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Renormalization Theory in Four-Dimensional Scalar Fields (I)

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Abstract. We present a renormalization group approach to the renormalization theory of Φ_4^4, using techniques that have been introduced and used in previous papers and that lead to very simple methods to bound the coefficients of the effective potential and of the Schwinger functions. The main aim of this paper is to show how one can in this way obtain the $n!$-bounds.

1. Introduction

Recently we developed a new technique to construct field theories without cutoffs [1] and our method came to play an important role in several papers, where they have been combined with new brilliant ideas [2, 3].

The possibility of treating situations as complex as that arising in the ultraviolet stability of the Coulomb gas [4] in two dimensions relies on the effectiveness of our method of performing the renormalization which allowed us to avoid completely the consideration of unboundedly large orders of perturbation theory in dealing with the construction of superrenormalizable field theories, in contrast with what was done in the previous breakthrough papers [11].

We shall illustrate in our work how this method can be naturally applied to derive some of the deepest mathematical results of the formal perturbation theory, namely Hepp's theorem [5] and the de Calan, Rivasseau $n!$-bounds [6]. The experts will recognize in the discussion below most of the intriguing difficulties met in [5, 6] and the ideas used to attack them; they will also recognize a somewhat different pattern of solution of the problems and several technical differences which, we hope, make our presentation something more than a rewriting of old results.

We perform our discussion in "coordinate space" rather than in the usual "momentum space" because we think that it is much easier and we study the

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"perturbative effective interaction" rather than the renormalized Schwinger functions, which is a new result. We follow the method outlined in [1], in the $d=3$ case, for third-order perturbation theory, showing that it leads to a very natural organization of the cancellations appearing automatically in a form which allows us to avoid the introduction of "Hepp sectors" to check finiteness. Our method is not special for scalar field theories, and we hope that it may be useful to treat renormalization problems in more structured theories, like gauge ones; it is also clearly built in order to be fit for a renormalization group study of the above models.

We present our discussion only for $d=4$; this, of course, implies that the same method would work for $d=2$ and $d=3$ with many complications disappearing. An explicit adaptation to the harder sine-Gordon problem will be, hopefully, presented elsewhere. A similar approach to the renormalization theory has been recently produced by Polchinski [7].

Let us now briefly describe the content of the next sections: Sect. 2 contains the formulation of the Φ_4^4-field theory problem, its perturbative solution and our results on the "effective potential" which implies both Hepp's theorem and the de Calan Rivasseau $n!$-bound. Sections 3–5 just set the definitions and the formalism to be used later on; although nearly purely definitory they cannot be skipped, as the concepts introduced there are a bit unusual; the notion of tree is given there in detail and it shows its resemblance to the notion of forest introduced by Zimmermann [14].

Sections 6 and 7 are the central ones. In Sect. 6 the general algorithm to build the counterterms of a generic order in g is given, the central notion of $\mathscr{R}$ and $\mathscr{L}$ operations is introduced and the general idea of the proof of our version of Hepp's theorem is presented. In Sect. 7 this proof is explicitly stated; then the remaining part of the section is devoted to the proof of the $n!$-bound.

The possibility of partially resumming the divergent perturbation series will be discussed in a second paper, hereafter referred as (II), where some more technical details on the problems discussed here will be also provided in the form of appendices. There also the proof of the Borel summability of the planar Φ_4^4-field theory [13] is sketched.

2. Notations and Formulation of the Renormalizability Problem

Consider the probability measure P on the space of distributions on R^4 ("fields") which is gaussian and has covariance

$$C_{\xi\eta}=\frac{1}{(2\pi)^4}\int_{R^4}\frac{e^{ip(\xi\cdot\eta)}}{1+p^2}d^4p\,. \tag{2.1}$$

The problem is to give a meaning to the expression

$$\mathscr{N}\left\{\exp-\int_{\Lambda}(\lambda:\varphi_\xi^4:+\mu:\varphi_\xi^2:+\alpha:(\partial\varphi_\xi)^2:)d^4\xi\right\}P(d\varphi)\,, \tag{2.2}$$

where the Wick product : : is defined as

$$:x^n:=(\sqrt{2\langle x^2\rangle})^n H_n(x/\sqrt{2\langle x^2\rangle})\,,\qquad n:0,1,\ldots, \tag{2.3}$$

where H_n are the Hermite polynomials, for any gaussian random variable with dispersion $\langle x^2\rangle$.

Obviously $\langle\varphi_\xi^2\rangle=+\infty$, $\langle(\partial\varphi_\xi)^2\rangle=+\infty$ if φ_ξ is distributed as a gaussian random field with covariance (2.1). The first step to give a meaning to (2.2) is to "regularize the field." We find it convenient to define a regularized cutoff field by using a Pauli-Villars type of regularization because it has a simple recursive structure. However, the simplicity of the recursive structure not being too important, one could use other regularizations if desired (e.g. the lattice regularization).

The regularization we shall use stems out of a few trivial identities. Let $\gamma>1$ and observe the following relations:

$$\begin{aligned}
\frac{1}{1+p^2} &= \sum_0^\infty{}_k\left(\frac{1}{p^2+\gamma^{2k}}-\frac{1}{p^2+\gamma^{2(k+1)}}\right)=\sum_0^\infty{}_k\frac{(\gamma^2-1)\gamma^{2k}}{(p^4+\gamma^{2k}(\gamma^2+1)p^2+\gamma^{4k+2})}\\
&=(\gamma^2-1)\sum_0^\infty{}_k\gamma^{2k}\sum_0^\infty{}_h\left(\frac{1}{p^4+\gamma^{2k}(\gamma^2+1)p^2+\gamma^{4k+2}\gamma^{4h}}\right.\\
&\qquad\left.-\frac{1}{p^2+\gamma^{2k}(\gamma^2+1)p^2+\gamma^{4k+2}\gamma^{4(h+1)}}\right)\\
&=\gamma^2(\gamma^2-1)(\gamma^4-1)\sum_0^\infty{}_k\sum_0^\infty{}_h\frac{\gamma^{2k}\gamma^{4(k+h)}}{(p^8+A(k,h)p^6+B(k,h)p^4+C(k,h)p^2+D(k,h))}\\
&=\gamma^2(\gamma^2-1)(\gamma^4-1)\sum_0^\infty{}_k\gamma^{2k}\sum_0^\infty{}_h\gamma^{4(k+h)}\\
&\qquad\cdot\sum_0^\infty{}_l\left(\frac{1}{p^8+Ap^6+Bp^4+Cp^2+\gamma^{8(k+h+l+1)}}\right.\\
&\qquad\left.-\frac{1}{p^8+Ap^6+Bp^4+Cp^2+\gamma^{8(k+h+l+2)}}\right)\\
&=\gamma^2(\gamma^2-1)(\gamma^4-1)\sum_0^\infty{}_q\sum_0^q{}_t\sum_0^t{}_l\gamma^{-2q}\gamma^{-2t}\gamma^{-4l}\\
&\qquad\cdot\left(\frac{1}{k^8+\tilde{A}k^6+\tilde{B}k^4+\tilde{C}k^2+\gamma^8}-\frac{1}{k^8+\tilde{A}k^6+\tilde{B}k^4+\tilde{C}k^2+\gamma^{16}}\right),
\end{aligned}\tag{2.4}$$

where

$$\begin{aligned}
A&=\gamma^{2q}(\gamma^{-2t}2(\gamma^2+1))\equiv\gamma^{2q}\tilde{A},\\
B&=\gamma^{4q}(\gamma^{-4l}\gamma^2(\gamma^4+1)+\gamma^{-4t}(\gamma^2+1)^2)\equiv\gamma^{4q}\tilde{B},\\
C&=\gamma^{6q}(\gamma^{-2t}\gamma^{-4l}\gamma^2(\gamma^2+1)(\gamma^4+1))\equiv\gamma^{6q}\tilde{C},\\
k&\equiv\gamma^{-q}p.
\end{aligned}\tag{2.5}$$

Therefore

$$\begin{aligned}
\frac{1}{(2\pi)^4}\int d^4p\frac{e^{ip\cdot(\xi-\eta)}}{p^2+1}&\equiv C_{\xi,\eta}^{(\infty)}\equiv\sum_0^\infty{}_q\tilde{C}^{(q)}(\xi,\eta)\equiv\sum_0^\infty{}_q\sum_0^q{}_t\hat{C}^{(q,t)}(\xi,\eta)\\
&\equiv\sum_0^\infty{}_q\sum_0^q{}_t\sum_0^t{}_l\hat{\hat{C}}^{(q,t,l)}(\xi,\eta),
\end{aligned}\tag{2.6}$$

where $\tilde{C}^{(q)}$, $\hat{C}^{(q,t)}$, $\hat{\hat{C}}^{(q,t,l)}$ can be easily written down explicitly looking at Eq. (2.4).

To put the cutoff means, here, to truncate the sum over q in (2.6) up to N, obtaining

$$C^{(N)}_{\xi,\eta}=\sum_{0}^{N}{}_q\,\gamma^{2q}\sum_{0}^{q}{}_t\,\gamma^{-2t}\sum_{0}^{t}{}_l\,\gamma^{-4l}\left\{\gamma^2(\gamma^2-1)(\gamma^4-1)\frac{1}{(2\pi)^4}\int d^4k e^{ik\gamma^q(\xi-\eta)}\right.$$
$$\cdot\left(\frac{1}{k^8+\tilde{A}k^6+\tilde{B}k^4+\tilde{C}k^2+\gamma^8}-\frac{1}{k^8+\tilde{A}k^6+\tilde{B}k^4+\tilde{C}k^2+\gamma^{16}}\right), \quad (2.7)$$

which defines the covariance of the cutoff field

$$\varphi^{[\leq N]}_\xi\equiv\sum_{0}^{N}{}_q\,\varphi^{[q]}_\xi\equiv\sum_{0}^{N}{}_q\sum_{0}^{q}{}_t\sum_{0}^{t}{}_l\sqrt{\gamma^{2q}\gamma^{-2t}\gamma^{-4l}}\,Z^{(t,l)}_{\gamma^q\xi}, \quad (2.8)$$

where the covariance of $Z^{(t,l)}_{\gamma^q\xi}$ is

$$\gamma^{-2q}\gamma^{2t}\gamma^{4l}\hat{\hat{C}}^{(q,t,l)}(\xi,\eta)\equiv\Gamma^{(t,l)}_{\gamma^q\xi,\gamma^q\eta}, \quad (2.9)$$

and

$$\Gamma^{(t,l)}_{\xi,\eta}=\frac{\gamma^2(\gamma^2-1)(\gamma^2-1)}{(2\pi)^4}\int d^4k e^{ik(\xi-\eta)}$$
$$\cdot\left(\frac{1}{k^8+\tilde{A}k^6+\tilde{B}k^4+\tilde{C}k^2+\gamma^8}-\frac{1}{k^8+\tilde{A}k^6+\tilde{B}k^4+\tilde{C}k^2+\gamma^{16}}\right). \quad (2.10)$$

It is clear that, being $\langle\cdot\rangle\equiv\int P(d\varphi^{[\leq N]})$, we have

$$\langle(\varphi^{[\leq N]}_\xi)^2\rangle,\quad \langle(\partial_i\varphi^{[\leq N]}_\xi)^2\rangle,\quad \langle(\partial^2_{ij}\varphi^{[\leq N]}_\xi)^2\rangle,\quad \langle(\partial^3_{ijk}\varphi^{[\leq N]}_\xi)^2\rangle,$$
$$\langle(\partial^4_{ijkl}\varphi^{[\leq N]}_\xi)^2\rangle,\quad \langle(\partial^5_{ijkls}\varphi^{[\leq N]}_\xi)^2\rangle,\quad \langle((1-\Delta)^{3(1-\varepsilon)}\varphi^{[\leq N]}_\xi)^2\rangle<\infty, \quad (2.11)$$
$$\forall\varepsilon>0,$$

and therefore it makes sense to define

$$\exp-\int_\Lambda\{\lambda_N:(\varphi^{[\leq N]}_\xi)^4:+\mu_N:(\varphi^{[\leq N]}_\xi)^2:+\alpha_N:(\partial\varphi^{[\leq N]}_\xi)^2:+\nu_N\}d^4\xi \quad (2.12)$$

with respect to the measure:

$$P(d\varphi^{[\leq N]})\equiv\prod_{0}^{N}{}_q\,P(d\varphi^{[q]})\equiv\prod_{0}^{N}{}_q\prod_{0}^{q}{}_t\prod_{0}^{t}{}_l\,\hat{\hat{P}}^{(q,t,l)}(dZ), \quad (2.13)$$

where $\hat{\hat{P}}^{(q,t,l)}$ is the gaussian measure with covariance $\Gamma^{(t,l)}_{\gamma^q\xi,\gamma^q\eta}$. In fact if $\lambda_N>0$, it makes sense to consider (2.12) as a density function with respect to the measure (2.13) because it can be shown by classical methods [8] that it is a L_1-function with integral $\equiv\mathcal{N}^{-1}<+\infty$.

Remarks. a) It should be clear that the regularization used is a "multiple" Pauli-Villars regularization; the necessity of such a generalization of the standard Pauli-Villars will be clear in Sect. 6 and comes from the need of having the second derivatives of the cutoff field $\varphi^{[\leq N]}$ well defined. Nevertheless a careful investigation of our proofs shows that whatever regularization one chooses everything works, provided it satisfies the following requests:

1) $\varphi^{[\leqq N]}$ is well defined with its derivatives at least up to the second order.

2) $C^{(N)}$, the covariance of $\varphi^{[\leqq N]}$, can be decomposed as $C^{(N)}=\sum_{0}^{N}{}_q \tilde{C}^{(q)}$ or $\varphi_\xi^{[\leqq N]}=\sum_{0}^{N}{}_q \varphi_\xi^{[q]}$ and:

I) $\varphi_\xi^{[q]}$ is regular on the scale γ^{-q},

II) $\varphi_\xi^{[q]}$ is nearly independent on the same scale.

This is fulfilled if $\tilde{C}^{(q)}(\xi\eta)$ is sufficiently regular and decays exponentially on this scale, but also a power decay strong enough would be sufficient.

The Pauli-Villars regularization turns out to be really useful in the constructive theory, see [4], where its Markov property is used. Although in this paper we only deal with formal perturbation theory we have preferred to use the Pauli-Villars regularization.

b) We shall assume, hereafter, periodic boundary conditions for the volume Λ. Therefore the field $\varphi^{[\leqq k]}$ has covariance:

$$C_p^{(k)}(\xi,\eta)=\sum_{n\in Z^4} C^{(k)}(\xi,\eta+nL),$$

where L is the linear size of Λ.

The main problem is to see if we can find λ_N, μ_N, α_N, ν_N such that

$$P_{\text{int}}^{(N)}\equiv \Big\{\exp-\int_\Lambda(\lambda_N:(\varphi_\xi^{[\leqq N]})^4:+\mu_N:(\varphi_\xi^{[\leqq N]})^2: \\ +\alpha_N:(\partial\varphi_\xi^{[\leqq N]})^2:+\nu_N)d_\xi^4\Big\}P(d\varphi^{[\leqq N]}) \tag{2.14}$$

admits a limit as $N\to\infty$ in the sense that

$$\int \exp\varphi^{[\leqq N]}(f)P_{\text{int}}^{(N)}(d\varphi^{[\leqq N]})\xrightarrow[N\to\infty]{}\int\exp\varphi(f)P_\infty(d\varphi) \tag{2.15}$$

for all $f\in\mathscr{S}(R^4)$ and for a suitable chosen measure P_∞ on $\mathscr{S}'(R^4)$. This problem has not yet been solved, see [9, 12]. However, it admits a perturbative solution; namely it can be shown [5, 6, 9].

Proposition 1. *There exist four formal power series in a parameter g, the physical coupling constant,*

$$\lambda_N=g+\sum_{2}^{\infty}{}_k l_k(N)g^k,\qquad \mu_N=\sum_{2}^{\infty}{}_k m_k(N)g^k, \\ \alpha_N=\sum_{2}^{\infty}{}_k a_k(N)g^k,\qquad \nu_N=\sum_{2}^{\infty}{}_k n_k(N)g^k, \tag{2.16}$$

such that the resulting formal power series expansion of

$$S_{(N)}^T(f;p)=\frac{\partial^p}{\partial\theta^p}\log\int\exp\theta\varphi^{[\leqq N]}(f)P_{\text{int}}^{(N)}(d\varphi^{[\leqq N]})|_{\theta=0} \tag{2.17}$$

has the form

$$S_{(N)}^T(f;p)=\sum_{0}^{\infty}{}_n g^n\bar{S}_{(N),n}^T(f;p) \tag{2.18}$$

and the limits

$$\lim_{N\to\infty}\bar{S}_{(N);n}^T(f;p)=\bar{S}_n^T(f,p) \tag{2.19}$$

exist $\forall f \in \mathscr{S}(R^4)$ *and satisfy*

$$\sup_p \frac{|\bar{S}_n^T(f;p)|}{(c^p p! \|f\|_1^p)} \leqq (\bar{\bar{C}})^n n!, \qquad \bar{S}_n^T(f;4) \neq 0, \tag{2.20}$$

where $\|\cdot\|_1$ *is the* L_1*-norm.*

In this paper we give a new proof of the above result and we study also the "effective potential" in a formal way. Namely remembering Eq. (2.8),

$$\varphi_\xi^{[\leqq m]} = \sum_{q=0}^{m} \varphi_\xi^{[q]}, \tag{2.21}$$

we wish to study

$$\int \exp \varphi^{[\leqq m]}(f) P_{\text{int}}^{(N)}(d\varphi^{[\leqq N]}), \tag{2.22}$$

and we ask whether there is a function $V^{(m)}(\varphi^{[\leqq m]})$ (the "effective potential") such that

$$\int P_{\text{int}}^{(N)} d\varphi^{[\leqq N]}) \exp \varphi^{[\leqq m]}(f) = \frac{\int P^{(m)}(d\varphi^{[\leqq m]}) \exp(V^{(m)}(\varphi^{[\leqq m]}) + \varphi^{[\leqq m]}(f))}{\int P^{(m)}(d\varphi^{[\leqq m]}) \exp V^{(m)}(\varphi^{[\leqq m]})}. \tag{2.23}$$

This problem is harder than the former one and is unsolved. However one can hope that the choice (2.16) leads to a perturbative solution of the problem of the computation of the effective potential $V^{(m)}$.

In fact in this paper we show not only that $V^{(m)}$ is well defined as a formal power series but also that it can be written in the following form:

$$V^{(m)}(\varphi) = \Sigma_p \sum_{\{\underline{a},\underline{b},\underline{c},\underline{d},\underline{e},\underline{f},\underline{g}\}} \int_{\Lambda^p} dX\, V_p^{(m)}(X;\underline{a},\underline{b},\underline{c},\underline{d},\underline{e},\underline{f},\underline{g})$$
$$\cdot : \varphi_X^{\underline{a}} (\partial\varphi_X)^{\underline{b}} D_X^{\underline{c}} S_X^{\underline{d}} D_X^{1\underline{e}} T_X^{\underline{f}} S_X^{1\underline{g}} :, \tag{2.24}$$

where

$$X = (\underline{x}_1, \ldots, \underline{x}_p), \qquad \underline{a} = (a_1, \ldots, a_p), \qquad \underline{b} = (b_1, \ldots, b_p),$$
$$\underline{c} = \{c_{ij}\}_{i<j}, \quad \underline{d} = \{d_{ij}\}_{i<j}, \quad \underline{e} = \{e_{ij}\}_{i<j}, \quad \underline{f} = \{f_{ij}\}_{i<j}, \quad \underline{g} = \{g_{ij}\}_{i<j}$$

where

$$a_i, b_i, c_{ij}, d_{ij}, e_{ij}, f_{ij}, g_{ij} \in Z \cap [0,4], \qquad \varphi = \varphi^{[\leqq m]}, \qquad \partial\varphi = \partial\varphi^{[\leqq m]},$$
$$D_{xy} = \varphi_x^{[\leqq m]} - \varphi_y^{[\leqq m]}, \qquad S_{xy} = (\varphi_x^{[\leqq m]} - \varphi_y^{[\leqq m]}) - (\underline{x}-\underline{y}) \cdot \underline{\partial}\varphi_y^{[\leqq m]},$$
$$T_{xy} = (\varphi_x^{[\leqq m]} - \varphi_y^{[\leqq m]}) - (\underline{x}-\underline{y}) \cdot \underline{\partial}\varphi_y^{[\leqq m]} - \tfrac{1}{2}(\underline{x}-\underline{y})(\underline{x}-\underline{y})\underline{\partial}^2 \varphi_y^{[\leqq m]},$$
$$D_{xy}^1 = \partial\varphi_x^{[\leqq m]} - \partial\varphi_y^{[\leqq m]}, \qquad S_{xy}^1 = \partial\varphi_x^{[\leqq m]} - \partial\varphi_y^{[\leqq m]} - (\underline{x}-\underline{y}) \cdot \underline{\partial}^2 \varphi_y^{[\leqq m]}, \tag{2.25}$$

and

$$: \varphi_X^{\underline{a}} (\partial\varphi_X)^{\underline{b}} D_X^{\underline{c}} S_X^{\underline{d}} D_X^{1\underline{e}} T_X^{\underline{f}} S_X^{1\underline{g}} :$$
$$= : \prod_{i=1}^{p} \varphi_{x_i}^{a_i} (\partial\varphi_{x_i})^{b_i} \prod_{i<j}^{(1,p)} D_{x_i x_j}^{c_{ij}} S_{x_i x_j}^{d_{ij}} D_{x_i x_j}^{1 e_{ij}} T_{x_i x_j}^{f_{ij}} S_{x_i x_j}^{1 g_{ij}} : \tag{2.26}$$

Furthermore we have the following proposition which is the central result in the proof of the renormalizability of the theory.

Proposition 2. *Let's denote $\eta_i, \ldots, \eta_r$ the r independent variables of the fields in the Wick monomials* $:\varphi_X^{\underline{a}}(\partial\varphi_X)^{\underline{b}} \ldots S_X^{1\underline{g}}:$ *then* $\exists\kappa>0$, $G>0$, *N-independent such that*

$$\begin{aligned}&\int_{\Delta_1\times\ldots\times\Delta_r} d\eta_1 \ldots d\eta_r \int_{\Delta^{p-r}} d(X\backslash\eta)|V_p^{(m)}(X;a,b,\ldots,g)|\\&\cdot\left(\prod_1^p{}_i(\gamma^m)^{a_i}(\gamma^{2m})^{b_i}\prod_{i<j}^{(1,p)}[(\gamma^m(\gamma^m|x_i-x_j|))^{c_{ij}}(\gamma^m(\gamma^m|x_i-x_j|)^2)^{d_{ij}}\right.\\&\cdot(\gamma^{2m}(\gamma^m|x_i-x_j|))^{e_{ij}}(\gamma^m(\gamma^m|x_i-x_j|)^3)^{f_{ij}}(\gamma^{2m}(\gamma^m|x_i-x_j|)^2)^{g_{ij}}]\\&\leqq(|\underline{a}|+\ldots+|\underline{g}|)!e^{-\kappa\gamma^m d(\Delta_1,\ldots,\Delta_r)}p!(gG)^p\end{aligned}\tag{2.27}$$

where $d(X\backslash\eta)$ means that we integrate over all the variables except the η's, Δ_i is a hypercubic tessera of linear size γ^{-m}, $i=1,\ldots,r$, $|\underline{a}|=\sum_1^p{}_i a_i$ and similar definitions for $|\underline{b}|, |\underline{c}|, \ldots, |\underline{f}|, |\underline{g}|$. $d(\Delta_1,\ldots,\Delta_r)$ is the length of the shortest polygonal joining the tesserae $\Delta_1,\ldots,\Delta_r$. Moreover

$$|V_p^{(m)}(X;a,b,\ldots,g)|=0 \quad \text{if} \quad |\underline{a}|+|\underline{b}|+\ldots+|\underline{g}|\geqq 4p\,.$$

This proposition will be proved in Sect. 7.

