Progress of Theoretical Physics, Vol. 67, No. 5, May 1982

Renormalized Field Theory of Weakly Anisotropic Antiferromagnets under Gaussian Random Fields

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(Received December 12, 1981)

Antiferromagnets with weak spin anisotropy under Gaussian random fields are shown to exhibit just the same bicritical and tetracritical behaviors as those without random fields. Second order transitions occur only for unphysical space dimensionality $4<d<6$ and multicritical points are also subject to $d\to d-2$ rule. It is predicted that weakly anisotropic antiferromagnets such as La-doped $GdAlO_3$ will have 2nd-order bicritical line surrounding 1st-order spin flop plane.

§ 1. Introduction

Nelson-Kosterlitz-Fisher[1)] showed that models of weakly anisotropic antiferromagnets are reduced to the following Hamiltonian density:

$$\mathcal{H}_0=\frac{1}{2}(\nabla\phi_1)^2+\frac{1}{2}m_1{}^2\phi_1{}^2+\frac{1}{2}(\nabla\phi_2)^2+\frac{1}{2}m_2{}^2\phi_2{}^2$$
$$+\frac{u_1}{4!}\phi_1{}^4+\frac{u_2}{4!}\phi_2{}^4+\frac{2u_3}{4!}\phi_1{}^2\phi_2{}^2\,. \tag{1·1}$$

Transverse spin field has n-components; $\phi_1=\phi_1(x)=(\phi_1{}^1, \phi_1{}^2, \cdots, \phi_1{}^n)$, and longitudinal spin field is continuous Ising-like; $\phi_2=\phi_2(x)$. In the previous paper,[2),*)] this model was analyzed by renormalized field theory and the results obtained by Nelson-Kosterlitz-Fisher were confirmed up to $O(\varepsilon)$ $[\varepsilon=4-d]$.

In this paper, we study the multicritical phenomena of weakly anisotropic antiferromagnets under Gaussian random fields described by the Hamiltonian density:

$$\mathcal{H}=\mathcal{H}_0-h_1\phi_1-h_2\phi_2\,, \tag{1·2}$$

where local random fields $h_1=h_1(x)=(h_1{}^1, h_1{}^2, \cdots, h_1{}^n)$ and $h_2=h_2(x)$ are assumed to be Gaussian random variables, that is, their Fourier transforms h_{1q}^{α} and h_{2q} into momentum space satisfy the following properties:

$$\langle h_{1q}^{\alpha}h_{1q'}^{\beta}\rangle=\zeta_1\delta_{\alpha\beta}\delta(\boldsymbol{q}+\boldsymbol{q}')\qquad(\alpha,\beta=1,2,\cdots,n) \tag{1·3a}$$

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[*)] Since Ref. 2) is unpublished, we add a brief summary of Ref. 2) as the Appendix.

and

$$\langle h_{2q} h_{2q'} \rangle = \zeta_2 \delta(\boldsymbol{q}+\boldsymbol{q}'). \tag{1·3b}$$

Here random average is represented by $\langle \cdots \rangle$.

In § 2 we extend the ordinary renormalized field theory to the system with random fields and determine renormalization constants up to $O(\varepsilon)$ [$\varepsilon=6-d$]. In § 3 we show that 2nd-order phase transitions occur only for $4<d<6$, but just the same multicritical phenomena as (1·1) are observed and well-known $d\to d-2$ rule for crossover exponents remain valid even in this case. Existence of bicritical line for weakly anisotropic antiferromagnets under Gaussian random fields is predicted in Discussion (§ 4).

§ 2. Renormalizations for the system with random fields

In order that renormalized masses $\bar{m}_i$ $(i=1, 2)$ are proportional to the deviation from a true transition temperature of $\phi_i(x)$ near multicritical points ($\bar{m}_i \simeq 0$), including the deviation of transition temperatures due to random fields, we should adopt the following renormalization procedures.

