η_μ, η_μ we shall give normalization conditions of course independent of β and γ - they will be, for instance, coupling constants for L_1 and L_2. Applying (64) to these normalization conditions which are required to be independent of β we find that b_1, b_2, b_5, b_6 must vanish. Thus we have proved that the connected Green functions containing no field ψ_ν, ψ, $\bar{\psi}_\nu$, $\bar{\psi}$ are independent of β and γ, which confirms what was expected from the heuristic approach of Section 2. A consequence is that the dilatation parameters λ coming together with β and γ in the Lagrangian can be forgotten, which makes manifest the dilatation invariance (at $s = 1$) in the tree approximation. However, dilatation invariance is broken at higher orders. Let us recall indeed that in Section 4 we have chosen the counterterms to absorb the right-hand side of the Slavnov identity. As we have explained, a contribution to this right-hand side came out of the Zimmermann identities used to reduce the "anisotropy" generated in the equations of motion by the mass terms proportional to λ^2. As a result, the counterterms do depend on λ, and dilatation invariance is broken at order $\hbar$.

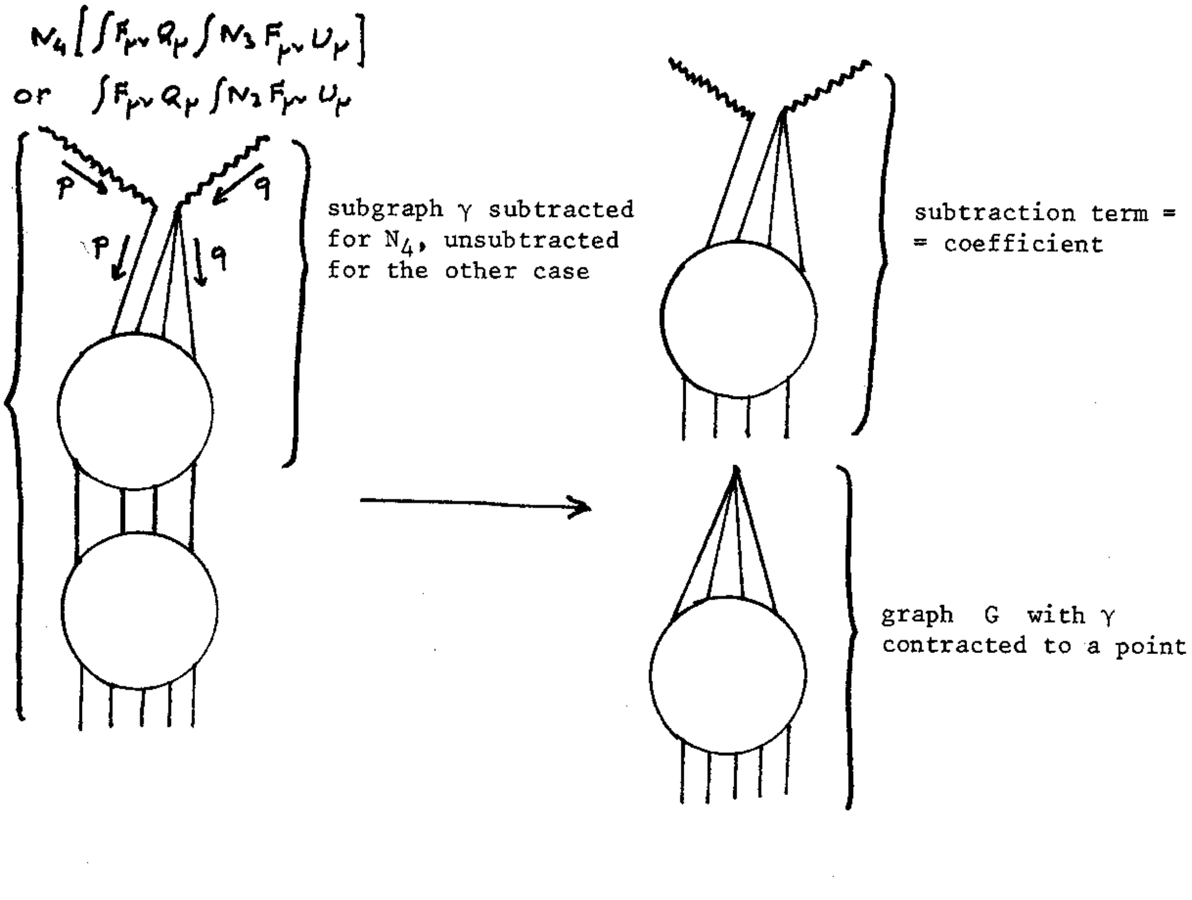
We cannot argue about the dependence on α in pure Yang-Mills theory. However, if we supply this model with a Higgs-Kibble mechanism we can obviously follow the arguments given in Ref. 6) for the perturbation theory developed around zero.

To summarize, we have rigorously defined a perturbation expansion of the Yang-Mills theory around the BPST instanton solution in the sense of a formal power series in $\hbar$. This expansion has, therefore, the same status as the perturbative one around the $A_\mu = 0$ solution (usual vacuum).

The method we introduce can be generalized to more complicated classical solutions. Its form will determine the zero modes that should be considered for each case.

Of course, we do not enter into the matter of whether or not the perturbative treatment around classical solutions can have any chance to exhaust the non-perturbative character of the theory.


$N_4(Q_\mu U_\mu)$
or $N_2(Q_\mu U_\mu)$
subgraph γ subtracted for N_4, unsubtracted for N_2
subtraction term (zero external momenta) = coefficient
Graph G with γ contracted to a point


$N_4[\int F_{\mu\nu} Q_\mu \int N_3 F_{\mu\nu} U_\mu]$
or $\int F_{\mu\nu} Q_\mu \int N_2 F_{\mu\nu} U_\mu$
p
q
p
q
subgraph γ subtracted for N_4, unsubtracted for the other case
subtraction term = = coefficient
graph G with γ contracted to a point

REFERENCES

1) A.A. Belavin et al., Phys. Letters 59B, 85 (1975).

2) A.M. Polyakov, Nuclear Phys. B120, 429 (1977).

3) G. 't Hooft, Phys. Rev. D14, 3432 (1976).

4) A formal attempt with a similar aim, but without the treatment of zero modes, has been performed by H. Kluberg-Stern and J.B. Zuber, Phys, Rev. D14, 3432 (1976).

5) J.L. Gervais and B. Sakita, Phys. Rev. D11, 2943 (1975).

6) C. Becchi, A. Rouet and R. Stora, Ann. Phys 98, 2 (1976).

7) W. Zimmermann, Ann. Phys. 77, 536 (1973); 77, 570 (1973).

8) J.H. Lowenstein and W. Zimmermann, Nuclear Phys. B86, 77 (1975).
J.H. Loweinstein, Comm. Math. Phys. 47, 53 (1976).
T. Clark and J.H. Lowenstein, Generalization of Zimmermann's normal product identity (to be published).

9) D. Amati and A. Rouet, Phys. Letters 73B, 39 (1979).

10) L.D. Faddeev, V.E Korepin, Phys. Letters 63B, 435 (1976).

11) L. Brown, R.D. Carlitz, D.B. Creamer and C. Lee, Phys. Letters 70B, 180 (1977).

12) J.H. Lowenstein, Comm. Math. Phys. 24, 1 (1971).
Y.M.P. Lam, Phys. Rev. D6, 2145 (1972); D7 2943 (1973).