The integrals (2.27) depend on N but they are uniformly bounded by (2.27) and have a limit as $N\to\infty$ if m is held fixed. Estimates like (2.27) have played a key role in our approach to the construction of the Φ^4-theory in 2 or 3 dimensions [1] as well as in the Sine-Gordon theory and therefore have, in our opinion, some interest.

3. Free Field Properties

It is useful to have in mind the following simple properties of the fields $\varphi^{[q]}$ and $\varphi^{[\leqq q]}$, which follow from standard arguments [10] based on the inequalities immediately derived from (2.4) ... (2.10),

$$\begin{aligned}&\langle(\varphi_\xi^{[q]})^2\rangle\leqq C_1(\gamma)\gamma^{2q},\\&\langle(\varphi_\xi^{[q]}-\varphi_\eta^{[q]})^2\rangle\leqq C_2(\gamma)\gamma^{2q}(\gamma^q|\xi-\eta|)^2e^{-R\gamma^q|\xi-\eta|},\\&\langle(\underline{\partial}\varphi_\xi^{[q]})^2\rangle\leqq C_3(\gamma)\gamma^{4q},\\&\langle(\underline{\partial}\varphi_\xi^{[q]}-\underline{\partial}\varphi_\eta^{[q]})^2\rangle\leqq C_4(\gamma)\gamma^{4q}(\gamma^q|\xi-\eta|)^2e^{-R\gamma^q|\xi-\eta|},\\&\langle(\varphi_\xi^{[q]}-\varphi_\eta^{[q]}-(\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta^{[q]})^2\rangle\leqq C_5(\gamma)\gamma^{2q}(\gamma^{2q}|\xi-\eta|^2)^2e^{-R\gamma^q|\xi-\eta|},\\&\langle(\varphi_\xi^{[q]}-\varphi_\eta^{[q]}-(\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta^{[q]}-\tfrac{1}{2}(\underline{\xi}-\underline{\eta})(\underline{\xi}-\underline{\eta})\cdot\underline{\underline{\partial}}^2\varphi_\eta^{[q]})^2\rangle\\&\quad\leqq C_6(\gamma)\gamma^{2q}(\gamma^{3q}|\xi-\eta|^3)^2e^{-R\gamma^q|\xi-\eta|},\end{aligned}\tag{3.1}$$

where $C_i(\gamma)$ are constants, depending on γ, which can be explicitly computed and R is >0. The probability that $\forall\xi,\eta\in\Lambda$, $\forall q\geqq 0$,

$$\begin{aligned}&|\varphi_\xi^{[\leqq q]}|\leqq Bq^2\gamma^q,\quad|\varphi_\xi^{[\leqq q]}-\varphi_\eta^{[\leqq q]}|\leqq Bq^2\gamma^q(\gamma^q|\xi-\eta|),\\&|\varphi_\xi^{[\leqq q]}-\varphi_\eta^{[\leqq q]}-(\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta^{[\leqq q]}|\leqq Bq^2\gamma^q(\gamma^q|\xi-\eta|)^2,\\&|\partial\varphi_\xi^{[\leqq q]}-\partial\varphi_\eta^{[\leqq q]}|\leqq Bq^2\gamma^{2q}(\gamma^q|\xi-\eta|),\\&|\varphi_\xi^{[\leqq q]}-\varphi_\eta^{[\leqq q]}-(\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta^{[\leqq q]}-\tfrac{1}{2}(\underline{\xi}-\underline{\eta})(\underline{\xi}-\underline{\eta})\cdot\underline{\underline{\partial}}^2\varphi_\eta^{[\leqq q]}|\\&\quad\leqq Bq^2\gamma^q(\gamma^q|\xi-\eta|)^3\end{aligned}\tag{3.2}$$

tends to 1 as $B\to\infty$ if Λ is kept fixed.

Therefore the free field will always be thought of as verifying inequalities (3.2) for some $B<+\infty$. This is a property which will not be used explicitly here but it is important to keep it in mind in order to understand the intuitive ideas behind the various techniques we introduce.

The proof of inequalities (3.2) goes back to Wiener; in this form it can be found in [10] and is a more or less straightforward consequence of the properties of the covariance of the field $\varphi_\xi^{[\leqq q]}$. The dependence on Λ of B is not important at this stage, but should be taken carefully into account in any constructive proof of the Φ_4^4-theory. The inequalities (3.2) play a key role in the actual construction of the P_∞ measure in the $d=2, 3$ cases. The reason why they seem to be irrelevant here is only due to the fact that we deal here only with formal perturbation theory and we do not really construct P_∞.

It is important to introduce some notations for the integrations with respect to the variables $\varphi^{[q]}$.

We denote

$$\begin{aligned}\mathscr{E}_{[q,p]}&\equiv\mathscr{E}_{(q,q+1,\dots,p)}\equiv\text{integration over } \varphi^{[q]},\varphi^{[q+1]},\dots,\varphi^{[q+p]},\\ \mathscr{E}^T_{[q,p]}&\equiv\mathscr{E}^T_{(q,q+1,\dots,p)}\equiv\text{truncated expectation over } \varphi^{[q]},\varphi^{[q+1]},\dots,\varphi^{[q+p]},\end{aligned}\tag{3.3}$$

where we recall that if $\mathscr{E}$ is an integration with respect to some variables the truncated expectations or "cumulants" are defined in general as follows: If $x_1, x_2, \dots, x_s$ are arbitrary random variables and $k_1, k_2, \dots, k_s$ are non-negative integers we set

$$\mathscr{E}^T(x_1,\dots,x_s;k_1,\dots,k_s)=\left[\frac{\partial^{k_1+\dots+k_s}}{\partial\theta_1^{k_1}\dots\partial\theta_s^{k_s}}\log\mathscr{E}\left(\exp\sum_{i=1}^{s}\theta_i x_i\right)\right]_{\theta_1=\dots=\theta_s=0}.\tag{3.4}$$

The truncated expectations verify a "Leibnitz formula"

$$\mathscr{E}^T\left(\sum_{i=1}^{p}\lambda_i x_i;k\right)=\sum_{\substack{k_1,\dots,k_p\\ k_1+\dots+k_p=k}}\frac{k!}{k_1!\dots k_p!}\lambda_1^{k_1}\dots\lambda_p^{k_p}\mathscr{E}^T(x_1,\dots,x_p;k_1,\dots,k_p).\tag{3.5}$$

Since (3.4) turns out to be linear combination of products of expectations of products of the x_i's, the notion of cumulant (3.4) makes sense even when $\exp\sum_{i=1}^{s}\theta_i x_i$ is not integrable provided $\mathscr{E}(|x_j|^q)<+\infty$, $\forall j$, $\forall q\geqq 1$. Then (3.4) can also be written formally as

$$\mathscr{E}(e^x)=\exp\sum_{k=1}^{\infty}\frac{\mathscr{E}^T(x;k)}{k!},\tag{3.6}$$

which gives a concrete way of building the formal power series expansion of the exponential of a random variable which is a formal power series in a parameter g with coefficients which are random variables with finite moments of any order.

We shall assume the reader is familiar with the notion of Wick ordering of products of gaussian variables: the basic property used over and over again is that given n gaussian random variables $x_1, \dots, x_n$ with covariance matrix C_{ij} the Wick ordered monomial of degree $(k_1, \dots, k_n)$ is a random variable denoted $:x_1^{k_1}\dots x_n^{k_n}:$ such that if $y_1, \dots, y_r$ are r gaussian variables

$$\mathscr{E}\left(:x_1^{k_1}\dots x_n^{k_n}:\prod_\alpha y_\alpha\right)=\sum_{\sigma\in\mathscr{S}}\prod_{\mu\in\sigma}C_\mu,\tag{3.7}$$

where $\mathscr{S}$ is the set of sets of pairs of indices (i, α), (α, α') such that in every element of $\sigma \in \mathscr{S}$ the index $i=1$ appears in k_1 pairs, the index $i=2$ in k_2 pairs, etc., and each index α appears in one pair ("set of all the possible pairings") and $C_\mu \equiv \langle x_i y_\alpha \rangle$ if $\mu=(i,\alpha)$ and $C_\mu = \langle y_\alpha y_\beta \rangle$ if $\mu=(\alpha,\beta)$.

Since (3.7) provides us with the value of the integral of every polynomial gaussian random variables times $:x_1^{k_1} \dots x_p^{k_p}$ it could be taken as definition of $:x_1^{k_1} \dots x_p^{k_p}:$. However the natural definition of $:x_1^{k_1} \dots x_p^{k_p}:$ is such that the following basic property stems naturally from it

$$\mathscr{E}^T(:x_{a_1^1}^{k_1^{(1)}} \dots x_{a_1^s}^{k_1^{(s)}}:, :x_{a_{n_1}^1}^{k_{n_1}^{(1)}} \dots x_{a_1^2}^{k_1^{(2)}}:, \dots, :x_{a_{n_2}^2}^{k_{n_2}^{(2)}} \dots x_{a_{n_s}^s}^{k_{n_s}^{(s)}}:) = \sum_G \prod_{\lambda \in G} C_\lambda, \tag{3.8}$$

where λ denotes a pair $(a_p^{(j)}, a_{p'}^{(j')})$ and

$$C_\lambda \equiv \mathscr{E}(x_{a_p^{(j)}} x_{a_{p'}^{(j')}}), \tag{3.9}$$

while G denotes a set of pairs λ such that no pair $\lambda \in G$ is built with indices $a_p^{(j)}, a_{p'}^{(j')}$ with $j=j'$, and furthermore given any pair j, j' one can find a sequence $j_1 \equiv j, j_2, j_3, \dots, j_m \equiv j'$ and pairs

$$\lambda_1 = (a_{\alpha_1}^{j_1}, a_{\alpha_2}^{j_2}), \quad \lambda_2 = (a_{\alpha_2}^{j_2}, a_{\alpha_3}^{j_3}), \dots, \lambda_{m-1} = (a_{\alpha_{m-1}}^{j_{m-1}}, a_{\alpha_m}^{j_m}),$$

which appear in G, i.e. G is a connected graph if the indices relative to the same Wick monomial are thought of as describing the same vertex.

The interest of (3.8) lies not only in its intrinsic elegance but also in the fact that it shows that the truncated expectation of Wick ordered monomials of gaussian variables contains very few terms compared to the number which one might naively expect from the general definition of truncated expectation.

4. Trees

In this paragraph we introduce some very useful definitions that will allow us to have a very general formalism to compute the formal power series for the non-renormalized effective potential. Draw on the (x, y) plane vertical lines at $x = -1, 0, 1, \dots, N+1$, the last one dotted. Given $k \geqq -1$ imagine all possible oriented trees starting with an horizontal line at $x = k$ having the other vertices on the vertical lines $x > k$ and no branches "coming back", i.e. forming an angle $\theta \geqq \frac{\pi}{2}$ or $\theta \leqq -\frac{\pi}{2}$ with the x-axis, and finally having terminal vertices lying on the $x = N+1$ dotted vertical line. We also suppose that at all vertices the branching is non-trivial except at the first and at the final vertices. These trees can be deformed by letting their vertices glide on the vertical lines on which they lie: trees obtained by such deformations will be considered equivalent and denoted generically by γ. The degree of γ will be the number $v(\gamma)$ of its endpoints. Given γ we consider its first "non-trivial" vertex, i.e. the vertex with the lowest abscissa larger than k. From it bifurcate s_0 trees $\gamma_1, \dots, \gamma_{s_0}$ of similar type (with a different k but with the same N); we divide the trees into q families of identical trees each containing $P_1, \dots, P_q$ copies of the same tree. We define for each tree a combinatorial weight inductively as

$$n(\gamma) = \prod_{1}^{q}{}_i P_i! \, n(\gamma_i)^{P_i}, \tag{4.1}$$

setting $n(\gamma)=1$ if γ is the "trivial tree", i.e. a tree with only one line (necessarily ending on $x=N+1$).

Denote $\Gamma(v,k)$ the set of the trees with degree v and "root" k (i.e. with the initial vertex on $x=k$). We imagine drawing the trees γ always in such a way that their lines do not overlap and choosing once and for all a representative tree from each equivalence class; we label by $1,2,\ldots,v$ the v terminal vertices of the tree. Each vertex of the tree γ bears a "frequency index" equal to the abscissa of the line on which it lies.

We fix an index α to label each vertex and we call k_α the frequency index of the vertex α. The label can be naturally defined to be $\alpha=0$ for the vertex after the root, $\alpha=00,01,\ldots,0q_0$ for the vertices arising from the first bifurcation, $\alpha=000,001,\ldots$ for the vertices arising from the bifurcation following the vertex 00, etc. The vertices on the dotted line $x=N+1$ will be labeled $1,2,\ldots$ from top to bottom. Figure 4.1 shows a possible tree.

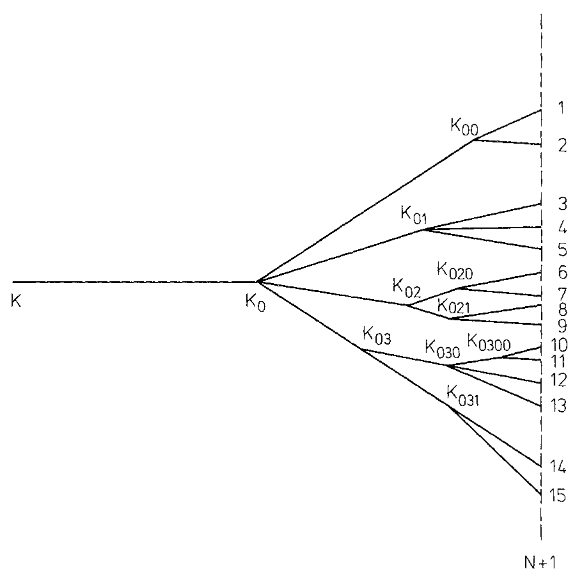


Fig. 4.1

5. Power Series for the Non-Renormalized Effective Potential

Let $\varphi\equiv\varphi^{[\leq N]}$, see (2.8), and let

$$V^{(N)}(\varphi)=-g\int_\Lambda :\varphi_\xi^4: d^4\xi\,. \tag{5.1}$$

Denoting $\mathscr{E}_{[n]}$ the expectation with respect to $\varphi^{[n]}$ we define $V^{(m)}$, $m<N$ by

$$\exp V^{(m)}(\varphi^{[\leq m]})\equiv\int \exp V^{(N)}(\varphi)P(d\varphi^{[m+1]})\ldots P(d\varphi^{[N]})\,. \tag{5.2}$$

It is easy to obtain $V^{(m)}$'s formal power series in g, starting from (3.6) and using (3.5).

In fact given $\gamma\in\Gamma(v,k)$ with frequencies k_0, k_{00}, $k_{001},\dots,k_{01}$, $k_{010},\dots,k_{0q_0}$, $k_{0q_00},\dots$, which are partially ordered as discussed in the previous section, we associate to it the following expression

$$V(\gamma)=\mathscr{E}_{[k+1,k_0-1]}\mathscr{E}^T_{[k_0]}(\mathscr{E}\mathscr{E}^T_{[k_{00}]}(\mathscr{E}\mathscr{E}^T_{[k_{000}]}(\dots),\mathscr{E}\mathscr{E}^T_{[k_{001}]}(\dots),\dots)\dots$$
$$\dots\mathscr{E}\mathscr{E}^T_{[k_{0q_0}]}(\mathscr{E}\mathscr{E}^T_{[k_{0q_00}]}(\dots),\dots),\dots), \tag{5.3}$$

where the $\mathscr{E}$ denote expectation with respect to all the $\varphi^{[h]}$ with h between the indices $k+1$ and $k'-1$, where k is the index of the $\mathscr{E}^T_{[\,]}$ following the first parentheses containing it and k' is the index of the one immediately former. For instance the first $\mathscr{E}$ in (5.3) denotes integration with respect to $\varphi^{[k_0+1]},\dots,\varphi^{[k_{00}-1]}$, the second concerns $\varphi^{[k_{00}+1]},\dots,\varphi^{[k_{000}-1]}$ etc.

As an example let's consider the tree γ in Fig. 5.1

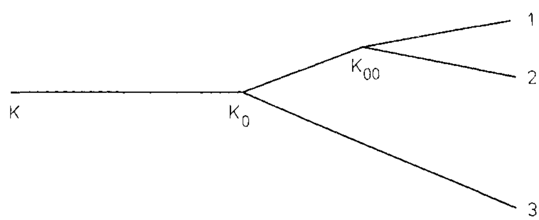


Fig. 5.1

To it we associate the factor

$$V(\gamma)=\mathscr{E}_{[k+1,k_0-1]}\mathscr{E}^T_{[k_0]}[\mathscr{E}_{[k_0+1]}\dots\mathscr{E}_{[k_{00}-1]}(\mathscr{E}^T_{[k_{00}]}(\mathscr{E}_{[k_{00}+1]}\dots\mathscr{E}_{[N]}V^{(N)},$$
$$\mathscr{E}_{[k_{00}+1]}\dots\mathscr{E}_{[N]}V^{(N)};1,1),\mathscr{E}_{[k_0+1]}\dots\mathscr{E}_{[N]}V^{(N)};1,1], \tag{5.4}$$

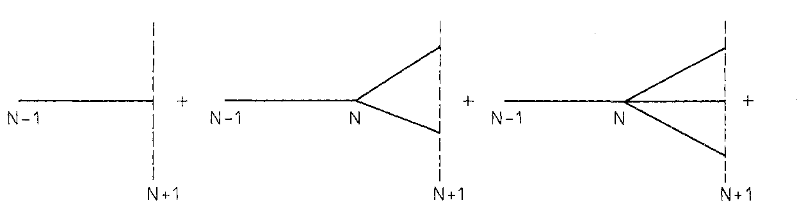
then the formalism is set up to write an explicit formula for $V^{(k)}(\varphi^{[\leqq k]})$,

$$V^{(k)}(\varphi^{[\leqq k]})=\sum_1^v{}_n\sum_{\gamma\in\Gamma(n,k)}\frac{V(\gamma)}{n(\gamma)}+O(g^{v+1}). \tag{5.5}$$

The proof of this formula is inductive on the index k. In fact (5.5) is obvious for $k=N-1$ being the same as (3.6),

$$V^{(N-1)}(\varphi^{[\leqq N-1]})=\sum_1^\infty{}_p\frac{\mathscr{E}^T_{[N]}(V^{(N)},p)}{p!}, \tag{5.6}$$

where the identity holds in the sense of the formal power series. Equation (5.6) can be regarded as a sum over the trees



Assuming the validity of (5.5) for $N,N-1,\dots,k+1$, it follows that

$$V^{(k)}(\varphi^{[\leqq k]})=\sum_1^v{}_p\frac{1}{p!}\mathscr{E}^T_{[k+1]}\left(\sum_1^v{}_n\sum_{\gamma\in\Gamma(n,k+1)}\frac{V(\gamma)}{n(\gamma)},p\right)+O(g^{v+1}), \tag{5.8}$$

and we observe that the expectation written in the right-hand side can be developed by (3.5) as

$$V^{(k)}=\left(\sum_1^v{}_p \sum_{s_{\gamma_1},\ldots,s_{\gamma_p}} \sum_{\substack{\gamma_1,\ldots,\gamma_p\\ \Sigma_j v(\gamma_j)s_{\gamma_j}\le v}} \frac{1}{s_{\gamma_1}!\ldots s_{\gamma_p}!} \frac{\mathscr{E}^T_{[k+1]}(V(\gamma_1),\ldots,V(\gamma_p),s_{\gamma_1},\ldots,s_{\gamma_p})}{n(\gamma_1)^{s_{\gamma_1}}\ldots n(\gamma_p)^{s_{\gamma_p}}}\right)$$

$$=\sum_1^v{}_p \sum_{\gamma\in\Gamma(p,k)} \frac{V(\gamma)}{n(\gamma)} \tag{5.9}$$

as the generic expectation written in the intermediate term of (5.9) is just $V(\gamma)/n(\gamma)$, where γ is the tree obtained by joining in a new vertex s_{γ_1} copies of $\gamma_1,\ldots,s_{\gamma_p}$ copies of γ_p and adding an extra horizontal line which ends with a vertex of frequency k ("the root of the tree").

6. Counterterms

We now discuss the algorithm allowing us to construct the series (2.16) or, in other words, the series (2.24). As it is well known, the idea is to modify $V^{(N)}(\varphi^{[\le N]})$ [see (5.1)] as

$$V_R^{(N)}=V^{(N)}+\Delta_2^{(N)}V+\Delta_3^{(N)}V+\Delta_4^{(N)}V+\ldots, \tag{6.1}$$

where $\Delta_j^{(N)}V$ is of order j in g and has the form

$$\Delta_j^{(N)}V=g^j\Sigma_\beta \int r_j(N,\beta):P_\beta(\varphi_\xi^{[\le N]}):d\xi \tag{6.2}$$

(Hereafter $d\xi$ will always mean $d^4\xi$), $\beta=0,2,2',4$ and

$$P_0(\varphi)=1\,,\qquad P_2(\varphi)=\varphi^2\,,\qquad P_{2'}(\varphi)=(\underline{\partial}\varphi)^2\,,\qquad P_4(\varphi)=\varphi^4\,. \tag{6.3}$$

The choice of the counterterms as well as their effects and the possibility of obtaining bounds on the effective potential [see inequalities (2.27)] become very natural and easy to understand only after one realizes the detailed and delicate combinatorial mechanism allowing us to express concisely the effective potential $V_R^{(k)}$, after $V^{(N)}$ is modified as in (6.1).

Of course (5.4) (5.5) are very general, and they hold unchanged if $V^{(N)}$ is replaced by (6.1); if we now consider $V^{(N)}+\Delta_2^{(N)}V$, and we develop a formula like (5.5) for $V_2^{(k)}$ defined by

$$\exp V_2^{(k)}(\varphi^{[\le k]})=\int P(d\varphi^{[k+1]})\ldots P(d\varphi^{[N]})\exp(V^{(N)}+\Delta_2^{(N)}V)\,, \tag{6.4}$$

we see that

$$V_2^{(k)}(\varphi^{[\le k]})=\sum_{\substack{\gamma\\ k(\gamma)=k}} \frac{V_2(\gamma)}{n(\gamma)}\,, \tag{6.5}$$

where $k(\gamma)$ is the root of γ (the lowest frequency of the tree).