In the first place, we renormalize masses and spin fields as usual:[3)]

$$\tilde{\phi}_i(x) = Z_{si}^{-1/2}\phi_i \qquad (i=1, 2) \tag{2·1}$$

and

$$\delta m_i^2 = Z_{si} m_i^2 - \bar{m}_i^2 \qquad (i=1, 2) \tag{2·2}$$

with spin field renormalization constants Z_{si} $(i=1, 2)$. Accordingly random fields are normalized: $\tilde{h}_i = Z_{si}^{1/2} h_i$ $(i=1, 2)$.

Next we eliminate $\tilde{h}_i\tilde{\phi}_i$ $(i=1, 2)$ terms by translating a variable $\tilde{\phi}_i$ by ψ_i: $\tilde{\phi}_i \to \tilde{\phi}_i + \psi_i$ $(i=1, 2)$. To eliminate $\tilde{h}_1{}^\alpha \phi_1{}^\alpha$, for example, there are two alternative ways:

$$-\nabla^2\psi_1{}^\alpha + \bar{m}_1{}^2\psi_1{}^\alpha = \tilde{h}_i{}^\alpha \qquad (\alpha=1, 2, \cdots, n) \tag{2·3a}$$

or

$$-\nabla^2\psi_1{}^\alpha + \bar{m}_1{}^2\psi_1{}^\alpha + \frac{1}{6}Z_{s1}Z_{s2}u_3\psi_1{}^\alpha\tilde{\phi}_2{}^2 + \frac{1}{6}Z_{s1}Z_{s2}u_3\psi_1{}^\alpha\tilde{\phi}^2\psi_2$$
$$+\frac{1}{6}Z_{s1}^2 u_1\psi_1{}^\alpha\psi_1{}^2 + \frac{1}{6}Z_{s1}Z_{s2}u_3\psi_1{}^\alpha\psi_2{}^2 = \tilde{h}_1{}^\alpha . \quad (\alpha=1, 2, \cdots, n) \tag{2·3b}$$

If we determine $\psi_1{}^\alpha$ via (2·3a), other terms in the left-hand side of (2·3b) are driven to the terms as thermal fluctuations. Then deviation of transition temperatures due to random fields is renormalized into mass counter term $\delta m_1{}^2$, so that renormalized mass $\bar{m}_1$ given by $\bar{m}_1{}^2 = Z_{s1} m_1{}^2 - \delta m_1{}^2$ becomes a natural definition of the deviation from a true transition temperature of $\phi_1(x)$, which involves the deviation of transition temperatures due to random fields. As a consequence, renormalized masses $\bar{m}_i$ $(i=1, 2)$ become zero at a true multicritical temperature.

On the other hand, if we choose (2·3b) to determine $\psi_1{}^\alpha$, renormalized mass $\bar{m}_1$ does not become zero at a true transition temperature. We must redefine $\bar{m}_1$ so as to include the deviation due to random fields. In order to avoid these inconveniences, we should select (2·3a) as the definition of $\psi_1{}^\alpha$. In this case $\psi_1{}^\alpha$ and ψ_2 are defined as Gaussian random variables:

$$\langle \psi^\alpha_{1q} \rangle = 0\,, \tag{2·4a}$$

$$\langle \psi^\alpha_{1q} \psi^\beta_{1q'} \rangle = \tilde{\zeta}_1 \delta_{\alpha\beta} \delta(\boldsymbol{q}+\boldsymbol{q}')/(q^2+\bar{m}_1{}^2)^2\,, \qquad (i=1, 2) \tag{2·4b}$$

$$\langle \psi_{2q} \rangle = 0 \tag{2·5a}$$

and

$$\langle \psi_{2q} \psi_{2q'} \rangle = \tilde{\zeta}_2 \delta(\boldsymbol{q}+\boldsymbol{q}')/(q^2+\bar{m}_2{}^2)^2 \tag{2·5b}$$

by the use of (1·3a) and (1·3b). Here the intensity of random fields is normalized as $\tilde{\zeta}_i = Z_{si} \zeta_i$ $(i=1, 2)$.