The sum runs over the trees of the same type as those used so far but with some of the terminal lines marked in some way, say with a little empty frame, ———○, recalling which term, in the expansion of (5.4), after substituting $V^{(N)}$ by $V^{(N)}+\Delta_2^{(N)}V$, and expanding the powers of $(V^{(N)}+\Delta_2^{(N)}V)$, is selected.

So typical trees of the new type will be like:

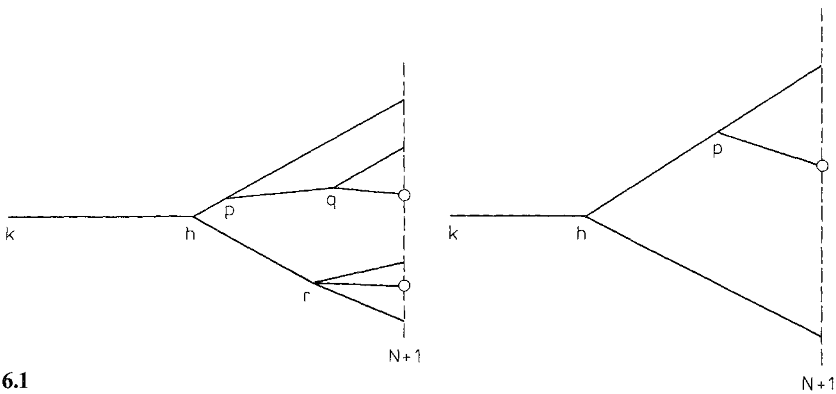


Fig. 6.1

For instance the second tree corresponds to

$$V(\gamma)=\mathscr{E}_{[k+1,N]}\mathscr{E}^T_{[h]}(\mathscr{E}\mathscr{E}^T_{[p]}(\mathscr{E}V^{(N)},\mathscr{E}\Delta_2^{(N)}V;1,1),\mathscr{E}V^{(N)},1,1)\,. \tag{6.6}$$

Note that the tree below, Fig. 6.2, would produce a term equal to (6.6 with $V^{(N)}$ and $\Delta_2^{(N)}V$ in the inner parenthesis interchanged.

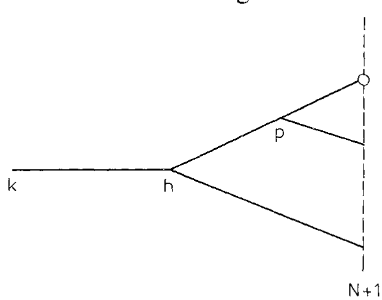


Fig. 6.2

On the other hand since the trees γ considered in Sect. 5 were always drawn in a standard way selecting one representative per equivalence class, it is clear that the second tree in Fig. 6.1 and that of Fig. 6.2 would have to be considered different. It is convenient, however, to keep consistent the convention that two trees, even if the new type, should be considered equivalent if they can be superimposed after letting their vertices glide, suitably, on the vertical lines on which they lie. This can be easily done provided the combinatorial factor $n(\gamma)$ is now defined exactly as before, taking into account the possible presence of little frames at the ends of the top lines [in this way the (combinatorial factor) of the second tree of Fig. 6.1 will be twice that of the unlabeled tree (Fig. 5.2), but this tree will be considered equivalent to the tree in Fig. 6.2 so that no counting error is made].

The formal degree of a tree γ with d decorated tops associated to $\Delta_2^{(N)}V$ is $|\gamma|+d$, where $|\gamma|$ is the number of final branches of γ; to keep track of the power counting we shall draw inside the little balls at the end of the tree two lines like ———⊜ so that the degree of the tree : $v(\gamma)$ can be read out of the tree itself by looking at how many end points there are in it, including those of the lines in frames.

It is now clear how to describe graphically $V_R^{(k)}(\varphi^{[\le k]})$ defined by

$$\exp V_R^{(k)}(\varphi^{[\le k]}) = \int P(d\varphi^{[k+1]}) \dots P(d\varphi^{[N]}) \exp(V^{(N)} + \Delta_2^{(N)} V + \Delta_3^{(N)} V + \dots) . \quad (6.7)$$

In fact one can use more general decorated trees in which at the end of a top line a frame appears with n lines in it,

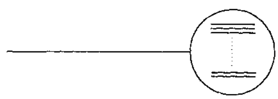


Fig. 6.3

corresponding to the term $\Delta_n^{(N)} V$ of the interaction (6.1).

An example of decorated tree is

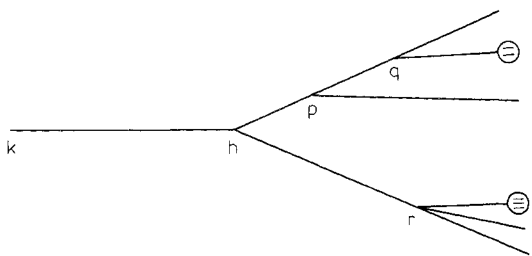


Fig. 6.4

which symbolizes the expression

$$\mathscr{E}_{[>k]} \mathscr{E}^T_{[h]} (\mathscr{E}(\mathscr{E}^T_{[p]}(\mathscr{E}\mathscr{E}^T_{[q]}(V; \Delta_2^{(n)} V); \mathscr{E} V)); \mathscr{E}\mathscr{E}^T_{[r]}(\mathscr{E}\Delta_3^{(N)} V; \mathscr{E} V; \mathscr{E} V)) , \quad (6.8)$$

where, to shorten the notations, we used $V = V^{(N)}$ and

$$\mathscr{E}^T(X, Y, Z) \equiv \mathscr{E}^T(X, Y, Z; 1, 1, 1) .$$

The combinatorial factor $n(\gamma)$ associated with a given tree γ with several decorated tops will be computed with the same rules used so far. With the above conventions we can represent the result of the integral (6.7) exactly with the same formula as before but with the new meaning of the trees

$$V_R^{(k)}(\varphi^{[\le k]}) = \sum_{\substack{\gamma \\ k(\gamma) = k}} \frac{V(\gamma)}{n(\gamma)} , \quad (6.9)$$

which also provides the formal series of $V_R^{(k)}(\varphi^{[\le k]})$ in powers of g, easily. In fact the terms of order g^n are exactly obtained by collecting together all the trees with n final lines, including those which appear inside the decorations.

It remains to choose appropriately the counterterms $\Delta_j^{(N)} V$; their choice will be dictated by the requirement that the sum of the terms of $V_R^{(k)}(\varphi^{[\le k]})$, of a given order in g, for k fixed, is "finite" as $N \to \infty$ (see Proposition 2 of Sect. 2). As usual in perturbation theory, this choice is done inductively: given $\Delta_2^{(N)} V, \Delta_3^{(N)} V, \dots, \Delta_m^{(N)} V$ one will define $\Delta_{m+1}^{(N)} V$ by requiring that the contribution to $V_R^{(k)}$ of order $(m+1)$ in g is "ultraviolet finite" i.e. does not diverge as $N \to \infty$. The first task is to define $\Delta_2^{(N)} V$. This is done by requiring that $V_R^{(k)}$ be ultraviolet finite to the order g^2. $V_R^{(k)}$ is given, up to second order in g, by the sum of the contributions associated with the following trees

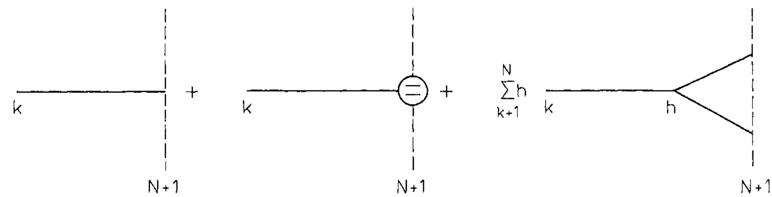


Fig. 6.5

or, in formulae, if $\varphi \equiv \varphi^{[\leq N]}$, $\tilde{\varphi} \equiv \varphi^{[\leq K]}$

$$-g \int :\tilde{\varphi}_\xi^4: d\xi + \mathscr{E}_{[>k]}(\Delta_2^{(N)} V) + \sum_{k+1}^{N} {}_h \frac{g^2}{2} \int \mathscr{E}_{[>k]} \mathscr{E}^T_{[h]}(\mathscr{E}(:\varphi_\xi^4:), \mathscr{E}(:\varphi_\eta^4:)) d\xi\, d\eta\,. \tag{6.10}$$

The first term is obviously ultraviolet finite; we rewrite the other ones as

$$\begin{aligned}\frac{g^2}{2} \sum_{k+1}^{N} {}_h \Bigg\{ & \binom{4}{1}^2 1! \int d\xi\, d\eta (C_{\xi\eta}^{(h)} - C_{\xi\eta}^{(h-1)}) : \tilde{\varphi}_\xi^3 \tilde{\varphi}_\eta^3 : \\ & + \binom{4}{2}^2 2! \int d\xi\, d\eta (C_{\xi\eta}^{(h)2} - C_{\xi\eta}^{(h-1)2}) : \tilde{\varphi}_\xi^2 \tilde{\varphi}_\eta^2 : \\ & + \binom{4}{3}^2 3! \int d\xi\, d\eta\, (C_{\xi\eta}^{(h)3} - C_{\xi\eta}^{(h-1)3}) : \tilde{\varphi}_\xi \tilde{\varphi}_\eta : \\ & + \binom{4}{4}^2 4! \int d\xi\, d\eta (C_{\xi\eta}^{(h)4} - C_{\xi\eta}^{(h-1)4}) \Bigg\} + \mathscr{E}_{[>k]}(\Delta_2^{(N)} V)\,,\end{aligned}$$

after summing over h,

$$\begin{aligned} = & \frac{g^2}{2}\binom{4}{1}^2 1! \int d\xi\, d\eta (C_{\xi\eta}^{(N)} - C_{\xi\eta}^{(k)}) : \tilde{\varphi}_\xi^3 \tilde{\varphi}_\eta^3 : \\ & + \frac{g^2}{2}\binom{4}{2}^2 2! \int d\xi\, d\eta (C_{\xi\eta}^{(N)2} - C_{\xi\eta}^{(k)2}) : \tilde{\varphi}_\xi^2 \tilde{\varphi}_\eta^2 : \\ & + \frac{g^2}{2}\binom{4}{3}^2 3! \int d\xi\, d\eta (C_{\xi\eta}^{(N)3} - C_{\xi\eta}^{(k)3}) : \tilde{\varphi}_\xi \tilde{\varphi}_\eta : \\ & + \frac{g^2}{2}\binom{4}{4}^2 4! \int d\xi\, d\eta (C_{\xi\eta}^{(N)4} - C_{\xi\eta}^{(k)4}) + \mathscr{E}_{[>k]}(\Delta_2^{(N)} V)\,, \end{aligned} \tag{6.11}$$

where $C_{\xi\eta}^{(h)}$ is the covariance of the random field $\varphi^{[\leq h]}$.

The combinatorial factors are not developed explicitly to help the reader to recognize their origin (from the Wick ordering computational rules). Clearly the second, third and fourth integrals are divergent with N logarithmically, quadratically and quartically respectively even if $\tilde{\varphi}_\xi$ is treated as a smooth field verifying, for some B, the inequalities (3.2). Inspection of the divergent terms of (6.11) shows clearly that to eliminate these divergences (let's concentrate mainly on the divergent parts which are field dependent, as the constant one is easy to cure) we would need for the logarithmic divergent part a zero in $(\xi - \eta)$ of order $\varepsilon > 0$ and for the quadratic divergent part a zero in $(\xi-\eta)$ of order >2. There is a simple choice of $\Delta_2^{(N)}V$ which produces exactly the needed zeroes and make the above expressions ultraviolet finite.

This choice stems out from the following trivial relationships,

$$\varphi_\xi^2\varphi_\eta^2-\varphi_\xi^4=\varphi_\xi^3D_{\eta\xi}+\varphi_\xi^2\varphi_\eta D_{\eta\xi},$$
$$\varphi_\xi\varphi_\eta-\tfrac{1}{2}(\varphi_\xi^2+\varphi_\eta^2)+\tfrac{1}{2}((\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta)^2=\tfrac{1}{2}S_{\xi\eta}^2-S_{\xi\eta}D_{\xi\eta}, \tag{6.12}$$

where

$$D_{\xi\eta}\equiv\varphi_\xi-\varphi_\eta,\qquad S_{\xi\eta}\equiv(\varphi_\xi-\varphi_\eta)-(\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta. \tag{6.13}$$

In fact these identities tell us which has to be $\Delta_2^{(N)}V$ to transform the Wick monomials of the dangerous terms of (6.11) in the right-hand-side terms of (6.12) which have the needed zeroes in $(\xi-\eta)$. Therefore we define

$$\begin{aligned}\Delta_2^{(N)}V=&-\frac{g^2}{2}\binom{4}{2}^2 2!\int d\xi\,d\eta C_{\xi\eta}^{(N)2}:\varphi_\xi^4:\\&-\frac{g^2}{2}\binom{4}{3}^2 3!\int d\xi\,d\eta C_{\xi\eta}^{(N)3}\left[:\varphi_\eta^2:-\frac{1}{2}:((\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta)^2:\right]\\&-\frac{g^2}{2}\binom{4}{4}^2 4!\int d\xi\,d\eta C_{\xi\eta}^{(N)4},\end{aligned} \tag{6.14}$$

which is useful to rewrite as

$$\begin{aligned}\Delta_2^{(N)}V=&\left[-\frac{g^2}{2}\binom{4}{2}^2 2!\int d\xi\,d\eta(C_{\xi\eta}^{(N)2}-C_{\xi\eta}^{(k)2}):\varphi_\xi^4:\right.\\&-\frac{g^2}{2}\binom{4}{3}^2 3!\int d\xi\,d\eta(C_{\xi\eta}^{(N)3}-C_{\xi\eta}^{(k)3})\left[:\varphi_\xi^2:-\frac{1}{2}:((\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta)^2:\right]\\&\left.-\frac{g^2}{2}\binom{4}{4}^2 4!\int d\xi\,d\eta(C_{\xi\eta}^{(N)4}-C_{\xi\eta}^{(k)4})\right]\\&+\left\{-\frac{g^2}{2}\binom{4}{2}^2 2!\int d\xi\,d\eta C_{\xi\eta}^{(k)2}:\varphi_\xi^4:\ldots\right\}.\end{aligned} \tag{6.15}$$

With this choice of $\Delta_2^{(N)}V$ the $O(g^2)$ part of the "effective potential" $V_R^{(k)}$ becomes

$$\begin{aligned}V_{R,(g^2)}^{(k)}=&\frac{g^2}{2}\binom{4}{1}^2 1!\int d\xi\,d\eta(C_{\xi\eta}^{(N)}-C_{\xi\eta}^{(k)}):\tilde\varphi_\xi^3\tilde\varphi_\eta^3:\\&+\frac{g^2}{2}\binom{4}{2}^2 2!\int d\xi\,d\eta(C_{\xi\eta}^{(N)2}-C_{\xi\eta}^{(k)2}):\tilde\varphi_\xi^3\tilde D_{\eta\xi}+\tilde\varphi_\xi^2\tilde\varphi_\eta\tilde D_{\eta\xi}:\\&+\frac{g^2}{2}\binom{4}{3}^2 3!\int d\xi\,d\eta(C_{\xi\eta}^{(N)3}-C_{\xi\eta}^{(k)3}):\frac{1}{2}\tilde S_{\xi\eta}^2-\tilde S_{\xi\eta}\tilde D_{\xi\eta}:\\&+\left\{-\frac{g^2}{2}\binom{4}{2}^2 2!\int d\xi\,d\eta C_{\xi\eta}^{(k)2}:\tilde\varphi_\xi^4:\ldots\right\}\end{aligned} \tag{6.16}$$

with obvious notations.

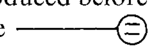
Let us describe a way of constructing $\Delta_2^{(N)}V$ which will have a natural generalization for the higher orders in g: let us introduce a fictitious field of frequency $-1: \varphi_\xi^{[-1]}$ and compute the order g^2 of $V^{(-1)}$ (starting from the potential $V^{(N)}$, the result being (6.11) without $\mathscr{E}_{[>-1]}(\Delta_2^{(N)}V)$, and with $\tilde{\varphi}=\varphi^{[-1]}$. Expand the result in Wick monomials and select those of order 0, 2, 4 in the fields, change sign of them and localize their expressions, i.e. replace the constants by themselves and

$$\begin{aligned} &:\tilde{\varphi}_\xi^2\tilde{\varphi}_\eta^2: \text{ by } \mathscr{L}(:\tilde{\varphi}_\xi^2\tilde{\varphi}_\eta^2:)=:\tilde{\varphi}_\xi^4: \\ &:\tilde{\varphi}_\xi\tilde{\varphi}_\eta: \text{ by } \mathscr{L}(:\tilde{\varphi}_\xi\tilde{\varphi}_\eta:)=\tfrac{1}{2}(:\tilde{\varphi}_\xi^2:+:\tilde{\varphi}_\eta^2)-\tfrac{1}{2}:((\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta)^2:, \end{aligned} \tag{6.17}$$

if $\xi \prec \eta$ and with ξ and η interchanged if $\eta \prec \xi$[1], then substitute everywhere $\tilde{\varphi}=\varphi^{[-1]}$ with $\varphi^{[\leq N]}$.

Before going to higher orders in g let's elaborate a graphical rule for the evaluation of $V_2^{(k)}$ as defined in (6.4). As previously discussed, see (6.5),

$$V_2^{(k)}=\sum_{\substack{\gamma \\ k(\gamma)=k}}\frac{V(\gamma)}{n(\gamma)},$$

where the sum runs over the decorated trees which are the trees introduced before with some of the terminal lines, possibly none, of the following type  (see, for instance, Fig. 6.1). It is easy to realize that to a final line

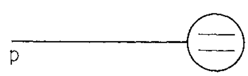


Fig. 6.6

corresponds, in the truncated expectation, the term

$$\begin{aligned} \mathscr{E}_{[>p]}\Delta_2^{(N)}V= &-\frac{g^2}{2}\binom{4}{2}^2 2!\sum_{p+1}^{N}{}_h\int d\xi\, d\eta(C_{\xi\eta}^{(h)2}-C_{\xi\eta}^{(h-1)2}):\varphi_\xi^{[\leq p]4}: \\ &-\frac{g^2}{2}\binom{4}{3}^2 3!\sum_{p+1}^{N}{}_h\int d\xi\, d\eta(C_{\xi\eta}^{(h)3}-C_{\xi\eta}^{(h-1)3}) \\ &\cdot\left[:\varphi_\xi^{[\leq p]2}:-\frac{1}{2}:((\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta^{[\leq p]})^2:\right] \\ &-\frac{g^2}{2}\binom{4}{4}^2 4!\sum_{p+1}^{N}{}_h\, d\xi\, d\eta(C_{\xi\eta}^{(h)4}-C_{\xi\eta}^{(h-1)4}) \\ &+\left\{-\frac{g^2}{2}\binom{4}{2}^2 2!\sum_{0}^{p}{}_h\int d\xi\, d\eta(C_{\xi\eta}^{(h)2}-C_{\xi\eta}^{(h-1)2}):\varphi_\xi^{[\leq p]4}:\dots\right\}. \end{aligned} \tag{6.18}$$

Let us call $\Delta_{2,\beta}^{(p)}$, $\beta=0,2,2',4$ the four monomials in the { } of (6.18) defined as in (6.3). The contribution of each of these four terms will be indicated, graphically, in the following way

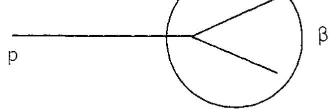


Fig. 6.7

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1 Where the ordering "$\prec$" will be defined later

Let's consider now the first sums of (6.18)

$$\sum_{p+1}^{N}{}_{j}\left[-\frac{g^2}{2}\binom{4}{2}^2 2!\int d\xi\, d\eta (C_{\xi\eta}^{(j)2}-C_{\xi\eta}^{(j-1)2}) : \varphi_\xi^{[\le p]4} : \dots\right]. \tag{6.19}$$

For each j the associated term of the sum can be put in a one to one correspondence to the contribution to $V(\gamma)$ from the subtree

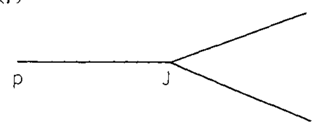


Fig. 6.8

where the lines which bifurcate from j are final branches.

Therefore collecting together the contribution associated to the subtree of Fig. 6.8 and the term of index j of (6.19) in any tree γ for any p and j we have the following result: each contribution $V(\gamma)$ to $V_2^{(k)}$, where γ has some final bifurcations as in Fig. 6.8, for some p and j is similar to the previous one for $V^{(k)}$, with the only difference that in the truncated expectation the bifurcation of the type of Fig. 6.8 corresponds now to a sum of Wick monomials which is modified according to the following rules:

$$\begin{aligned} &:\tilde{P}: \Rightarrow :\tilde{P}: \text{ if } \tilde{P} \text{ is a monomial of order } >4, \\ &:\tilde{\varphi}_\xi\tilde{\varphi}_\eta: \Rightarrow \mathscr{R}(:\tilde{\varphi}_\xi\tilde{\varphi}_\eta:) = :\tfrac{1}{2}\tilde{S}_{\xi\eta}^2 - \tilde{S}_{\xi\eta}\tilde{D}_{\xi\eta}:, \\ &:\tilde{\varphi}_\xi^2\tilde{\varphi}_\eta^2: \Rightarrow \mathscr{R}(:\tilde{\varphi}_\xi^2\tilde{\varphi}_\eta^2:) = :\tilde{\varphi}_\xi^3\tilde{D}_{\xi\eta} + \tilde{\varphi}_\xi^2\tilde{\varphi}_\eta\tilde{D}_{\xi\eta}:. \end{aligned} \tag{6.20}$$

if $\xi \prec \eta$, otherwise with ξ and η interchanged. To remember this an index R will be appended to these bifurcations.

This prescription, together with the introduction of framed parts (see Fig. 6.7), tells us that the contribution of order n to $V_2^{(k)}$ will be described by all the undecorated trees with n final lines but with a letter R hung to each of vertices of the tree which bifurcate in two final lines, plus the trees obtained by a dressing operation. The dressing operation consists in drawing a frame around some (possibly none or all) of the branches of the tree which arise at a final bifurcation in two lines, then erasing the frequency index of that vertex while writing near it an index $\beta = 0, 2, 2', 4$, which selects the term in the { } of (6.18).

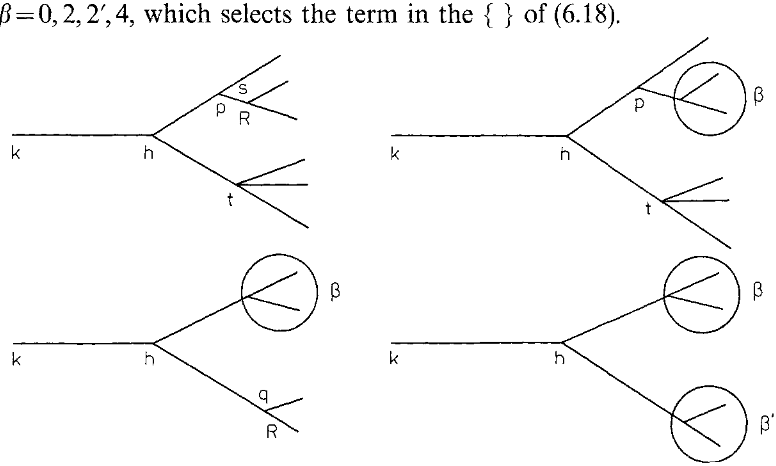


Fig. 6.9

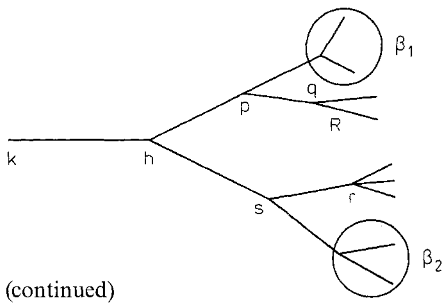


Fig. 6.9 (continued)

"dressed" trees to second order

A tree, dressed to second order, will look like those of Fig. 6.9, the meaning to attribute to it should be clear from the above discussion; the combinatorial factor to be given to each dressed tree is computed with the usual rule, taking into account all the various labels and frames to recognize identical trees or subtrees.

It is now easy to build $\Delta_3^{(N)}V$; we discuss it now in detail as it will turn out that this case has nearly all the features of the general one. Let's examine the various steps to construct it:

1) The order g^3 of $V_2^{(k)}$ [see (6.4)] corresponds to the sum of the contributions of the following dressed trees:

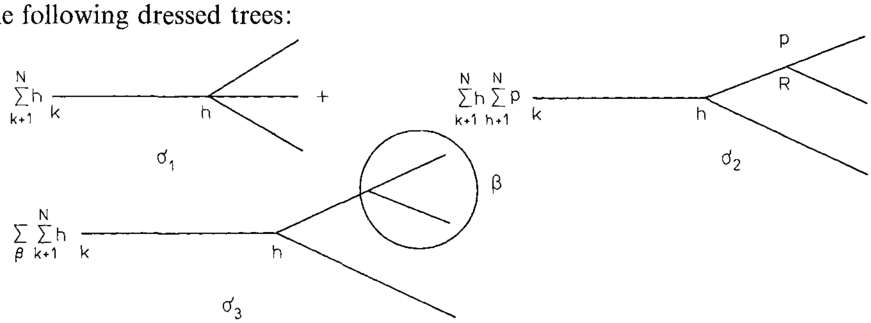


Fig. 6.10

where with σ_i, $i: 1, 2, 3$ we indicate the "shape" of the trees irrespective of their frequencies. Therefore let's compute the order g^3 of $V_2^{(k)}$ for $k = -1$.

2) Isolate from these contributions the monomials of order 0, 2, 4, change their sign and "localize" each one of them with an operation $\mathscr{L}$ [see Eq. (6.17)] that we are going to define.

3) Substitute the "fictitious" fields $\tilde{\varphi} = \varphi^{[-1]}$ with $\varphi^{[\leq N]}$.

4) Let γ be a dressed tree (with well defined frequencies) of shape σ_1 or σ_2 or σ_3; let's call $\Delta_3^{(N)}V(\gamma)$ the expression obtained applying the rules 1), 2), 3) to that part of $V_2^{(k)}$ associated to γ. Then let us define

$$\Delta_3^{(N)}V = \Sigma_i \frac{\Delta_3^{(N)}V(\sigma_i)}{n(\sigma_i)},$$

where

$$\frac{\Delta_3^{(N)}V(\sigma_i)}{n(\sigma_i)} = \sum_{\substack{\sigma(\gamma)=\sigma_i \\ k(\gamma)=-1}} \frac{\Delta_3^{(N)}V(\gamma)}{n(\gamma)}, \tag{6.21}$$

and $\sum_{\sigma(\gamma)=\sigma_i} {}_\gamma$ means the sum over all the trees with different frequencies but the same shape, and $n(\sigma)$ is computed exactly in the same way as $n(\gamma)$ with the prescription that two trees have to be considered different if and only if their shapes are different.

Exactly as it has been done for $V_2^{(k)}$, the contribution from $\Delta_3^{(N)}V$ can be split in two parts, one giving rise to new framed parts, the other producing an index R to the lower vertex of the subtrees of shape $\sigma_i, \sigma_2, \sigma_3$ which are not framed.

Therefore the graphical rules to represent $V_3^{(k)}$ to a given order in g are the following:

($V_3^{(k)}$ is defined in the same way as $V_2^{(k)}$ namely [see Eq. (6.4)])

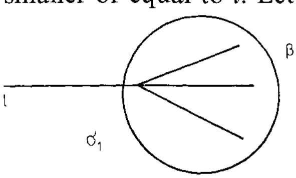
$$\exp V_3^{(k)}(\varphi^{[2k]}) = \int \exp(V^{(N)} + \Delta_2^{(N)}V + \Delta_3^{(N)}V) P(d\varphi^{[k+1]}) \dots P(d\varphi^{[N]}). \qquad (6.22)$$

One just has to consider all the dressed trees built in the analysis of $V_2^{(k)}$, then look at all the final branchings of these trees which have one of the shapes σ_i of Fig. 6.10, surround some of them by frames while writing an index R at the lower vertex of the others; one does it in all the possible ways obtaining the dressed trees to the third order. One erases all the frequency indices inside the new frames and introduces in it a new β index which again can assume the values $0, 2, 2', 4$.

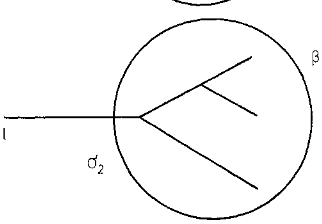
Once the family of all the dressed trees to third order are defined we have to specify the computational prescription associated with them. For the subtrees of order 3 enclosed in frames their truncated expectation has to be substituted by

$$\frac{1}{n(\sigma)} \Delta_{3,\beta}^{(l)}(\sigma; N) = \sum_{\substack{\sigma(\gamma)=\sigma \\ h(\gamma)\leq l \\ k(\gamma)=-1}} {}_\gamma\, \mathscr{E}_{[>l]} \frac{\Delta_3^{(N)} V(\gamma)}{n(\gamma)}, \qquad (6.23)$$

where σ is the shape of this subtree and the index β tells us which monomial one has to select. $\sigma(\gamma)=\sigma$ means that the sum is over all those subtrees with the same shape σ, $h(\gamma) \leq l$ that the frequency of the last bifurcation of these subtrees has to be smaller or equal to l. Let's give a more explicit expression of these terms

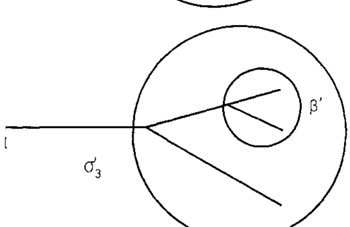


$$\equiv \frac{\Delta_{3,\beta}^{(l)}(\sigma_1; N)}{n(\sigma_1)} = \frac{1}{n(\sigma_1)} \sum_0^l {}_h\, \mathscr{E}_{[>l]} \mathscr{L} \mathscr{E}_{[h]}^T(\mathscr{E}V; 3)|_\beta,$$



$$\equiv \frac{\Delta_{3,\beta}^{(l)}(\sigma_2; N)}{n(\sigma_2)} = \frac{1}{n(\sigma_2)} \sum_0^l {}_h \sum_{h+1}^N {}_q\, \mathscr{E}_{[>l]} \mathscr{L} \mathscr{E}_{[h]}^T$$

$$\cdot (\mathscr{E}V, \mathscr{E}\mathscr{R}\mathscr{E}_{[q]}^T(\mathscr{E}V; 2); 1, 1)|_\beta, \qquad (6.24)$$



$$\beta \equiv \frac{\Delta_{3,\beta}^{(l)}(\sigma_3, N)}{n(\sigma_3)} = \frac{1}{n(\sigma_3)} \sum_0^l {}_h \sum_0^h {}_q\, \mathscr{E}_{[>l]} \mathscr{L} \mathscr{E}_{[h]}^T$$

$$\cdot (\mathscr{E}V, \mathscr{E}\mathscr{L}\mathscr{E}_{[q]}^T(\mathscr{E}(V,2)|_{\beta'}; 1; 1)|_\beta,$$

where the way in which $\mathscr{L}$ operates over all the Wick monomials which can be produced at order g^3 has still to be defined. The index β tells us which monomial one has to select among the ones produced by $\mathscr{L}$, and the N is to remember that $\Delta^{(l)}_{3,\beta}(\sigma; N)$ has still an N-dependence but is uniformly bounded as $N\to\infty$. The $\mathscr{R}$ appearing in the second line of (6.24) has been previously defined by Eqs. (6.20) as it refers to a bifurcation of order g^2.

We have now to specify the $\mathscr{L}$ operation and simultaneously to give a meaning to the index R appended to the vertices from which unframed subtrees with shape $\sigma_1, \sigma_2, \sigma_3$ start. This index will correspond again [see Eq. (6.20) for the order g^2] to a well defined operation we still call $\mathscr{R}$ to be applied to the Wick monomials associated to these order g^3 subtrees.

The $\mathscr{R}$ and $\mathscr{L}$ Operations

The $\mathscr{L}$ and $\mathscr{R}$ operations discussed in dealing with the order g^2 of $V^{(k)}$ were produced by the presence of the appropriate counterterms and had the effect of making the order g^2 of $V^{(k)}$ finite in the limit $N\to\infty$. The same has to happen at all the higher orders. In fact the counterterms are constructed once we have defined the appropriate $\mathscr{L}$ and $\mathscr{R}$ operations. To do that we have to realize from general arguments which Wick monomials in the expansion of $V^{(k)}$ have coefficients diverging as $N\to\infty$.

Therefore we discuss the general ideas behind all that and the general prescriptions for these operations. The rigorous proof that with these prescriptions everything really works will be given in the next section.

$V^{(N)}$ is a adimensional quantity: $[g]=l^0$, $[\varphi^{[\leqq N]}]=l^{-1}$; therefore each term of any order in the formal series defining $V^{(k)}$ has still to be of dimension zero. Nevertheless what really matters is the dimension of the Wick monomial with respect to a certain scale. Let's give the following

Definition. Let $F(\varphi^{[\leqq k]})$ be a function of the field $\varphi^{[\leqq k]}$, we'll call "k-dimension" of $F(\varphi^{[\leqq k]})$, and we denote $d_k(F)$ the coefficient α appearing in the following expression

$$\langle (F(\varphi^{[\leqq k]}))^2\rangle \sim \gamma^{2k\alpha}.^2$$

For instance $d_k(\varphi^{[\leqq k]})=1$, $d_k(\underline{\partial}\varphi^{[\leqq k]})=2$. In $V^{(k)}$ the terms of order g^2 have the following dimensional structure [see Eq. (6.11)], written symbolically,

$$V^{(k)}_{(g^2)}=\sum_{\beta} V^{(k)}_{(g^2),\beta}\sim |\Lambda| \sum_{\beta}\sum_{k+1}^{N}{}_h F^{(h)}_{\beta} :(\varphi^{[\leqq k]})^{\beta}: . \tag{6.25}$$

β is the order of the Wick monomial. The (length) dimension of $F^{(h)}$ must be $4-\alpha$, where α satisfies

$$\alpha+\beta=4\cdot 2=8 . \tag{6.26}$$

---

2 If $F(\varphi^{[\leqq k]})=\varphi^{[\leqq k]}_{\xi}-\varphi^{[\leqq k]}_{\eta}$, we have

$$\langle (F(\varphi^{[\leqq k]}))^2\rangle=\langle (\varphi^{[\leqq k]}_{\xi}-\varphi^{[\leqq k]}_{\eta})^2\rangle \sim \gamma^{2k}|\xi-\eta|^2 ,$$

and therefore $d_k(\varphi^{[\leqq k]}_{\xi}-\varphi^{[\leqq k]}_{\eta})=2$

The only (length) frequency dependent dimensional quantity in $F_\beta^{(h)}$ is γ^{-h}, and in $:(\varphi^{[\leqq k]})^\beta:$ is γ^{-k}. Therefore each term of (6.25) can be estimated in the following way:

$$|V^{(k)}_{(g^2),\beta}| \sim |\Lambda| \sum_{k+1}^{N}{}_h (\gamma^h)^{\alpha-4}(\gamma^k)^\beta . \tag{6.27}$$

In general the terms in $V^{(k)}$ of order g^n can be written as (6.25) but now the dimension of $F_\beta^{(h)}$ is [see the proof of Theorem 1 in Appendix A of (II)]

$$[F_\beta^{(h)}] = 4(n-1) - \alpha , \tag{6.28}$$

where α satisfies

$$\alpha + \beta = 4n , \tag{6.29}$$

and the analogue of (6.27) is

$$|V^{(k)}_{(g^n),\beta}| \sim |\Lambda| \sum_{k+1}^{N}{}_h (\gamma^h)^{\alpha-4(n-1)}(\gamma^k)^\beta . \tag{6.30}$$

It is clear that, if $\beta > 4$, (6.27) and (6.30) give finite contributions in the $N \to \infty$ limit, while of $\beta \leqq 4$ their contributions are divergent. The counterterms transform the Wick monomial $:(\varphi^{[\leqq k]})^\beta:$ into $\mathscr{R}(:(\varphi^{[\leqq k]})^\beta:)$ [see Eq. (6.20)], which has the same dimension but a greater "k-dimension" which will imply in Eqs. (6.27) and (6.30) a greater value of β and a smaller value of α. Then if the "k-dimension" of $\mathscr{R}(:(\varphi^{[\leqq k]})^\beta:)$ is >4 it follows that the estimates (6.27) and (6.30) are now finite. From (6.20) it is clear that

$$d_k(S_{\xi\eta}^{[\leqq k]2}) = 6 , \qquad d_k(S_{\xi\eta}^{[\leqq k]} D_{\xi\eta}^{[\leqq k]}) = 5 . \tag{6.31}$$

To extend this kind of argument to build $\Delta_3^{(N)}V$ we have to recall that the presence of $\Delta_2^{(N)}V$ has produced in $V_2^{(k)}$ new fields: $S_{\xi\eta}^{[\leqq k]}$ and $D_{\xi\eta}^{[\leqq k]}$, which cannot be undone as they bring the appropriate zeroes in $(\xi - \eta)$ needed to make the order g^2 terms finite. These new fields can produce, in the lower frequency truncated expectations, new monomials with "k-dimension" $\leqq 4$, which therefore have again to be modified with new $\mathscr{R}$ and $\mathscr{L}$ operations which in their turn will imply a well defined counterterm $\Delta_3^{(N)}V$. Let us therefore examine which are the monomials which can be produced with these new fields, with "k-dimension" $\leqq 4$ and how we have to modify them.

Monomials of second order	*"k-dimension"*
$:\varphi_\xi^{[\leqq k]}\varphi_\eta^{[\leqq k]}:$	2
$:\underline{\partial}\varphi_\xi^{[\leqq k]}\varphi_\eta^{[\leqq k]}:$	3
$:S_{\xi\eta}^{[\leqq k]}\varphi_\tau^{[\leqq k]}:$	4
$:D_{\xi\eta}^{[\leqq k]}\varphi_\tau^{[\leqq k]}:$	3
$:D_{\xi\eta}^{[\leqq k]}D_{\tau\zeta}^{[\leqq k]}:$	4
$:D_{\xi\eta}^{[\leqq k]}S_{\tau\zeta}^{[\leqq k]}:$	5
$S_{\xi\eta}^{[\leqq k]}S_{\tau\zeta}^{[\leqq k]}:$	6

Therefore the counterterm $\Delta_3^{(N)}V$ has to modify only the first five monomials; the first one has been found already at the order g^2, and therefore the $\mathscr{L}$ operation, and also the $\mathscr{R}$ one, have been defined before. We have to do the same for the next four monomials. Let's discuss explicitly the third one: one has to proceed, as in Eq. (6.12), starting from the following relationships:

$$S_{\xi\eta}\varphi_\tau = S_{\xi\eta}D_{\tau\eta} + S_{\xi\eta}\varphi_\eta\,. \tag{6.32}$$

As the "k-dimension of $S_{\xi\eta}D_{\tau\eta}$ is 5 we have to worry only about the last term

$$\begin{aligned} S_{\xi\eta}\varphi_\eta &= (\varphi_\xi - \varphi_\eta - (\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta - \tfrac{1}{2}(\underline{\xi}-\underline{\eta})^2\cdot\underline{\underline{\partial}}^2\varphi_\eta)\varphi_\eta \\ &\quad + \tfrac{1}{2}(\underline{\xi}-\underline{\eta})^2\cdot\underline{\underline{\partial}}^2\varphi_\eta\varphi_\eta \equiv T_{\xi\eta}\varphi_\eta + \tfrac{1}{2}(\underline{\xi}-\underline{\eta})^2\cdot\underline{\underline{\partial}}^2\varphi_\eta\varphi_\eta\,, \end{aligned} \tag{6.33}$$

where

$$(\underline{\xi}-\underline{\eta})^2\cdot\underline{\underline{\partial}}^2\varphi_\eta = \sum_i\sum_j(\xi-\eta)_i(\xi-\eta)_j\partial^2_{ij}\varphi_\eta\,. \tag{6.34}$$

Therefore as the "k-dimension" of $T^{[\leq k]}_{\xi\eta}$ is 4 we have

$$\begin{aligned} \mathscr{L}(:S^{[\leq k]}_{\xi\eta}\varphi^{[\leq k]}_\eta:) &= :\tfrac{1}{2}(\underline{\xi}-\underline{\eta})^2\cdot\underline{\underline{\partial}}^2\varphi_\eta\varphi_\eta:) \\ \mathscr{R}(:S^{[\leq k]}_{\xi\eta}\varphi^{[\leq k]}_\eta:) &= :T^{[\leq k]}_{\xi\eta}\varphi^{[\leq k]}_\eta:. \end{aligned} \tag{6.35}$$

Let us collect all the results one finds, proceeding in a similar way, for the other monomials to "renormalize", omitting the index $[\leq k]$:

$$\begin{aligned} &\mathscr{R}(:\varphi_\xi\varphi_\eta:) = \tfrac{1}{2}:S^2_{\xi\eta}: - :S_{\xi\eta}D_{\xi\eta}:, \\ &\mathscr{R}(:\underline{\partial}\varphi_\xi\varphi_\eta:) = :\underline{\partial}\varphi_\xi S_{\eta\xi}:, \\ &\mathscr{R}(:S_{\xi\eta}\varphi_\tau:) = :S_{\xi\eta}D_{\tau\eta}: + \mathscr{R}(:S_{\xi\eta}\varphi_\eta:)\,, \\ &\mathscr{R}(:S_{\xi\eta}\varphi_\eta:) = :T_{\xi\eta}\varphi_\eta:, \\ &\mathscr{R}(:D_{\xi\eta}\varphi_\tau:) = \mathscr{R}(:D_{\xi\eta}D_{\tau\eta}:) + \mathscr{R}(:D_{\xi\eta}\varphi_\eta:)\,, \\ &\mathscr{R}(:D_{\xi\eta}\varphi_\eta:) = :T_{\xi\eta}\varphi_\eta:, \\ &\mathscr{R}(:D_{\xi\eta}D_{\tau\zeta}:) = :S_{\xi\eta}D_{\tau\zeta}: + :S_{\tau\zeta}D_{\xi\eta}: - :S_{\xi\eta}S_{\tau\zeta}: - (\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta(\underline{\tau}-\underline{\zeta})\cdot\underline{D}^1_{\eta\zeta}\,, \end{aligned} \tag{6.36}$$

where

$$\underline{D}^1_{\eta\zeta} \equiv (\underline{\partial}\varphi_\eta - \underline{\partial}\varphi_\zeta)\,. \tag{6.37}$$

As $\mathscr{L} = 1 - \mathscr{R}$, it follows:

$$\begin{aligned} &\mathscr{L}(:\varphi_\xi\varphi_\eta:) = \tfrac{1}{2}(:\varphi^2_\xi: + :\varphi^2_\eta:) - \tfrac{1}{2}:((\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta)^2:, \\ &\mathscr{L}(:\underline{\partial}\varphi_\xi\varphi_\eta:) = :((\underline{\eta}-\underline{\xi})\cdot\underline{\partial}\varphi_\xi)\underline{\partial}\varphi_\xi: + :\varphi_\xi\underline{\partial}\varphi_\xi:, \\ &\mathscr{L}(:S_{\xi\eta}\varphi_\tau:) = \mathscr{L}(:S_{\xi\eta}\varphi_\eta:) = \tfrac{1}{2}:(\underline{\xi}-\underline{\eta})^2\cdot\underline{\underline{\partial}}^2\varphi_\eta\varphi_\eta:, \\ &\mathscr{L}(:D_{\xi\eta}\varphi_\tau) = \mathscr{L}(:D_{\xi\eta}D_{\tau\eta}:) + \mathscr{L}(:D_{\xi\eta}\varphi_\eta:)\,, \\ &\mathscr{L}(:D_{\xi\eta}\varphi_\eta:) = :(\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta\varphi_\eta: + \tfrac{1}{2}:(\underline{\xi}-\underline{\eta})^2\cdot\underline{\underline{\partial}}^2\varphi_\eta\varphi_\eta:, \\ &\mathscr{L}(:D_{\xi\eta}D_{\tau\zeta}:) = :(\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta(\underline{\tau}-\underline{\zeta})\cdot\underline{\partial}\varphi_\eta:. \end{aligned} \tag{6.38}$$

Looking at Eqs. (6.36) and (6.38) the careful reader will notice that the $\mathscr{R}$ operation is not defined in a symmetric way. For instance $\mathscr{R}(:\varphi_\xi\varphi_\eta:)$ is different from $\mathscr{R}(:\varphi_\eta\varphi_\xi:)$; this ambiguity is eliminated by giving a well defined prescription of which coordinate of the generic couple (x, y) has to play the role of ξ and which that of η in the prescription on the first line of (6.36) and for all the other ones which, otherwise, are ambiguously defined.

Let us remember that the contribution of order g^n to the effective potential is made by a sum over dressed trees with n final branches, each one associated to $-g\int d_\xi^4 : \varphi_\xi^{[\leq N]4}:$. Call the corresponding coordinates $\xi_1, \xi_2, \ldots, \xi_n$ which we consider ordered by their indices assigned in a definite way, from top to bottom, to the final branches of the tree which are not allowed to change their relative position. Therefore we say $\xi_i \prec \xi_j$ if $i<j$. Now the prescriptions (6.36) and consequently (6.38) have no ambiguity if we assume $\xi \prec \eta$ in $\mathscr{R}(:\varphi_\xi\varphi_\eta:)$ and in $\mathscr{R}(:D_{\xi\eta}\varphi_\tau:)$, and $\xi \prec \eta$, $\tau \prec \zeta$ in $\mathscr{R}(:D_{\xi\eta}D_{\tau\zeta}:)$. If the monomial to which we apply $\mathscr{R}$ differs by a sign from one listed above then the $\mathscr{R}$ operation commutes with the $(-)$. For instance let's assume $\xi \succ \eta$, then

$$\begin{aligned}\mathscr{R}(:D_{\xi\eta}\varphi_\eta:) = \mathscr{R}(-:D_{\eta\xi}\varphi_\eta:) &\equiv -\mathscr{R}(:D_{\eta\xi}\varphi_\eta:)\\ &= -[\mathscr{R}(:D_{\eta\xi}D_{\eta\xi}:) + \mathscr{R}(:D_{\eta\xi}\varphi_\xi:)]\,.\end{aligned}$$

Some remarks are now appropriate: it should be clear that the $\mathscr{R}$ and $\mathscr{L} = 1 - \mathscr{R}$ operations must satisfy these constraints:

a) They cannot destroy the zeroes of the monomials over which they are applied.

b) The $\mathscr{L}$ operation must always produce a local term between these possible ones : $\varphi_\xi^4:$, $:\varphi_\xi^2:$, $:(\partial\varphi_\xi)^2:$, 1.

(a) is a technical property which will be clear later on during the proof of finiteness (Theorem 1). The idea is that we want to estimate separately the coefficient of any Wick monomial and prove that it is bounded in N. As these zeroes are crucial for that, we cannot destroy them, once they have been produced. (b) follows from the general properties the counterterms must satisfy. For the second order monomials in Eqs. (6.38) it is clear that (b) is satisfied as, after all the integrations are performed, these counterterms will be of the following type:

$$\int d\eta \{T_{ij}(\eta) : \partial_i\varphi_\eta\partial_j\varphi_\eta : + T_i(\eta) : \partial_i\varphi_\eta\varphi_\eta : + T(\eta) : \varphi_\eta^2 :\} \tag{6.39}$$

and using rotation covariance (in the restricted sense valid on a torus) and integrating by parts, we get the expected result.

c) Let us observe that different prescriptions from that in (6.36), (6.38) could have been given, also suitable for renormalizing the theory. For instance one could choose a more symmetric prescription which does not require any choice of ordering between the coordinates; let us give an example:

$$\mathscr{R}_s(:\varphi_\xi\varphi_\eta:) = \tfrac{1}{2}(:S_{\xi\eta}^2 - S_{\xi\eta}D_{\xi\eta}:) + \tfrac{1}{2}(:\tfrac{1}{2}S_{\eta\xi}^2 - S_{\eta\xi}D_{\eta\xi}:)\,,$$

which implies ($\mathscr{L} = 1 - \mathscr{R}$),

$$\mathscr{L}_s(:\varphi_\xi\varphi_\eta:) = \tfrac{1}{2}(:\varphi_\xi^2: + :\varphi_\eta^2:) - \tfrac{1}{4}[((\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\eta)^2 + ((\underline{\xi}-\underline{\eta})\cdot\underline{\partial}\varphi_\xi)^2]\,.$$

Fourth Order Monomials

Looking at the "k-dimension" of the new fields produced $D_{\xi\eta}^{[\leq k]}$, $S_{\xi\eta}^{[\leq k]}$, it is clear that if a fourth order monomial has a D or an S between its fields, its "k-dimension" is >4, and therefore the $\mathscr{R}$ operation acts as the identity ($\mathscr{L} = 1 - \mathscr{R} = 0$). The only difference for the fourth order monomials produced at the order g^3 or higher with respect to those produced to the order g^2 is that the φ's in it can be all computed at different points. Let therefore ξ, η, ζ, θ be such that, with the ordering discussed in remark (c), $\xi \prec \eta \prec \zeta \prec \theta$, then

$$\begin{aligned}
&\mathscr{R}(:\varphi_\xi^3\varphi_\eta:) = :\varphi_\xi^3 D_{\eta\xi}:; \qquad \mathscr{R}(\varphi_\xi^2\varphi_\eta^2) = :\varphi_\xi^3 D_{\eta\xi}: + :\varphi_\xi^2\varphi_\eta D_{\eta\xi}:,\\
&\mathscr{R}(:\varphi_\xi\varphi_\eta^3:) = :\varphi_\xi^3 D_{\eta\xi}: + :\varphi_\xi^2\varphi_\eta D_{\eta\xi}: + :\varphi_\xi\varphi_\eta^2 D_{\eta\xi},\\
&\mathscr{R}(:\varphi_\xi^2\varphi_\eta\varphi_\zeta:) = :\varphi_\xi^2 D_{\eta\xi}\varphi_\zeta: + :\varphi_\xi^3 D_{\zeta\xi}:,\\
&\mathscr{R}(:\varphi_\xi\varphi_\eta^2\varphi_\zeta:) = :\varphi_\xi^2 D_{\eta\xi}\varphi_\zeta: + :\varphi_\xi^3 D_{\zeta\xi}: + :\varphi_\xi\varphi_\eta\varphi_\zeta D_{\eta\xi}:,\\
&\mathscr{R}(:\varphi_\xi\varphi_\eta\varphi_\zeta^2:) = :\varphi_\xi^2 D_{\eta\xi}\varphi_\zeta: + :\varphi_\xi^3 D_{\zeta\xi}: + :\varphi_\xi\varphi_\eta\varphi_\zeta D_{\zeta\xi}:,\\
&\mathscr{R}(:\varphi_\xi\varphi_\eta\varphi_\zeta\varphi_\theta) = :\varphi_\xi^2 D_{\zeta\xi}\varphi_\theta: + :\varphi_\xi^3 D_{\theta\xi}: + :\varphi_\xi\varphi_\zeta\varphi_\theta D_{\eta\xi}:,\\
&\mathscr{L}(:\varphi_\xi\varphi_\eta\varphi_\zeta\varphi_\theta:) = :\varphi_\xi^4:.
\end{aligned} \tag{6.40}$$

Remark. Also this prescription is highly non-unique; for instance one could completely forget about the previous ordering between the coordinates and define

$$\begin{aligned}
&\mathscr{R}(:\varphi_\xi^3\varphi_\eta:) = :\varphi_\xi^3 D_{\eta\xi}:,\\
&\mathscr{R}(:\varphi_\xi^2\varphi_\eta^2:) = :\varphi_\xi^3 D_{\eta\xi}: + :\varphi_\xi^2\varphi_\eta D_{\eta\xi}:, \qquad \xi > \eta,\\
&\mathscr{R}(:\varphi_\xi^2\varphi_\eta\varphi_\zeta:) = :\varphi_\xi^2 D_{\eta\xi}\varphi_\zeta: + :\varphi_\xi^3 D_{\zeta\xi}:,\\
&\mathscr{R}(:\varphi_\xi\varphi_\eta\varphi_\zeta\varphi_\theta) = :\varphi_\xi^2 D_{\zeta\xi}\varphi_\theta: + :\varphi_\xi^3 D_{\theta\xi}: + :\varphi_\xi\varphi_\zeta\varphi_\theta D_{\eta\xi}:.
\end{aligned}$$

The possibility of changing in many ways the prescriptions just discussed is the reflection of the possibility one always has of changing the renormalization procedure by a "finite renormalization".

Equations (6.37), ..., (6.40) with the associated remarks, nearly conclude the definition of the $\mathscr{R}$ and the $\mathscr{L}$ operations; at higher orders in g there are still some monomials of second order produced with the new fields that have appeared now, namely $T_{\xi\eta}$ and $\underline{D}_{\xi\eta}^1$ that still need the application of $\mathscr{R}$. Nevertheless they are in a finite number and they don't produce under the $\mathscr{R}$ operation new fields which require any renormalization. Let's indicate these monomials and let's define the $\mathscr{R}$ operation on them.

$$\begin{aligned}
&\mathscr{R}(:\underline{D}_{\xi\eta}^1\varphi_\tau:) = :\underline{D}_{\xi\eta}^1 D_{\tau\eta}: + \mathscr{R}(:\underline{D}_{\xi\eta}^1\varphi_\eta:), \qquad \xi \prec \eta,\\
&\mathscr{R}(:\underline{D}_{\xi\eta}^1\varphi_\eta:) = :\underline{S}_{\xi\eta}^1\varphi_\eta:,\\
&\mathscr{R}(:\underline{\partial}\varphi_\xi\underline{\partial}\varphi_\eta:) = :\underline{\partial}\varphi_\xi\underline{D}_{\eta\xi}^1:, \qquad \xi \prec \eta,\\
&\mathscr{R}(:\underline{\partial}\varphi_\xi D_{\eta\tau}:) = :\underline{D}_{\xi\tau}^1 D_{\eta\tau}: + :\underline{\partial}\varphi_\tau S_{\eta\tau}:, \qquad \eta \prec \tau,
\end{aligned} \tag{6.41}$$

where

$$\underline{S}_{\xi\eta}^1 = \underline{\partial}\varphi_\xi - \underline{\partial}\varphi_\eta - (\underline{\xi} - \underline{\eta}) \cdot \underline{\underline{\partial}}^2\varphi_\eta,$$

and again if the monomial $\mathscr{M}$ is $-\mathscr{N}$, where $\mathscr{N}$ has been listed above, then $\mathscr{R}(\mathscr{M}) \equiv -\mathscr{R}(\mathscr{N})$,

$$\begin{aligned}
&\mathscr{L}(:\underline{D}_{\xi\eta}^1\varphi_\tau:) = :(\underline{\xi} - \underline{\eta}) \cdot \underline{\underline{\partial}}^2\varphi_\eta\varphi_\eta:,\\
&\mathscr{L}(:\underline{\partial}\varphi_\xi\underline{\partial}\varphi_\eta:) = :\underline{\partial}\varphi_\xi\underline{\partial}\varphi_\xi:,\\
&\mathscr{L}(:\underline{\partial}\varphi_\xi D_{\eta\tau}:) = :\underline{\partial}\varphi_\tau(\underline{\eta} - \underline{\tau}) \cdot \underline{\partial}\varphi_\tau:.
\end{aligned} \tag{6.42}$$

Remarks. a) The "k-dimension" of $S_{\xi\eta}^{1[\le k]}$ is 4 and therefore it will not produce any new monomial of second or higher order in the fields to be renormalized.

b) The $\mathscr{R}$ operation can be thought of as defined over any Wick monomial, being the identity when not explicitly stated differently.

c) Due to periodic boundary conditions, all the integrations are on a four dimensional torus and for the choice of the covariance, see remark (b) after Eq. (2.13) our expressions are translation invariant. Therefore this property has not to be spoiled by the $\mathscr{R}$ and the $\mathscr{L}$ operations. As $(\underline{\xi}-\underline{\eta})$ is not translation invariant on the torus, each time it appears it has to be interpreted as a symbolic expression for

$$\sum_{\underline{n}\in Z^4}(\underline{\xi}-\underline{\eta}+\underline{n}L)f(|\underline{\xi}-\underline{\eta}+\underline{n}L|)\equiv``(\underline{\xi}-\underline{\eta})", \tag{6.43}$$

where $\underline{n}=(n_1,\dots,n_4)$ and $f(r)$ is a function $\in C_0^\infty(R)$ with support $[-\delta,\delta],=1$ in $\left[-\frac{\delta}{2},\frac{\delta}{2}\right]$, and $L/2<\delta<L$.

The above discussion immediately leads to the general formulation of the rules for computing the counterterms as well as to find the formal series of $V_R^{(N)}=V^{(N)}+\Delta_2^{(N)}V+\Delta_3^{(N)}V+\dots$. In fact to compute $\Delta_s^{(N)}V$, $\forall s$, the procedure is the following: starting from $V_{s-1}^{(N)}$ compute all the dressed trees of order g^s with root $k=-1$, then to the final monomials in $\tilde{\varphi}=\varphi^{[-1]}$, $\partial\tilde{\varphi}=\partial\varphi^{[-1]}$, $\tilde{D}=D^{[-1]},\dots,\tilde{T}=T^{[-1]}$, apply the $\mathscr{L}$-operation, change the sign and finally substitute $\tilde{\varphi},\partial\tilde{\varphi},\tilde{D},\tilde{S}\dots$ with $\varphi^{[\leq N]}$, $\partial\varphi^{[\leq N]}$, $D^{[\leq N]}$, $S^{[\leq N]}\dots$. The sum of the contributions of all these trees, with these prescriptions, is $\Delta_s^{(N)}V$. Recall also that all these trees of order g^s have at any bifurcation, except the final one, either an index R or a frame with a label β.

In general, given a shape σ, we have, for the corresponding framed subtree [see (6.21)]

$$\begin{aligned}\frac{\Delta^{(k)}(\sigma;N)}{n(\sigma)}&=\Sigma_\beta\frac{\Delta_\beta^{(k)}(\sigma,N)}{n(\sigma)}=\sum_{\substack{\gamma\\ \sigma(\gamma)=\sigma\\ h(\gamma)\leq k\\ k(\gamma)=-1}}\frac{\Delta^{(N)}V(\gamma)}{n(\gamma)}\\&=\sum_\beta r_\beta^{(N)}(\sigma;k):P_\beta(\varphi^{[\leq k]}):,\end{aligned}\tag{6.44}$$

where $h(\gamma)$ is the frequency of the lowest bifurcation of γ, and $n(\sigma)$ is defined as after (6.21) for any possible shape.

Let us discuss now, on more general grounds, the way of computing the formal series of $V_R^{(k)}(\varphi^{[\leq k]})$ in terms of dressed trees just recollecting all that has been explicitly done for $V_2^{(k)}$ and $V_3^{(k)}$. A dressed tree is obtained from a simple tree γ by enclosing into frames indexed by an index $\beta=0,2,2',4$ final branches of γ in some arbitrary ways; after drawing the above "first generation frames" one can draw again new frames enclosing terminal branches of any of the trees constructed in the first step and appending an index β' to each of them etc. The frames are drawn in a hierarchical fashion so that no frame ever overlaps with another one. Once a frame is drawn all the frequency indices inside it are erased. Then one proceeds to build the "third generation" frames, etc. All the above framings are done in all possible ways. Once a tree with several framed parts is built one puts indices R on each unframed vertex, the resulting trees are called "dressed" trees. The contribution to $V_R^{(k)}$ from a dressed tree γ_0 is given by the following rule: To each frame with inside a shape σ (which itself may contain frames with shapes inside, but which is not enclosed in a larger frame) we associate a term, depending on the index β, of the fourth order polynomial [see Eq. (6.44)]

$$\frac{\Delta^{(k)}(\sigma; N)}{n(\sigma)} = \Sigma_\beta r_\beta^{(N)}(\sigma; k) : P_\beta(\varphi^{[\leq k]}) :, \tag{6.45}$$

and we replace with it the truncated expectation which corresponds to σ in the formula we would build for $V(\gamma_0)$ [see (5.5)] if γ_0 were an undressed tree with the shape σ. Furthermore we modify (proceeding in the ordered way prescribed by the tree, from top to the root) the truncated expectations corresponding to the bifurcations with an index R by the rules explained in (6.37), ..., (6.42), after having developed them (the expectations) into Wick monomials. This completes the description of the counterterms and of the "renormalized effective potential" $V_R^{(k)}$.

Remarks. a) As it should be clear the notion of "decorated" trees does not play any role and has been introduced just to make clear that the renormalization can be thought of as due to a change in the action, i.e. it is a "Lagrangian renormalization" as opposed to other possible ways of removing the divergences.

b) The dimensional argument to build the counterterms is nearly a proof of the finiteness of the coefficients of the formal power series of $V_R^{(k)}$, as $N \to \infty$; this will be clear looking at the proof of Theorem 1 of Sect. 7.

To prove the estimates of Proposition 2 more work is still needed.

7. Finiteness and the *n*! Factorial Bound

7.1

The goal of this section is to prove estimates (2.27); to achieve it we still need various preliminary notions. In particular we are going to introduce the Feynman graphs to compute the truncated expectations; nevertheless it should be clear at the end that their role is here completely auxiliary and also that their use could be completely avoided paying the price of being more formal.

We want to study the formal series (6.9)

$$V_R^{(k)}(\varphi^{[\leq k]}) = \sum_n \sum_{\substack{\gamma \\ k(\gamma)=k \\ v(\gamma)=n}} \frac{V(\gamma)}{n(\gamma)}, \tag{7.1}$$

where γ is a dressed tree, $k(\gamma)$ is the "root" of γ (its lowest frequency), $v(\gamma)$ is the number of final branches of γ including those encircled by frames; therefore selecting all the γ with $v(\gamma) = n$ amounts to fixing the order in g of these contributions to $V_R^{(k)}$.

Let a Feynman graph $\mathscr{G}$ be given: $\mathscr{G}$ has to be thought of as a completely labelled graph (vertices and lines), connected and with no line emerging from a vertex and entering on the same vertex.

The use of F-graphs is mainly the following, the generic term $V(\gamma)$ is made by a truncated expectation of truncated expectations of truncated expectations ..., each of them being a Wick polynomial times a function which is a product of covariances; at the end $V(\gamma)$ can be expressed as a linear sum of Wick monomials each multiplied by products of covariances. Each F-graph will select one of these contributions.

We make this statement more precise introducing the notion of compatibility between a F-graph $\mathscr{G}$ and a dressed tree γ. Given a graph $\mathscr{G}$ with n labelled vertices $1, 2, \ldots, n$ and a tree γ (hereafter we will omit the adjective dressed as all the trees are dressed) with n labelled final branches $1, 2, \ldots, n$ let us associate each vertex of $\mathscr{G}$ with label j with the top line j of γ. Having made this correspondance one associates to each tree-vertex (bifurcation) a "box" B surrounding a subgraph of $\mathscr{G}$, specifically encircling those vertices of $\mathscr{G}$ which correspond to the branches of γ which merge into this bifurcation, and drawing it in such a way that any line connecting two vertices in the box is also contained in the box.

These boxes are drawn with an iterative procedure, first by looking at the "first generation" bifurcations of γ, i.e. to the tree vertices from which the highest frequency lines bifurcate and drawing the corresponding boxes, then we proceed to build the "second generation" boxes by considering the second generation of bifurcations of γ etc., these new boxes possibly containing in their interior some of the previous ones.

The bifurcations which correspond to a framed part of the tree γ also produce boxes which will be drawn "marked" and will bring the same index of the framed part. Once this is done, a family of boxes, possibly marked and possibly enclosing other boxes and all enclosed in the largest one corresponding to the final bifurcation (of lowest frequency), has been built. For a more uniform notation we will consider as marked boxes also the vertices of $\mathscr{G}$ which are not enclosed in any marked box.

If B is a box we call ξ that vertex of $\mathscr{G}$ contained in B which is the first in the $\prec$ ordering previously introduced. Let us give an example of all that in Fig. 7.1.

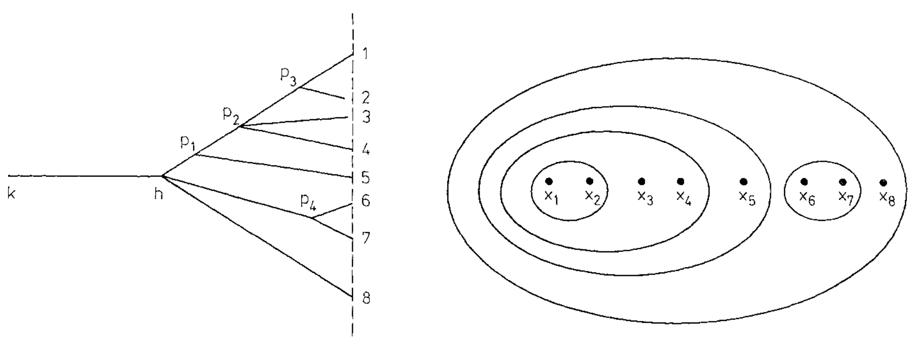


Fig. 7.1

The boxes corresponding to the tree in Fig. 7.1 are described next to it, without drawing $\mathscr{G}$ explicitly but drawing only the hierarchically ordered clusters of vertices as prescribed by γ. If some frames are present in the tree the only difference would be that the corresponding boxes will be marked. It is also clear that the hierarchy of boxes around the vertices of a F-graph, induced by the tree γ, depends only on its shape $\sigma(\gamma)$ and not on its frequency indices. The tree γ has therefore produced a family of boxes $B_1, B_2, \ldots, B_s$; each of them enclosing a subgraph of $\mathscr{G}$, $\mathscr{G}_{B_1}, \mathscr{G}_{B_2}, \ldots, \mathscr{G}_{B_s}$; *then* $\mathscr{G}$ *and* γ *will be called compatible if all the subgraphs* $\mathscr{G}_{B_1}, \mathscr{G}_{B_2}, \ldots, \mathscr{G}_{B_s}$ *are connected.*

In Fig. 7.2 we provide a more complicated example to illustrate the above concepts.

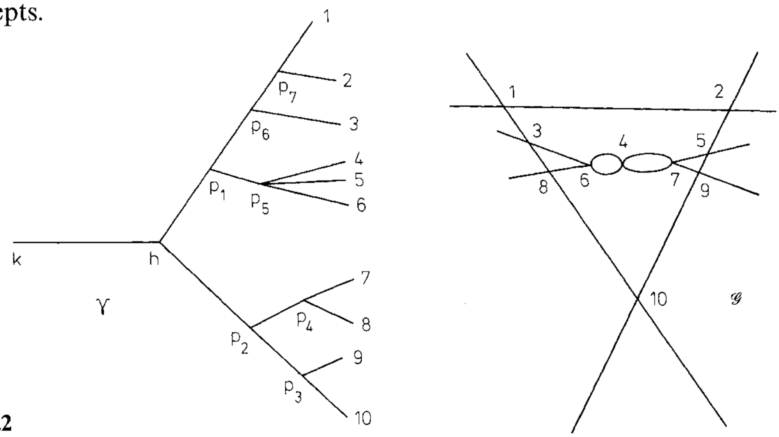


Fig. 7.2

The graph $\mathscr{G}$ is not compatible with the tree γ above: the graph $\mathscr{G}'$ obtained by replacing 5 by 7, 7 by 5, 8 by 9, and 9 by 8 is however compatible: a simple way to check this is to draw on the graph $\mathscr{G}$ boxes enclosing points as prescribed by the tree's structure [i.e. one box enclosing (1, 2), one containing the former one enclosing (1, 2, 3), one containing the former one enclosing (1, 2, 3, 4, 5, 6), another enclosing (7, 8) and one on (9, 10), one more enclosing (7, 8, 9, 10) and one enclosing every vertex]; then one checks if the graphs cut out of $\mathscr{G}$ by the innermost boxes are connected (when two endpoints of a line are in a box, the box is supposed to be drawn so that the whole line lies in it); then one thinks of the innermost boxes as points into which all the lines converge with only one vertex in the box and one proceeds to check the boxes next to the innermost ones to see if the graphs (that they cut out of the new graph obtained after the above identifications) are connected etc.

It is quite clear that:

a) this notion of compatibility depends only on $\sigma(\gamma)$ and not on γ;

b) quite a few graphs are not compatible with a given tree, but there are some very special trees, those of Fig. 7.3, which are compatible with all the the graphs with the same number of vertices:

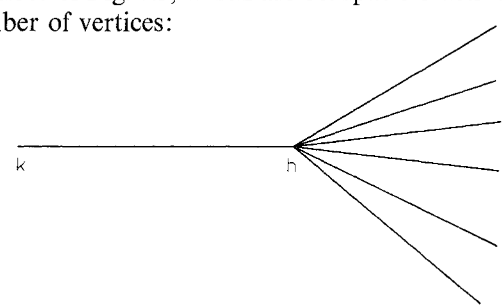


Fig. 7.3

To use the notion of F-graphs to construct explicitly the terms which contribute to $V(\gamma)$, for a fixed γ, we have to attach to each box an index and a frequency in the following way:

a) The boxes associated with "framed" bifurcations will keep the same index β of the frame.

b) The unmarked boxes not enclosed in any marked box will bring a frequency borrowed by the bifurcations which have generated them.

No fixed frequency index has to be assigned to the marked boxes or to those contained in a marked one because inside a frame of γ no frequency index is present.

c) The frequency of a line of $\mathscr{G}$ is the frequency of the first box into which it is enclosed and to any external line we associate the frequency of the root of $\gamma : k(\gamma)$.

Definition. We shall call a F-graph $\mathscr{G}$ and a tree γ completely compatible when they are compatible and from every marked box with index $\beta=0,2,2',4$ go out $0,2,2,4$ external lines respectively and we shall write in this case $\chi(\gamma,\mathscr{G})=1$, otherwise $\chi(\gamma,\mathscr{G})=0$, the function χ being implicitly defined.

We can now describe the contribution to $V(\gamma)$ of a F-graph $\mathscr{G}$ with $\chi(\gamma,\mathscr{G})=1$; each internal line of $\mathscr{G}$ corresponds to a covariance at two points (vertices) of $\mathscr{G}$ and can be thought of as the junction of two half lines (fields) going out from different boxes. Given $\mathscr{G}$ it is possible to fix a rule which assign to each half line a field φ, $\underline{\partial}\varphi$, D, S, T, $\underline{D}^1$, $\underline{S}^1$ with a specified frequency so that when two half lines are connected the covariance associated to the resulting line is completely determined. If a half line is not connected to any other half line it will go out from every box of $\mathscr{G}$ and will be a "external line".

We fix now the rules for the association of the lines with the fields: we classify the boxes according to how many box boundaries a line, starting inside a given box and going to infinity, has to cross and call such number the order of the box. Then we assign a frequency index to all the boxes in the following way: if the box is unmarked and not enclosed in any marked box it takes the frequency of the bifurcation which has generated it, as defined before in *b*). If the box is a marked one or an unmarked one but contained in some larger marked box we assign to it a generic frequency q over which we are going to sum later, so that at the end no fixed frequency will be assigned to these boxes. In this way every box has a frequency. We start with the higher order boxes which by convention are all marked; if $\beta=2,4$ the half lines (we omit the "half" hereafter) going out from the marked box $\bar{B}$ represent the field $\varphi_\xi^{[\leq h]}$ and if $\beta=2'$ the field $\underline{\partial}\varphi_\xi^{[\leq h]}$, where ξ is the point inside $\bar{B}$ prescribed by the rules defining $\mathscr{R}$ and $\mathscr{L}=1-\mathscr{R}$ and h is the frequency of the first box enclosing the line formed by this half line joined with some other half line. We remark that all the lines going out from the marked box $\bar{B}$ are associated to fields computed at the same point, but they can have different frequencies depending on which unmarked box contains them. The lines inside $\bar{B}$ keeps the frequency q of $\bar{B}$. If $\beta=0$ there are no lines going out from $\bar{B}$. After the innermost boxes we go out to the second generation of boxes which are marked or unmarked: for the marked ones we proceed as before; if B is an unmarked one the meaning of a line emerging from it is also clear if the line starts from an innermost marked box inside B, in fact it has the meaning assigned to it before. If the line emerges from a vertex η contained in B and not in the innermost boxes it has the meaning $\varphi_\eta^{[\leq \bar{h}]}$ if it is completely contained in a box of frequency $\bar{h}$ and if more than four lines emerge from B. If 2 or 4 lines come out from B, which means that this box corresponds to a bifurcation where the $\mathscr{R}$-operation is, in general, not the identity, [see Eqs. (6.37), ..., (6.42)], then also to this box we append an index β_{G_B} which tells us which elements in the right-hand side of (6.37), ..., (6.42) has to be selected. More precisely if B has only four external lines emerging from two, three or possibly four distinct points, ξ, μ, θ, ζ, we attribute to these lines a meaning which depends on the index β_{G_B}: it tells us which element in the right-hand side of (6.40) to select. Suppose then that B has only two vertices ξ, η with one external line each. This box again has one index whose different values describe which of the terms produced by the $\mathscr{R}$-operation, has to be selected. For instance if $\beta_{G_B}=1$ both the lines emerging from ξ and η have to be associated to $\frac{1}{\sqrt{2}} S_{\xi\eta}^{[\leq \bar{h}]}$.

When this procedure is iterated up to the last box enclosing the whole graph G we have drawn a graph whose lines have well defined "names" and frequencies and with some of its subgraphs encircled with boxes which also have definite frequencies. Recall that to each subgraph with all the internal lines specified we associate a well defined part of the truncated expectation referring to that bifurcation. Iterating this procedure, we can associate a well defined function of the covariances and of the fields associated with the external lines to the whole graph. We have now only to sum over the frequencies $q_1, \ldots, q_l$ of the boxes which are marked or unmarked but contained in a marked box with the following rule: if q is the frequency of a marked box one sums over q from 0 to q', where q' is the frequency of the first box enclosing it. If q is the frequency of an unmarked box one has to sum over q from $q'+1$ to N, q' being the same as before.

The contribution of a graph $\mathscr{G}$ with labels $\beta_{\mathscr{G}}$ represent a particular choice of such contractions and its contribution to $V(\gamma)/n(\gamma)$ will be denoted:

$$\chi(\gamma, \mathscr{G}) \frac{1}{n(\gamma)} \int K_{\gamma, \mathscr{G}, \beta_{\mathscr{G}}}(X_{\mathscr{G}}; k) : P_{\mathscr{G}, \beta_{\mathscr{G}}}^{[\leq k]}(\varphi) : dX_{\mathscr{G}}, \tag{7.2}$$

where $P_{\mathscr{G}, \beta_{\mathscr{G}}}^{[\leq k]}(\varphi)$ is a Wick monomial in the external fields divided by the appropriate factors such that the $P_{\mathscr{G}, \beta_{\mathscr{G}}}^{[\leq k]}$ has no zeroes in the coordinates and has "k-dimension" $=0$. To obtain this result we divide the fields $\varphi_x^{[\leq k]}, \underline{\partial}\varphi_x^{[\leq k]}, D_{xy}^{[\leq k]}, S_{xy}^{[\leq k]}, \underline{D}_{xy}^{1[\leq k]}, T_{xy}^{[\leq k]}, \underline{S}_{xy}^{1[\leq k]}, \gamma^k, \gamma^{2k}, \gamma^k(\gamma^k|x-y|), \gamma^k(\gamma^k|x-y|)^2, \gamma^{2k}(\gamma^k|x-y|)$ by $\gamma^k(\gamma^k|x-y|)^3, \gamma^{2k}(\gamma^k|x-y|)^2$ respectively. $dX_{\mathscr{G}}$ means integration over all the vertex coordinates of $\mathscr{G}$ and $K_{\gamma, \mathscr{G}, \beta_{\mathscr{G}}}(X_{\mathscr{G}}; k)$ is a function that we are going to estimate carefully.

Finally with $\beta_{\mathscr{G}}$ we indicate the family of indices attached to the marked and to the unmarked boxes, with a slight abuse of notation as the indices of the marked box are already fixed in the definition of the dressed tree γ. From (6.9)

$$V_R^{(k)}(\varphi^{[\leq k]}) = \sum_{k(\gamma)=k}{}^{\gamma} \sum_{\mathscr{G}, \beta_{\mathscr{G}}} \chi(\gamma, \mathscr{G}) \frac{1}{n(\gamma)} \int K_{\gamma, \mathscr{G}, \beta_{\mathscr{G}}}(X_{\mathscr{G}}; k) : P_{\mathscr{G}, \beta_{\mathscr{G}}}^{[\leq k]}(\varphi) : dX_{\mathscr{G}}. \tag{7.3}$$

follows.

The last thing to do before starting with the estimates of the generic term in the sum (7.3) is to modify the $\Sigma_{\mathscr{G}}$ making use of the factor $1/n(\gamma)$.

7.2

In the formula (7.3) the sum $\Sigma_{\mathscr{G}}$ is made over all the F-graphs compatible with a fixed γ, with labelled vertices and with the fields associated to the internal and to the external lines specified.

It is easy to show that we can reduce $\Sigma_{\mathscr{G}}$ to a sum over the topological graphs $\tilde{\mathscr{G}}$ (completely unlabelled, except for the names of the external lines which are specified).

We prove the following lemma:

Lemma 1. *Let $\tilde{\mathscr{G}}$ be a topological Feynman graph with n vertices and let γ be a tree with n endpoints. Then the number $N(\tilde{\mathscr{G}},\gamma,\{n_B\}_\gamma)$ of labellings of $\tilde{\mathscr{G}}$ compatible with γ and such that for every bifurcation B of γ, the subgraph of $\tilde{\mathscr{G}}$ corresponding to B has n_B external lines, is bounded above by $C_\varepsilon^n n(\sigma)\exp\varepsilon\sum_B n_B$, for all $\varepsilon>0$ and some constant C_ε, if σ is the shape of the tree γ.*

Proof. The proof is in the Appendix A of (II) and it is due to Giovanni Felder (Zürich).

We can rewrite Eq. (7.3) in the following way

$$V_R^{(k)}(\varphi^{[\leqq k]})=\sum_{\tilde{\mathscr{G}}}\sum_{k(\gamma)=k}\nolimits_\gamma \int \bar{K}_{\gamma,\tilde{\mathscr{G}}}(X_{\tilde{\mathscr{G}}};k):P_{\tilde{\mathscr{G}}}^{[\leqq k]}(\varphi):dX_{\tilde{\mathscr{G}}}\,, \tag{7.4}$$

where

$$\bar{K}_{\gamma,\tilde{\mathscr{G}}}(X_{\tilde{\mathscr{G}}};k)=\sum_{\{n_B\}_\gamma}\sum_{\substack{\tilde{\mathscr{G}}\text{ fixed}\\ \{n_B\}_\gamma\text{ fixed}}}\nolimits_{\mathscr{G},\beta_{\mathscr{G}}}\frac{\chi(\gamma,\mathscr{G})}{n(\gamma)}K_{\gamma;\mathscr{G},\beta_{\mathscr{G}}}(X_{\mathscr{G}};k)\,. \tag{7.5}$$

Using the obvious relationship $\Sigma_\gamma\dfrac{1}{n(\gamma)}=\sum_\sigma\sum_{\underline{h}}\dfrac{1}{n(\sigma)}$ where $\underline{h}$ describes the set of frequencies of the bifurcations of γ, and Lemma 1 we have the following inequality

$$\begin{aligned}&\Sigma_\gamma\int dX_{\tilde{\mathscr{G}}}|\bar{K}_{\gamma,\tilde{\mathscr{G}}}(X_{\tilde{\mathscr{G}}};k)|\\ &\leqq\sum_{(\sigma,\underline{h})}C_\varepsilon^n\sum_{\{n_B\}_\gamma}\prod_{\{B\}_\gamma}e^{\varepsilon n_B}\sup_{\substack{\mathscr{G},\beta_{\mathscr{G}}\\ \tilde{\mathscr{G}}\text{ fixed}\\ \{n_B\}_\gamma\text{ fixed}}}\int dX_{\mathscr{G}}|\chi(\gamma,\mathscr{G})K_{\gamma;\mathscr{G},\beta_{\mathscr{G}}}(X_{\mathscr{G}};k)|\end{aligned} \tag{7.6}$$

which will be used in the proof of Theorem 1.

7.3

All the preliminaries are now settled and we can go into the heart of the matter. Our goal in the remaining part of this section will be to prove the estimates (2.27).

To estimate the generic term of (7.5) we decompose $\sum_\gamma$ in the following way: we call $f(\gamma)$ the number of frames in a dressed tree γ [inside a frame there can be other frames which are also counted by $f(\gamma)$], $v(\gamma)$, the number of final branches of γ [if $v(\gamma)=n$, $V(\gamma)$ is of order g^n]. $f(\gamma)=f$ means also that any $\mathscr{G}$ compatible (we omit hereafter the adjective completely) with γ has f marked boxes. Of course

$$f(\gamma) \leqq v(\gamma) - 1 . \tag{7.7}$$

We still want to decompose Σ_γ depending on how these frames are included one into the other and how many final branches (vertices in graph language) are in each frame. This can be done in the following way: consider f spheres and n points, and any two spheres can be disjoint or included one into the other. We say that a family of f spheres is completely determined once we fix a) how many external spheres there are, how many inside each of them and so on, b) how many points (vertices) are in the i^{th} sphere and not in any sphere inside it, $\forall i$. For fixed f and n the number of families of f spheres with n points inside: $\mathcal{N}(f, n)$, is finite. Order these families arbitrarily with an index w running from 1 to $\mathcal{N}(f, n)$.

We rewrite (7.4) in the following way

$$\begin{aligned} V_R^{(k)}(\varphi^{[\leqq k]}) &= \sum_n \sum_f \sum_w \sum_{\tilde{\mathscr{G}}} V^{(k)}(n, f, w, \tilde{\mathscr{G}}, \varphi^{[\leqq k]}) \\ &= \sum_n \sum_f \sum_w \sum_{\tilde{\mathscr{G}}} \int dX_{\tilde{\mathscr{G}}} K_{(n,f,w,\tilde{\mathscr{G}})}(X_{\tilde{\mathscr{G}}}; k) : P_{\tilde{\mathscr{G}}}^{[\leqq k]}(\varphi) : , \end{aligned} \tag{7.8}$$

where

$$K_{(n,f,w,\tilde{\mathscr{G}})}(X_{\tilde{\mathscr{G}}}; k) \equiv \sum_{\begin{cases} k(\gamma)=k \\ v(\gamma)=n \\ f(\gamma)=f \\ w(\gamma)=w \end{cases}} {}_{\gamma} \, K_{\gamma, \tilde{\mathscr{G}}}(X_{\tilde{\mathscr{G}}}; k)$$

where $w(\gamma) = w$ means that Σ_γ is over those trees whose $f(\gamma) = f$ frames satisfy, together with the final branches, the relation prescribed by the family labelled by the index w.

Remark. It is clear that from the tree point of view a sphere is a frame and a point is a final branch, while from the graph point of view a sphere is a marked box and a point is a vertex.

We are now in a better position to give a general idea of the estimates we are going to prove. The central point is to prove that

$$\begin{aligned} &\int_{\Delta_1 \times \ldots \times \Delta_r} d\eta_1 \ldots d\eta_r \int_{\Lambda^{n-r}} d(X_{\tilde{\mathscr{G}}} \backslash \eta) |K_{(n,f,w,\tilde{\mathscr{G}})}(X_{\tilde{\mathscr{G}}}; k)| \\ &\leqq (\text{const})^n \mu(w) f! \sum_{j=0}^{f} \frac{(bk)^j}{j!} e^{-\kappa \gamma^k d(\Delta_1, \ldots, \Delta_r)} , \end{aligned} \tag{7.9}$$

where $\kappa > 0$, $\eta_1, \ldots, \eta_r$ are the coordinates of the external lines of $\tilde{\mathscr{G}}$, $\Delta_1, \ldots, \Delta_r$ are tesserae of linear size γ^{-k}, $\mu(w)$ is a bounded function of w and $b > 0$, which will be specified later. Then one has to prove the easier results,

$$\begin{aligned} &\sum_w 1 \leqq (\text{const})^{n+f} , \\ &\sum_{\substack{\tilde{\mathscr{G}} \\ (w, \text{fixed})}} 1 \leqq (\text{const})^n \frac{n!}{f!} , \\ &\sum_f 1 \leqq (\text{const})^n , \end{aligned} \tag{7.10}$$

and from (7.8) and (7.9) the estimates (2.27) follow.

7.4. Technical Part

We want to estimate

$$V^{(k)}(n,f,w,\tilde{\mathscr{G}};\varphi^{[\leqq k]})=\sum_{\substack{\gamma\\ \left\{\begin{smallmatrix}k(\gamma)=k\\ v(\gamma)=n\\ f(\gamma)=f\\ w(\gamma)=w\end{smallmatrix}\right.}} V^{(k)}_{\gamma,\mathscr{G}}(\varphi), \tag{7.11}$$

$$V^{(k)}_{\gamma,\mathscr{G}}(\varphi)=\int dX_{\mathscr{G}}K_{\gamma,\mathscr{G}}(X_{\mathscr{G}};k):P^{[\leqq k]}_{\mathscr{G}}(\varphi):. \tag{7.12}$$

To evaluate $V^{(k)}_{\gamma,\mathscr{G}}(\varphi)$ we have to proceed as previously discussed. If $f(\gamma)=f$, this means that γ has f frames and $w(\gamma)$ tells us how they are distributed. Let's call s the number of external frames, $s\leqq f$, or, in the $\tilde{\mathscr{G}}$-language, the number of external marked boxes. They correspond to localized terms; in fact remembering Sect. 6 it should be clear that: an external frame of γ whose next bifurcation frequency is h, which has at its interior a subtree of shape σ and which brings an index $\beta=0', 2, 2', 4$ has to be considered in the computation of the truncated expectation as a "form factor" times a local term. Remembering Eq. (6.44) we have

$$\begin{aligned}
&\beta=0' \text{ (trivial marked box = simple vertex)}\leftrightarrow :\varphi_\xi^{[\leqq h]4}: ,\\
&\beta=2\leftrightarrow r_2^{(N)}(\sigma,\tilde{\mathscr{G}};h)\gamma^{2h}:\varphi_\xi^{[\leqq h]2}:,\\
&\beta=2'\leftrightarrow r_{2'}^{(N)}(\sigma,\tilde{\mathscr{G}},h):(\underline{\partial}\varphi_\xi^{[\leqq h]})^2:,\\
&\beta=4\leftrightarrow r_4^{(N)}(\sigma,\tilde{\mathscr{G}};h):\varphi_\xi^{[\leqq h]4}:.
\end{aligned} \tag{7.13}$$

Remark. $r_\beta^{(N)}(\sigma,\tilde{\mathscr{G}};h)$ introduced here differs from $r_\beta^{(N)}(\sigma,h)$ of (6.44) for the extra dependence on $\mathscr{G}$, but their relation is

$$\begin{aligned}
&r_\beta^{(N)}(\sigma,h)=\sum_{\tilde{\mathscr{G}}} r_\beta^{(N)}(\sigma,\tilde{\mathscr{G}};h)\gamma^{2h\varepsilon(\beta)},\\
&\varepsilon(\beta)=1 \text{ if } \beta=2,\ =0 \text{ otherwise}.
\end{aligned} \tag{7.14}$$

Let us rewrite the sum in (7.11) in the following way

$$\begin{aligned}
&V^{(k)}(n,f,w,\tilde{\mathscr{G}},\varphi^{[\leqq k]})\\
&=\sum_{\substack{\gamma_a\\ \left\{\begin{smallmatrix}k(\gamma_a)=k\\ v(\gamma_a)=m+s\\ f(\gamma_a)=0\end{smallmatrix}\right.}}\Bigg[\sum_{\tilde{\mathscr{G}}_a}{}^{*}\int dX_{\tilde{\mathscr{G}}_a}\bar{K}_{\gamma_a,\tilde{\mathscr{G}}_a}(X_{\tilde{\mathscr{G}}_a};k):P^{[\leqq k]}_{\tilde{\mathscr{G}}}(\varphi):\\
&\quad\cdot\Bigg(\sum_{\substack{\sigma_1\ldots\sigma_s\\ \left\{\begin{smallmatrix}v(\sigma_i)=n_i\\ f(\sigma_i)=f_i\\ w(\sigma_i)=w_i\end{smallmatrix}\right.}}\prod_1^s{}_i\, r_{\beta_i}^{(N)}(\sigma_i,\tilde{\mathscr{G}}_i;h_i)\Bigg)\Bigg],
\end{aligned} \tag{7.15}$$

where $\sum_1^s{}_i n_i+m=n$, $\sum_1^s{}_i f_i=f$ and $\sum^{*}_{\tilde{\mathscr{G}}_a}$ is the sum over the possible topological graphs one obtains by shrinking the s external marked boxes of the $\mathscr{G}$'s topologically equivalent to $\tilde{\mathscr{G}}$; the sum over the subtrees γ_i with shapes σ_i is inside the $r_{\beta_i}^{(N)}(\sigma_i,\tilde{\mathscr{G}}_i;h_i)$ (see the discussion about the counterterms in Sect. 6) and $\sum_{\gamma_a}$ is defined hereafter $\equiv\sum_\sigma\sum_{\underline{h}}$, where $\underline{h}$ describes the set of frequencies of the bifurcations of γ_a, where $\sigma_1, \sigma_2, \ldots, \sigma_s$ are the shapes of the subtrees inside the s external frames: $\tilde{\mathcal{G}}_1, \ldots, \tilde{\mathcal{G}}_s$ are the subgraphs of $\tilde{\mathcal{G}}$ for a fixed $\mathcal{G}_a$ contained in the corresponding s external marked boxes, $h_1, \ldots, h_s$ are the frequencies of the next unmarked boxes, γ_a is the previous γ amputated of its s framed parts which have to be substituted with points with the appropriate index β, $\tilde{\mathcal{G}}_a$ is, analogously, $\tilde{\mathcal{G}}$ where the marked boxes are shrunk to points. A graphical example is the following one;

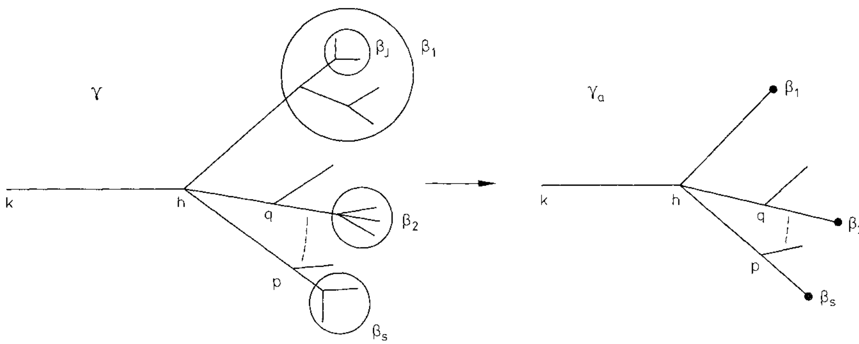


Fig. 7.4

Of course $\int dX_{\mathcal{G}_a}$ is an integration over less variables, part of the integration being factorized inside the $r_\beta^{(N)}(\ )$'s. The first result that we must obtain is an estimate of

$$\sum_{\substack{\tilde{\mathcal{G}}_a \\ \tilde{\mathcal{G}}\,\mathrm{fixed}}}^{*} V^{(k)}_{\gamma_a, \tilde{\mathcal{G}}_a}(\varphi) = \sum_{\substack{\tilde{\mathcal{G}}_a \\ \tilde{\mathcal{G}}\,\mathrm{fixed}}}^{*} \int dX_{\tilde{\mathcal{G}}_a} \bar{K}_{\gamma_a, \tilde{\mathcal{G}}_a}(X_{\tilde{\mathcal{G}}_a}, k) : P^{[\leq k]}_{\tilde{\mathcal{G}}}(\varphi) :. \tag{7.16}$$

This is the content of

Theorem 1. *Let $\tilde{\mathcal{G}}$ have r external lines starting from $\bar{r} \leq r$ vertices, then, given $\varepsilon > 0$:*

$$\begin{aligned}
&\sum_{\substack{\tilde{\mathcal{G}}_a \\ \tilde{\mathcal{G}}\,\mathrm{fixed}}}^{*} \int_{\Lambda^{m+s-\bar{r}} \times (\Delta_1 \times \ldots \times \Delta_{\bar{r}})} dX_{\tilde{\mathcal{G}}_a} |\bar{K}_{\gamma_a, \tilde{\mathcal{G}}_a}(X_{\tilde{\mathcal{G}}_a}; k)| \\
&\leq \sum_{\{n_B\}_{\gamma_a}} \sum_{\substack{\mathcal{G}_a \\ \left\{\begin{subarray}{l} \tilde{\mathcal{G}}\,\mathrm{fixed} \\ \{n_B\}_{\gamma_a}\,\mathrm{fixed}\end{subarray}\right.}} \frac{\chi(\gamma_a, \mathcal{G}_a)}{n(\sigma_a)} \\
&\quad \cdot \int_{\Lambda^{m+s-\bar{r}} \times (\Delta_1 \times \ldots \times \Delta_{\bar{r}})} dX_{m+s-\bar{r}} d\eta_1 \ldots d\eta_{\bar{r}} |K_{\gamma_a; \mathcal{G}_a, \beta_{\mathcal{G}_a}}(X_{\mathcal{G}_a}; k)| \\
&\leq \bar{C}_\varepsilon^{m+s} e^{-\kappa \gamma^k d(\Delta_1, \ldots, \Delta_{\bar{r}})} \prod_{\{B\}_{\gamma_a}} \gamma^{(1-\varepsilon)(h(B) - h(B'))}
\end{aligned} \tag{7.17}$$

where

a) *Σ_V runs over the seven kinds of fields we have introduced φ, $\partial\varphi$, D, S, D^1, S^1, T, and $d(V)$ is their "k-dimension", where here k should be, more appropriately, $h(B')$, but this is unimportant as the "k-dimension" is k-independent, therefore $d(\varphi) = 1$, $d(\partial\varphi) = 2$, $d(D) = 2$, $d(S) = d(D^1) = 3$, $d(S^1) = d(T) = 4$, $n_B(V)$ is the number of external lines of type V going out from the box B.*

b) *m is the number of vertices of $\mathcal{G}$ which are not enclosed in any marked box, S is the number of external marked boxes, $(m+s) \leq n$.*

c) *The* $^\varepsilon$ *dependence originates from Lemma 1.*

d) $e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_{\bar{r}})}$, $\kappa>0$, *is part of the exponential decay factors of the covariances;* $\Delta_1,\ldots,\Delta_{\bar{r}}$ *are tesserae of linear size* γ^{-k}.

e) $\{B\}_{\gamma_a}$ *is the set of unmarked boxes drawn on* $\mathcal{G}_\alpha$ *following the prescription given by* γ_a.

f) B' *is the first box containing* B, $h(B)$, *and* $h(B')$, *which satisfy* $h(B)>h(B')$ *are the associated frequencies. If* B *is the largest box* $h(B')=k(\gamma_a)$.

g) *In the last inequality of* (7.17) *we used* $\sum_{\substack{\mathcal{G}_a \\ \{\tilde{\mathcal{G}}\text{ fixed} \\ \{n_B\}_{\gamma_a}\text{ fixed}}} \frac{\chi(\gamma_a,\mathcal{G}_a)}{n(\sigma_a)} \leqq C_\varepsilon^{m+s} e^{\varepsilon \sum_{B\in\gamma_a} n_B}$ *for an arbitrary* $\varepsilon>0$ *and an appropriate* C_ε, *which follows immediately from Lemma 1.*

The proof of the theorem is in Appendix A of (II).

Remarks. a) The crucial information of the theorem is that, given an amputated tree γ_a and a compatible graph $\mathcal{G}_a$, the contribution to $V(\gamma_a)$ of $(\gamma_a,\mathcal{G}_a,B_{\mathcal{G}_a})$ can be easily estimated just associating to each internal line of γ_a (except the final ones) a convergence factor

$$\gamma^{-(h(B)-h(B'))\left[\sum_V d(V)n_B(V)-4\right]}, \tag{7.18}$$

where $h(B)$ and $h(B')$ are the frequencies of the two ends of the line and $d(V)$ is the "$h(B')$-dimension" of V and $\left[\sum_V d(V)n_B(V)-4\right]$ is always $\geqq 1$.

b) Remembering Eqs. (7.8), …, (7.14), Theorem 1 and the definition of $r_\beta^{(N)}(\sigma,h)$, it should be clear that we have proved the finiteness, when $N\to\infty$, of the terms of the formal series in g of $V_R^{(k)}(\varphi^{[\leqq k]})$, if we prove that the $r_\beta^{(N)}(\sigma,h)$'s are bounded uniformly in N, by a polynomial in h. In fact we are going to prove a more refined bound on the $r_\beta^{(N)}(\sigma,h)$'s which allows us to get the more accurate estimates of Eq. (2.24). This is a more complicated task which still requires some work.

We define

$$\bar{r}_{\beta_i}^{(N)}(w_i,\tilde{\mathcal{G}}_i;h_i)\equiv \sum_{\substack{\sigma_i \\ \{v(\sigma_i)=n_i \\ f(\sigma_i)=f_i \\ w(\sigma_i)=w_i}} r_{\beta_i}^{(N)}(\sigma_i,\tilde{\mathcal{G}}_i;h_i), \tag{7.19}$$

where w_i is an index implicitly defined, see Eq. (7.14), which plays the role of w relative to the shape σ_i of the subtree γ_i of γ. The second crucial theorem we need gives us an estimate of $\bar{r}_{\beta_i}^{(N)}(\ )$.

Theorem 2.

$$|\bar{r}_{\beta_i}^{(N)}(w_i,\tilde{\mathcal{G}}_i;h_i)|\leqq \bar{\mu}(w_i)f_i!\sum_{j=0}^{f_i}\frac{(bh_i)^j}{j!}, \tag{7.20}$$

where $\bar{\mu}(w_i)$ *is a function bounded by a constant independent of* N, $\tilde{\mathcal{G}}_i$, β_i, h_i, f_i *is* $1+\#$ *(frames inside the* i^{th} *one),* b *is* >0 *and will be specified later, insuring the proof of this theorem which will be postponed for a while.*

Remark. The $r_\beta^{(N)}(\ )$'s are the coefficients of the framed parts of the trees; these parts control the most divergent behaviour in the order n of g^n of $V_R^{(k)}$. Theorem 2 is essentially an estimate, not very crude, of the dependence of these terms on their frequencies (the log of the momenta in the more standard Feynman graph language) and on the number of the vertices.

We shall discuss these matters in a greater detail in (II).

From Theorems 1 and 2

$$\int_{\Delta_1\times\ldots\times\Delta_{\bar r}} d\eta_1\ldots d\eta_{\bar r}\int_{\Delta^{n-\bar r}} d(X_{\tilde{\mathscr{G}}}\backslash\eta)\,|K_{(n,f,w,\tilde{\mathscr{G}})}(X_{\tilde{\mathscr{G}}};k)|$$
$$\leqq e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_{\bar r})}\sum_{\substack{\gamma_a\\ \left\{\begin{array}{l}k(\gamma_a)=k\\ v(\gamma_a)=m+s\\ f(\gamma_a)=0\end{array}\right.}}\bar C_\varepsilon^{m+s}$$
$$\cdot\left\{\prod_{\{B\}_{\gamma_a}}\gamma^{-(1-\varepsilon)(h(B)-h(B'))}\prod_1^s{}_i\left(\bar\mu(w_i)f_i!\sum_0^{f_i}{}_j\frac{(bh_i)^j}{j!}\right)\right\} \tag{7.21}$$

We want to estimate the right hand side of (7.25), obtained observing that, remembering that every unframed bifurcation has an R, the following inequality holds, for all B and $\beta_{\mathscr{G}_a}$:

$$\left[\sum_V d(V)n_B(V)-4\right]\geqq 1 \tag{7.22}$$

[see the proof of Theorem 1 in Appendix A of (II)]. Moreover, remembering the definition of γ_a, summing over all the possible γ_a with a fixed number of final lines: $v(\gamma_a)$ means to choose an arbitrary set of frequencies with a partial order between them (which fixes the shape of the tree apart from the position of the final lines), to sum over all the possible values of the frequencies respecting this partial ordering, over all the possible sets and, finally, over all the possible ways of arranging the $v(\gamma_a)$ final lines.

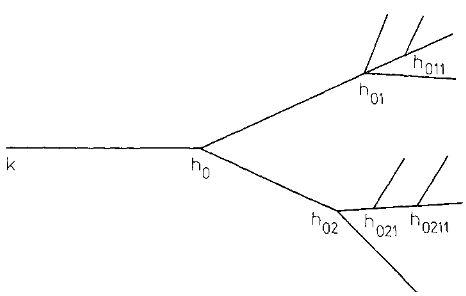


Fig. 7.5

The tree of Fig. 7.5 is completely defined assigning the partial order between the frequencies, the value of the frequency at each of its bifurcations and how the $v(\gamma_a)=8$ final lines are distributed. γ_a is a tree (amputated) with s final marked ends and m normal ones.

Perform the $\sum_{\gamma_a}$ in the following way: call $\tilde\gamma$ a tree (without frames) with only s marked ends; for fixed $\tilde\gamma$ consider all the γ_a which can be obtained adding in all the possible ways other branches so as to have m normal final lines in addition to the s marked ones. Given γ_a, call $\varrho(\gamma_a)=\tilde\gamma$ the "marked" tree obtained deleting all the subtrees which end with unmarked final lines (Fig. 7.6).

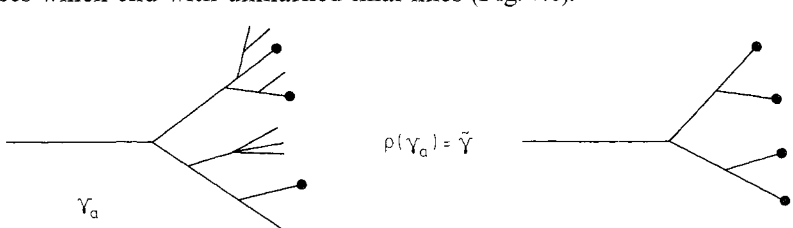


Fig. 7.6

We can decompose $\sum_{\gamma_a}$ in the following way,

$$\sum_{\substack{k(\gamma_a)=k\\ v(\gamma_a)=m+s\\ f(\gamma_a)=0}}{}_{\gamma_a} = \sum_{\substack{k(\tilde\gamma)=k\\ v(\tilde\gamma)=s\\ f(\tilde\gamma)=0}}{}_{\tilde\gamma} \sum_{\substack{\varrho(\gamma_a)=\tilde\gamma\\ v(\gamma_a)=m+s}}{}_{\gamma_a}, \tag{7.23}$$

where $\tilde\gamma$ fixed implies that the frequencies at its bifurcations are fixed. We call these bifurcations $\{p(B)\}$. Those $p(B)$'s which bifurcate into final branches are labelled p_i. Therefore

$$\int_{\Delta_1\times\ldots\times\Delta_{\bar r}} d\eta_1\ldots d\eta_r \int_{\Delta^{n-\bar r}} d(X_{\tilde{\mathscr G}}\backslash\eta)|K_{(n,f,w,\tilde{\mathscr G})}(X_{\tilde{\mathscr G}};k)|$$
$$\leqq \bar c_\varepsilon^{m+s} e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_r)} \sum_{\substack{k(\gamma)=k\\ v(\gamma)=s\\ f(\gamma)=0}}{}_{\tilde\gamma} \left[\prod_{\{B\}_{\tilde\gamma}} \gamma^{-1/4(p(B)-p(B'))}\right.$$
$$\left.\cdot\left(\prod_1^s{}_i\, \bar\mu(w_i) f_i! \sum_0^{f_i}{}_j \frac{(bp_i)^j}{j!}\right)\left(\sum_{\substack{\gamma_a\\ \varrho(\gamma_a)=\tilde\gamma\\ v(\gamma_a)=m+s}} \prod_{\{B\}_{\gamma_a}} \gamma^{-1/4(q(B)-q(B'))}\right)\right], \tag{7.24}$$

where the right hand side is independent from $\tilde{\mathscr G}$. Looking at the $\left(\sum_{\gamma_a}\ldots\right)$ term of the right-hand side of (7.24) we want to estimate it, removing the constraint $\varrho(\gamma_a)=\tilde\gamma$. This is provided by the following lemma:

Lemma 2. *Let γ be an undressed tree with at most n final branches, then*

$$\sum_\gamma \prod_{\{B\}_\gamma} \gamma^{-1/4(q(B)-q(B'))} \leqq c_2^n. \tag{7.25}$$

The proof of this lemma is in Appendix A of (II).

Using this lemma we have

$$\int_{\Delta_1\times\ldots\times\Delta_{\bar r}} d\eta_1\ldots d\eta_{\bar r} \int_{\Delta^{n-\bar r}} d(X_{\tilde{\mathscr G}}\backslash\eta)|K_{(n,f,w,\tilde{\mathscr G})}(X_{\tilde{\mathscr G}};k)|$$
$$\leqq (\bar c_\varepsilon c_2)^{m+s} \sum_{\substack{k(\tilde\gamma)=k\\ v(\tilde\gamma)=s\\ f(\tilde\gamma)=0}}{}_{\tilde\gamma} \left[\prod_{\{B\}_{\tilde\gamma}} \gamma^{-1/4(p(B)-p(B'))}\right.$$
$$\left.\cdot\prod_1^s{}_i\left(\bar\mu(w_i) f_i! \sum_0^{f_i}{}_j \frac{(bp_i)^j}{j!}\right)\right] e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_{\bar r})}. \tag{7.26}$$

Remark. The advantage of the bound (7.26) with respect to (7.21) is that now the sum on the right-hand side is restricted to those trees with marked ends only.

We want to get an estimate of the right-hand side of (7.26) more useful to us. Rewrite it in the following way, denoting, as usual, by σ the shape of a tree

$$[\text{r.h.s. }(7.26)] = \sum_{v(\sigma)=s}{}_\sigma \sum_{\substack{k(\tilde\gamma)=k\\ \sigma(\tilde\gamma)=\sigma\\ f(\tilde\gamma)=f}}{}_{\tilde\gamma} (\bar c_\varepsilon c_2)^{m+s}\left[\prod_{\{B\}_{\tilde\gamma}}\ldots\right]$$
$$\leqq \sum_{\substack{v(\sigma)=s\\ f(\sigma)=0}}{}_\sigma \left\{\sum_{\substack{k(\tilde\gamma)=k\\ \sigma(\tilde\gamma)=\sigma}}{}_{\tilde\gamma} \prod_1^s{}_i\left(\bar\mu(w_i) f_i! \sum_0^{f_i}{}_j \frac{(bh_i)^j}{j!}\right)\prod_{\{B\}_{\tilde\gamma}} \gamma^{-1/8(h(B)-h(B'))}\right.\ .$$
$$\left.\cdot (c_1c_2)^{m+s}\left\{\sum_{\substack{k(\tilde\gamma)=k\\ \sigma(\tilde\gamma)=\sigma}}{}_{\tilde\gamma} \prod_{\{B\}_{\tilde\gamma}} \gamma^{-1/8(h(B)-h(B'))}\right\}\right) \tag{7.27}$$

(all the terms of the sum are >0). The two { }'s of (7.27) have to be estimated separately

Lemma 3.

$$\left\{\sum_{\substack{k(\tilde{\gamma})=k\\ \sigma(\tilde{\gamma})=\sigma}}\tilde{\gamma}\left(\prod_1^s{}_i\,\bar{\mu}(w_i)f_i!\sum_0^{f_i}{}_{j_i}\frac{(bh_i)^{j_i}}{j_i!}\right)\prod_{\{B\}\tilde{\gamma}}\gamma^{-1/8(h(B)-h(B'))}\right\}$$

$$\leqq C_3^s\prod_1^s{}_i\,\bar{\mu}(w_i)\left(\sum_1^s{}_i f_i\right)!\sum_0^{\sum_1^s{}_i f_i}{}_j\frac{(bk)^j}{j!}. \tag{7.28}$$

The proof of this lemma is in Appendix A of (II).

Remark. The bound of inequality (7.28) is independent from σ.

Collecting together Eq. (7.27), Lemmas 2 and 3 we get

$$\int_{\Delta_1\times\ldots\times\Delta_{\bar{r}}}d\eta_1\ldots d\eta_{\bar{r}}\int_{\Lambda^{n-\bar{r}}}d(X_{\tilde{\mathscr{G}}}\backslash\eta)|K_{(n,f,w,\tilde{\mathscr{G}})}(X_{\tilde{\mathscr{G}}};k)|$$

$$\leqq c_4^m c_5^s\left[\left(\prod_1^s{}_i\,\bar{\mu}(w_i)\right)\left(\sum_1^s{}_i f_i\right)!\sum_0^{\sum_1^s{}_i f_i}{}_j\frac{(bk)^j}{j!}\right]e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_r)}, \tag{7.29}$$

which is essentially the inequality (7.9). We are now in the right position to prove Theorem 2, which is the last result of this section.

Proof of Theorem 2. The proof will be by induction. We have

$$\bar{r}_{\beta_i}^{(N)}(w_i,\tilde{\mathscr{G}}_i;q_i)=\sum_{\substack{v(\sigma_i)=v_i\\ f(\sigma_i)=f_i\\ w(\sigma_i)=w_i}}{}_{\sigma_i}\;r_{\beta_i}^{(N)}(\sigma_i,\tilde{\mathscr{G}}_i;q_i)\,. \tag{7.19}$$

As $w(\sigma_i)$ is fixed $=w_i$ it follows that the number of external frames in σ_i is fixed $=s_i$ (also the number of frames inside these frames is fixed).

Remembering the definition of a "amputated" tree γ_a, (see Fig. 7.6) we can write

$$\bar{r}_{\beta_i}^{(N)}(w_i,\tilde{\mathscr{G}}_i;q_i)=\sum_{\substack{v(\sigma_{i,a})=m_i+s_i\\ f(\sigma_{i,a})=0}}{}_{\sigma_{i,a}}\left[\sum_{\substack{\sigma_{i,a}\,\text{fixed}\\ f(\sigma_i)=f_i\\ w(\sigma_i)=w_i}}{}_{\sigma_i}\;r_{\beta_i}^{(N)}(\sigma_i,\tilde{\mathscr{G}}_i;q_i)\right]. \tag{7.30}$$

Graphically

$$:P_{\beta_i}(\varphi^{[\leqq q_i]}):\bar{r}_{\beta_i}^{(N)}(w_i,\tilde{\mathscr{G}}_i;q_i)=\sum_{\substack{\sigma_{i,a}\\ \{\}}}\sum_{\substack{\sigma_i\\ \{\}}}$$

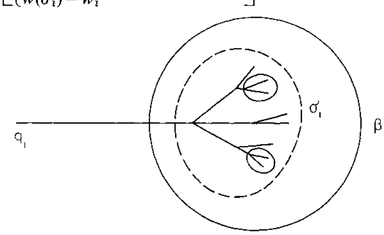


Fig. 7.7

where $\sigma_{i,a}$ is, in the picture where we drew a possible σ_i,

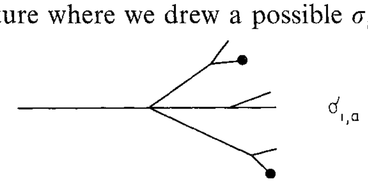


Fig. 7.8

To compute (7.30) we have to proceed in this way:

a) Remove the external frame, call $\hat{\gamma}_i$ the tree with shape σ_i and without the letter R appended to its last bifurcation of frequency h (due to the presence of the frame we have just erased) and with root $k(\hat{\gamma}_i) = -1$. Then compute $V(\hat{\gamma}_i)$ and erase the final Wick polynomial.

b) Perform the sum over $\hat{\gamma}_i$ as suggested by (7.30), namely first over the $\hat{\gamma}_i$ with $\hat{\gamma}_{i,a}$ fixed and then over the $\hat{\gamma}_{i,a}$. As to sum over $\hat{\gamma}_i$ with $\hat{\gamma}_{i,a}$ fixed means to sum over the subtrees inside the frames $1, 2, \ldots, s_i$ in this computation each line ending with a frame has to be estimated by $\bar{r}^{(N)}_{\beta_j}(w_{i,j}, \tilde{\mathscr{G}}_{i,j}; p_j)$, which by the inductive assumption satisfies Theorem 2, where $w_{i,j}$ is the index referring to the subtree contained in the j^{th} framed part of $\hat{\gamma}_i$ and $\tilde{\mathscr{G}}_{i,j}$ is the corresponding part of the graph $\tilde{\mathscr{G}}_i$. Therefore

$$\bar{r}^{(N)}_{\beta_i}(w_i, \tilde{\mathscr{G}}_i; q_i) = \sum^*_{\substack{\hat{\gamma}_{i,a} \\ \left\{\begin{array}{l} v(\hat{\gamma}_{i,a}) = m_i + s_i \\ f(\hat{\gamma}_{i,a}) = 0 \\ k(\hat{\gamma}_{i,a}) = -1 \end{array}\right.}} \left[\sum^*_{\tilde{\mathscr{G}}_{i,a}} \int_{\Lambda^{m_i+s_i}} dX_{\tilde{\mathscr{G}}_{i,a}} \bar{K}_{\hat{\gamma}_{i,a}, \tilde{\mathscr{G}}_{i,a}}(X_{\tilde{\mathscr{G}}_{i,a}}; -1) \cdot \left(\prod_{j=1}^{s_i} \bar{r}^{(N)}_{\beta_j}(w_{i,j}, \tilde{\mathscr{G}}_{i,j}; p_j) \right) \right], \tag{7.31}$$

where $\sum^*$ over $\hat{\gamma}_{i,a}$ means to sum with the condition that the frequency h of the last bifurcation of $\hat{\gamma}_{i,a}$ has to be summed from 0 to q_i instead of from 0 to N and $\sum^*_{\tilde{\mathscr{G}}_{i,a}}$ is defined.

It should be clear, after a moment of reflection, that apart from this condition on the last sum and from the fact that the last bifurcation of $\hat{\gamma}_{i,a}$ is not "renormalized" this computation is exactly of the same type as that performed to estimate the left-hand side (7.21)

$$\int dX_{\tilde{\mathscr{G}}} |K_{(n,f,w,\tilde{\mathscr{G}})}(X_{\tilde{\mathscr{G}}}; k)|, \tag{7.21}$$

where now $k = -1$ and $(n, f, w, \tilde{\mathscr{G}}) = (n_i, f_i, w_i, \tilde{\mathscr{G}}_i)$. Therefore one can repeat all the steps done to estimate (7.21), except the last sum over h and taking care of the missing R, obtaining, as it should be clear by looking at the proof of Lemma 3,

$$\bar{r}^{(N)}_{\beta_i}(w_i, \tilde{\mathscr{G}}_i, q_i) \leqq \sum_{h=0}^{q_i} c_4^{m_i} e_5^{\bar{s}_i} \left(\prod_{j=1}^{\bar{s}_i} \bar{\mu}(w_{i,j}) \right) (\bar{f}_1! \ldots \bar{f}_{\bar{s}_i}!) \cdot \left(\sum_{j_1=0}^{\bar{f}_i} \frac{(bh)^{j_1}}{j_1!} \ldots \sum_{j_{\bar{s}_i}=0}^{\bar{f}_{\bar{s}_i}} \frac{(bh)^{j_{\bar{s}_i}}}{j_{\bar{s}_i}!} \right), \tag{7.32}$$

where m_i is the number of vertices of $\tilde{\mathscr{G}}_i$ not enclosed in any marked box and $\bar{s}_i \leqq s_i$ is the number of lines with marked ends which merge into the last bifurcation of frequency h, Fig. 7.9.

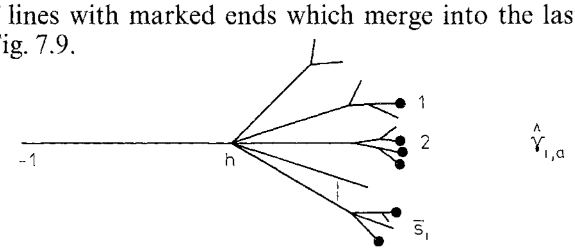


Fig. 7.9

Moreover remembering the definition of f_i we have

$$\bar{f}_l = \sum_{j \in \mathscr{B}_l} f_{i,j}, \tag{7.33}$$

where $\mathscr{B}_l$ is the set of external frames which end in the l^{th} line which bifurcates from "h". Therefore

$$\sum_{1}^{\bar{s}_i}{}_l \bar{f}_l = \sum_{1}^{s_i}{}_j f_{i,j} = f_i - 1 , \tag{7.34}$$

where $f_i - 1$ is the total number of frames inside the external one associated to $\bar{r}_{\beta_i}^{(N)}(w_i, \tilde{\mathscr{G}}_i; q_i)$. Then, omitting the index i,

$$\begin{aligned}
\bar{r}_{\beta}^{(N)}(w, \tilde{\mathscr{G}}; q) &\leqq c_4^m c_5^s \left(\sum_{1}^{s}{}_j \bar{\mu}(w_j) (f_1 + \dots + f_s)! \right. \\
&\quad \left. \cdot \sum_{0}^{q}{}_h \sum_{0}^{(f_1 + \dots + f_s)}{}_r \frac{(bh)^r}{r!} \left\{ \sum_{\substack{j_1 \dots j_{\bar{s}} \\ j_1 + \dots + j_{\bar{s}} = r}}^{\bar{f}_1, \bar{f}_{\bar{s}}} \frac{(j_1 + \dots + j_{\bar{s}})!}{j_1! \dots j_{\bar{s}}!} \frac{\bar{f}_1! \dots \bar{f}_{\bar{s}}!}{(\bar{f}_1 + \dots + \bar{f}_{\bar{s}})!} \right\} \right. \\
&\leqq [\text{with the use of the simple Lemma 3a discussed in (II)}] \\
&\leqq c_4^m c_5^s \left(\prod_{1}^{s}{}_j \bar{\mu}(w_j) \right) (f_1 + \dots + f_s)! \sum_{0}^{q}{}_h \sum_{0}^{(f_1 + \dots + f_s)}{}_r \frac{(bh)^r}{r!} \\
&\leqq c_6^{m+s} \left(\sum_{1}^{s}{}_j \bar{\mu}(w_j) \right) (f_1 + \dots + f_s)! \sum_{0}^{(f_1 + \dots + f_s + 1)}{}_r \frac{(bq)^r}{r!} \\
&= [\text{see (7.34)}] = c_6^{m+s} \left(\prod_{1}^{s}{}_j \bar{\mu}(w_j) \right) \frac{f!}{f} \sum_{0}^{f}{}_j \frac{(bq)^j}{j!} .
\end{aligned} \tag{7.35}$$

In order to complete the proof of Theorem 2, the following inequality must hold

$$\bar{\mu}(w) \leqq c_6^{m+s} \prod_{1}^{s}{}_j \bar{\mu}(w_j) . \tag{7.36}$$

Iterating this relation we get

$$\bar{\mu}(w) \leqq c_6^{(n - n_{\text{int}}) + f - 1} \prod_{\substack{k \\ \text{(innermost)} \\ \text{boxes}}} \bar{\mu}(w_k) , \tag{7.37}$$

where n is the total number of final branches (vertices of the associated graph) prescribed by w, n_{int} is the number of "vertices" contained in the innermost boxes or the number of final branches contained in the innermost framed parts, in the tree language, f is the total number of frames. If we make the following inductive assumption

$$\bar{\mu}(w_j) \leqq c_7^{n_j + f_j - 1} , \tag{7.38}$$

as for the innermost frames $f_j = 1$, it follows that

$$\bar{\mu}(w) \leqq c_6^{(n - n_{\text{int}}) + f - 1} c_7^{n_{\text{int}}} \leqq c_7^{(f-1)+n} \tag{7.39}$$

choosing $c_7 > c_6$ which shows that the inductive assumption

$$\bar{\mu}(w) \leqq c_7^{n + f - 1} \tag{7.40}$$

is reproduced provided it is true that $\bar{\mu}(\tilde{w}) \leqq c_7^n$ if this $\tilde{w}$ refers to a frame which inside has the contribution of all the possible trees without frames ($f = 1$) and n final lines. This is true and follows from inequality (7.32) with $\bar{s}_i = 0$, $\bar{f}_1 = \dots = \bar{f}_{\bar{s}_i} = 0$ and $m_i = n$. Theorem 2 is therefore proved. □

We have the following inequality

$$\begin{aligned}&\int_{\Delta_1\times\ldots\times\Delta_{\bar r}} d\eta_1\ldots d\eta_{\bar r}\int_{\Lambda^{n-\bar r}} d(X_{\bar{\mathscr{G}}}\backslash\eta)K_{(n,f,w,\bar{\mathscr{G}})}(X_{\mathscr{G}};k)\\ &\leqq\left(c_4^m c_5^s c_7^{n+f-1}f!\sum_0^f{}_j\frac{(bk)^j}{j!}\right)e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_r)}\\ &\leqq\left(c_8^n f!\sum_0^f{}_j\frac{(bk)^j}{j!}\right)e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_{\bar r})},\end{aligned}\tag{7.41}$$

where C_8 is a constant uniform in N, k, $\mathscr{G}$, $\beta_{\mathscr{G}}$. This is the inequality (7.9) we promised to prove.

To complete this "technical section" we have still to prove the easy estimates (7.9) which we state below.

Lemma 4.

$$\sum_{\substack{\mathscr{G}\\ w\,\text{fixed}}} 1\leqq c_9^n(n-f)!\tag{7.42}$$

The proof is in the Appendix A of (II).

Lemma 5.

$$\sum_{\substack{w\\ (n,f\,\text{fixed})}} 1\leqq c_{12}^{n+f}.\tag{7.43}$$

The proof is in Appendix A of (II).

Inequality (7.41), Lemma 4, Lemma 5 together allow us to write the following estimate: We rewrite Eq. (7.8) as

$$\begin{aligned}V_R^{(k)}(\varphi)=\sum_n\sum_{\{\underline{a},\underline{b},\underline{c},\underline{d},\underline{e},\underline{f},\underline{g}\}}\int_{\Lambda^n} dX\,V_n^{(k)}(X;\underline{a},\ldots,\underline{g})\\ \cdot\,\varphi_X^{\underline{a}}(\partial\varphi_X)^{\underline{b}}D_X^{\underline{c}}S_X^{\underline{d}}D_X^{1\underline{e}}T_X^{\underline{f}}S_X^{1\underline{g}}:,\end{aligned}\tag{7.44}$$

where $\varphi=\varphi^{[\leqq k]}$ (the same for $\partial\varphi, D, S, \ldots$), $X=(\underline{x}_1,\ldots,\underline{x}_n)$,

$$\begin{aligned}&\underline{a}=(a_1,\ldots,a_n),\quad \underline{b}=(b_1,\ldots,b_n),\quad \underline{c}=\{c_{ij}\}_{i<j},\\ &\underline{d}=\{d_{ij}\}_{i<j},\quad \underline{e}=\{e_{ij}\}_{i<j},\quad \underline{f}=\{f_{ij}\}_{i<j},\quad \underline{g}=\{g_{ij}\}_{i<j}.\end{aligned}\tag{7.45}$$

i,j vary over $(1,\ldots,n)$, $a_i, b_i, c_{ij},\ldots,g_{ij}\in Z\cap[0,4]$

$$\begin{aligned}&:\varphi_X^{\underline{a}}(\partial\varphi_X)^{\underline{b}}D_X^{\underline{c}}S_X^{\underline{d}}D_X^{1\underline{e}}T_X^{\underline{f}}S_X^{1\underline{g}}:\\ &=:\prod_1^n{}_i\,\varphi_{x_i}^{a_i}\partial\varphi_{x_i}^{b_i}\prod_{i<j}D_{x_ix_j}^{c_{ij}}S_{x_ix_j}^{d_{ij}}D_{x_ix_j}^{1e_{ij}}T_{x_ix_j}^{f_{ij}}S_{x_ix_j}^{1g_{ij}}:,\end{aligned}\tag{7.46}$$

then the following result holds. Let us denote $\eta_1,\ldots,\eta_r$ the r independent variables of the fields in the Wick monomial $:\varphi_X^{\underline{a}}(\partial\varphi_X)^{\underline{b}}\ldots S_X^{1\underline{g}}:$. Then $\exists\kappa>0$, $G>0$ N-independent such that

$$\begin{aligned}&\int_{\Delta_1\times\ldots\times\Delta_r} d\eta_1\ldots d\eta_r\int_{\Lambda^{n-r}} d(X\backslash\eta)|V_n^{(k)}(X;\underline{a},\underline{b},\ldots,\underline{g})|\\ &\cdot\left[\prod_1^n{}_i(\gamma^k)^{a_i}(\gamma^{2k})^{b_i}\prod_{i<j}^{(1,n)}(\gamma^k(\gamma^k|x_i-x_j|))^{c_{ij}}(\gamma^k(\gamma^k|x_i-x_j|)^2)^{d_{ij}}\right.\\ &\left.\cdot(\gamma^{2k}(\gamma^k|x_i-x_j|))^{e_{ij}}(\gamma^k(\gamma^k|x_i-x_j|)^3)^{f_{ij}}(\gamma^{2k}(\gamma^k|x_i-x_j|)^2)^{g_{ij}}\right]\\ &\leqq(|\underline{a}|+\ldots+|\underline{g}|)!\,e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_2)}n!\,(gG)^n,\end{aligned}\tag{7.47}$$

which is the estimate (2.27) and

$$|V_n^{(k)}(X;\underline{a},\dots,\underline{g})|=0 \quad \text{if} \quad |\underline{a}|+\dots+|\underline{g}|>4n\,. \tag{7.48}$$

7.5. The Formal Power Series for the Schwinger Functions

Let $\varphi=\varphi^{[\leq N]}$, f a test function. The truncated Schwinger function is

$$\begin{aligned}
S^T_{(N)}(f;p)&=\mathscr{E}^T_{\mathrm{int}}(\varphi(f);p)=\frac{\partial^p}{\partial\theta^p}\log\frac{\mathscr{E}(e^{\theta\varphi(f)+V_R^{(V)}})}{\mathscr{E}(e^{V_R^{(N)}})}\Bigg|_{\theta=0}\\
&=\frac{\partial^p}{\partial\theta^p}\log\mathscr{E}(e^{\theta\varphi(f)+V_R^{(N)}})|_{\theta=0}\\
&=\sum_0^\infty{}_s\frac{1}{s!}\frac{\partial^{p+s}}{\partial\theta^p\partial\tau^s}\log\mathscr{E}(e^{\theta\varphi(f)+\tau V_R^{(N)}})|_{\theta=\tau=0}\\
&=\sum_0^\infty{}_s\frac{1}{s!}\mathscr{E}^T_{[0,N]}\left(\left(\sum_0^N{}_l\,\varphi^{[l]}(f)\right),\,V_R^{(N)};p,s\right)\\
&=\sum_0^\infty{}_s\frac{1}{s!}\sum^{(0,p)}_{\substack{j_0,\dots,j_N\\ \sum_0^N{}_l j_l=p}}\frac{p!}{j_0!\dots j_N!}\mathscr{E}^T_{[0,N]}(\varphi^{[0]}(f),\dots,\varphi^{[N]}(f),V_R^{(N)},j_0,\dots,j_N,s)\\
&=p!\sum_1^p{}_q\sum^{(0,N)}_{k_i<\dots<k_q}\left\{\sum_0^\infty{}_s\sum^{(1,p)}_{\substack{l_1,\dots,l_q\\ \sum_1^q{}_i l_i=p}}\frac{1}{s!l_1!\dots l_q!}\mathscr{E}^T_{[0,N]}(\varphi^{[k_1]}(f),\dots,\varphi^{[k_q]}(f),\,V_R^{(N)};l_i,l_q;s)\right\}.
\end{aligned} \tag{7.49}$$

To express {(7.49)} in the tree formalism we proceed as follows: The following relation holds in general:

We define

$$\begin{aligned}
\tilde V^{(N)}&=V^{(N)}+(V_1+\dots+V_q)\,,\\
e^{\tilde V^{(k)}}&=\mathscr{E}_{[k+1,N]}(e^{\tilde V^{(N)}})\,,
\end{aligned} \tag{7.50}$$

$$\begin{aligned}
\tilde V^{(k)}&=\log\mathscr{E}_{[k+1,N]}(e^{\tilde V^{(N)}})\\
&=\sum^{(0,\infty)}_{l_1\dots l_q,s}\frac{1}{l_1!\dots l_q!s!}\frac{\partial^{(l_1+\dots+l_q+s)}}{\partial\theta_1^{l_1}\dots\partial\theta_q^{l_q}\partial\tau^s}\\
&\quad\cdot\log\mathscr{E}_{[k+1,N]}(e^{\theta_1V_1+\dots+\theta_qV_q+\tau V^{(N)}})|_{\theta_i=\tau=0}\\
&=\sum^{(0,\infty)}_{l_1,\dots,l_q,s}\frac{1}{s!l_1!\dots l_q!}\mathscr{E}^T_{[k+1,N]}(V_1,\dots,V_q,V^{(N)},l_1,\dots,l_q,s)\,.
\end{aligned} \tag{7.51}$$

On the other hand integrating frequency by frequency

$$\tilde V^{(k)}=\sum_{k(\gamma)=k}{}_\gamma\frac{1}{n(\gamma)}\tilde V(\gamma)\,, \tag{7.52}$$

where $\tilde V(\gamma)$ is the contribution of the tree γ to the truncated expectation, remembering that the final lines can be associated to $V_1,\dots,V_q$ and $V^{(N)}$ arbitrarily.

Therefore

$$\sum_{0}^{\infty}{}_s \sum_{\substack{l_1,\dots,l_q\\ \sum_1^q l_i=p}}^{(1,p)} \frac{1}{s!\,l_1!\dots l_q!}\,\mathscr{E}^T_{[k+1,N]}(V_1,\dots,V_q,V^{(N)},l_1,\dots,l_q,s)=\sum_{\substack{k(\gamma)=k\\ p\,\text{final lines}\\ \text{of type}\,(1,2,\dots,q)}}{}_\gamma\ \frac{\tilde V(\gamma)}{n(\gamma)}\,, \tag{7.53}$$

and

$$\frac{1}{l_1!\dots l_q!}\sum_{0}^{\infty}{}_s\frac{1}{s!}\,\mathscr{E}^T_{[k+1,N]}(V_1,\dots,V_q,V^{(N)};l_1,\dots,l_q,s)=\sum_{\substack{k(\gamma)=k\\ \{l_1,\dots,l_q\}}}{}_\gamma\ \frac{\tilde V(\gamma)}{n(\gamma)}\,, \tag{7.54}$$

where $\{l_1,\dots,l_q\}$ means that there are l_1 final lines of type $V_1,\dots,l_q$ of type V_q which we draw wavy.

Define now

$$V_i=\varphi^{[k_i]}(f)\,,\qquad i:1,\dots,q\,,\qquad V^{(N)}=V_R^{(N)}\,, \tag{7.55}$$

then

$$S^T_{(N)}(f,p)=p!\sum_{1}^{p}{}_q\sum_{k_1<\dots<k_q}^{(0,N)}\ \sum_{\substack{l_1,\dots,l_q\\ \sum_1^q l_i=p}}^{(1,p)}\left\{\sum_{\substack{k(\gamma)=-1\\ \{l_1,\dots,l_q\}}}{}_\gamma\ \frac{1}{n(\gamma)}\tilde V(\gamma)\right\}. \tag{7.56}$$

γ are dressed trees, the potential being $V_R^{(N)}+\sum_1^q{}_i\,\varphi^{[k_i]}(f)$ which does not require any more counterterms. The tree γ has p wavy lines and an index δ running from 1 to q. The bifurcations where they merge and the following ones do not require any $\mathscr{R}$-operation, as we are going to prove.

Before that, expand $S^T_{(N)}(f;p)$ in a formal power series of g,

$$\sum_{\substack{k(\gamma)=-1\\ \{l_1,\dots,l_q\}}}{}_\gamma\ \frac{1}{n(\gamma)}\tilde V(\gamma)=\sum_{0}^{\infty}{}_n\sum_{\substack{k(\gamma)=-1\\ v(\gamma)=n\\ \{l_1,\dots,l_q\}}}{}_\gamma\ \frac{1}{n(\gamma)}\tilde V(\gamma)\,, \tag{7.57}$$

where $v(\gamma)$ means the number of lines (final) associated to $-g\int d\xi:\varphi_\xi^4:$, counting also those included in frames. Then

$$\begin{aligned} S^T_{(N)}(f;p)&=\sum_{0}^{\infty}{}_n\,g^n\bar S^T_{(N),n}(f;p)\\ &=\sum_{0}^{\infty}{}_n\,g^n\left\{p!\sum_{1}^{p}{}_q\sum_{k_1<\dots<k_q}^{(0,N)}\ \sum_{\substack{l_1,\dots,l_q\\ \sum_1^q l_i=p}}^{(1,p)}\left[\frac{1}{g^n}\sum_{\substack{k(\gamma)=-1\\ v(\gamma)=n\\ \{l_1,\dots,l_q\}}}{}_\gamma\ \frac{1}{n(\gamma)}\tilde V(\gamma)\right]\right\}. \end{aligned} \tag{7.58}$$

We turn now to the estimate of $\bar S^T_{(N),n}(f;p)$: The generic tree of (7.58) is a dressed tree with p final wavy lines

associated to $\varphi^{[k_1]},\dots,\varphi^{[k_q]}$. These lines will merge in normal internal lines of γ between two adjacent bifurcations or in a bifurcation together with other straight lines.

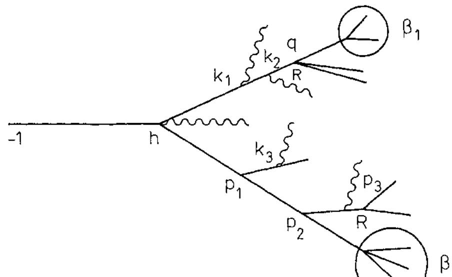


Fig. 7.10

Consider the bifurcation made by a wiggled and a normal line

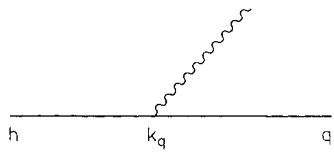


Fig. 7.11

It corresponds to the truncated expectation $\mathscr{E}^T_{[k_q]}$ between $\varphi^{[k_q]}$ and a Wick polynomial made by the sum of monomials of any order, to those of order $\leqq 4$ being the $\mathscr{R}$-operation applied. The contribution of this truncated expectation is proportional to a sum of terms of this kind:

$$\int dx f(x) \int d\xi_1 \dots d\xi_p F(q, k_q; \xi_1, \dots, \xi_p) \mathscr{E}^T_{[k_q]}(\varphi_x^{[k_q]} : P^{[\leqq k_q]}(\varphi, \partial\varphi, \dots, S^1; \xi_1, \dots, \xi_p) :), \tag{7.59}$$

where $P^{[\leqq k_q]}(\ )$ is "normalized" to have "k-dimension" $=0$ and $F(q, k_q; \xi_1, \dots, \xi_p)$ satisfies

$$|F(q, k_q; \xi_1, \dots, \xi_p)| \leqq \gamma^{-(q-k_q)\left[\sum_V d_{k_q}(V) n(V) - 4\right]}. \tag{7.60}$$

$\varphi_x^{[k_q]}$ when contracted with another field of $P^{[\leqq k]}$ gives rise to a covariance proportional to $\gamma^{k+\sigma k}$, the $\gamma^{\sigma k}$ being then cancelled by the $\gamma^{\sigma k}$ present in the denominator of the Wick monomial due to its "normalization." Therefore it remains a factor γ^k but also an exponential factor $e^{-\gamma^k d(x,\xi)}$ which produces a volume factor γ^{-4k}. The net result is a factor γ^{-3k} and the monomial $P^{[\leqq k]}$ has now the degree in fields lowered by one. Nothing really changes if we have a bifurcation of this kind

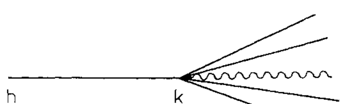


Fig. 7.12

It is easy to recognize that this factor γ^{-3k} is such that along the whole line from the k_q bifurcation to the lowest frequencies there is no longer need of the $\mathscr{R}$ operation.

Therefore extracting from each of these factors, one for each wiggled line, a factor $\gamma^{-\varepsilon k}$ with $\varepsilon > 0$ we can easily deduce the following estimate:

$$\left| \sum_{\substack{k(\gamma) = -1 \\ v(\gamma) = n \\ \{l_1, \dots, l_q\}}}^{\gamma} \frac{1}{n(\gamma)} \tilde{V}(\gamma) \right| \leqq \gamma^{-\varepsilon(k_1 + k_2 + \dots + k_q)} (\bar{c})^n n!, \tag{7.61}$$

which implies

$$|\bar{S}^T_{(N),n}(f, p)| \leqq (p! e^p \|f\|_1^p)(\bar{c})^n n!, \tag{7.62}$$

which is the promised result (2.20).

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