Thus Hamiltonian density (1·2) becomes

$$\begin{aligned}
\mathcal{H} = {} & \frac{1}{2}(\nabla \tilde{\phi}_1)^2 + \frac{1}{2}\bar{m}_1{}^2 \tilde{\phi}_1{}^2 - \frac{1}{2}\tilde{h}_1 \phi_1 + \frac{1}{2}(\nabla \tilde{\phi}_2)^2 + \frac{1}{2}\bar{m}_2{}^2 \tilde{\phi}_2{}^2 - \frac{1}{2}\tilde{h}_2 \phi_2 \\
& + \frac{Z_{s1}^2 u_1}{4!}\tilde{\phi}_1{}^4 + \frac{Z_{s2}^2 u_2}{4!}\tilde{\phi}_2{}^4 + \frac{2 Z_{s1} Z_{s2} u_3}{4!}\tilde{\phi}_1{}^2 \tilde{\phi}_2{}^2 \\
& + \frac{Z_{s1}^2 u_1}{6}\left[\tilde{\phi}_1{}^2 (\tilde{\phi}_1 \psi_1) + (\tilde{\phi}_1 \psi_1)^2 + \frac{1}{2}\tilde{\phi}_1{}^2 \psi_1{}^2 + (\tilde{\phi}_1 \psi_1)\psi_1{}^2 + \frac{1}{4}\psi_1{}^4 \right] \\
& + \frac{Z_{s2}^2 u_2}{6}\left[\tilde{\phi}_2{}^3 \psi_2 + \frac{3}{2}\tilde{\phi}_2{}^2 \psi_2{}^2 + \tilde{\phi}_2 \psi_2{}^3 + \frac{1}{4}\psi_2{}^4 \right] \\
& + \frac{Z_{s1} Z_{s2} u_3}{6}\left[\tilde{\phi}_1{}^2 \tilde{\phi}_2 \psi_2 + \frac{1}{2}\tilde{\phi}_1{}^2 \psi_2{}^2 + (\tilde{\phi}_1 \psi_1)\tilde{\phi}_2{}^2 + \frac{1}{2}\psi_1{}^2 \tilde{\phi}_2{}^2 \right. \\
& \left. + 2(\tilde{\phi}_1 \psi_1)\tilde{\phi}_2 \psi_2 + (\tilde{\phi}_1 \psi_1)\psi_2{}^2 + \psi_1{}^2 \tilde{\phi}_2 \psi_2 + \frac{1}{2}\psi_1{}^2 \psi_2{}^2 \right] \\
& + \frac{1}{2}(Z_{s1}-1)(\nabla \tilde{\phi}_1)^2 + \frac{1}{2}\delta m_1{}^2 \tilde{\phi}_1{}^2 + \frac{1}{2}(Z_{s2}-1)(\nabla \tilde{\phi}_2)^2 + \frac{1}{2}\delta m_2{}^2 \tilde{\phi}_2{}^2 \\
& + (Z_{s1}-1)(\nabla \tilde{\phi}_1)(\nabla \psi_1) + \delta m_1{}^2 \tilde{\phi}_1 \psi_1 + (Z_{s2}-1)(\nabla \tilde{\phi}_2)(\nabla \psi_2) + \delta m_2{}^2 \tilde{\phi}_2 \psi_2
\end{aligned}$$

$$+\frac{1}{2}(Z_{s1}-1)(\nabla\psi_1)^2+\frac{1}{2}\delta m_1{}^2\psi_1{}^2+\frac{1}{2}(Z_{s2}-1)(\nabla\psi_2)^2+\frac{1}{2}\delta m_2{}^2\psi_2{}^2 . \quad (2\cdot6)$$

Since we assume that random fields are weak and of the same strength, we set $\zeta_i \simeq \zeta$ $(i=1, 2)$. According to the dimensional analysis,[4),5)] kinematical dimensions of u_i $(i=1, 2, 3)$ and ζ are $\dim[u_i]=4-d$ $(i=1, 2, 3)$ and $\dim[\zeta]=2$. Hence perturbational parameters for weak random fields are ζu_i whose dimensions are all $\varepsilon=6-d$.[6),7)] Diagrams such as Fig. 1(a) become irrelevant and diagrams as Fig. 1(b) are relevant. Therefore from the viewpoint of universality, assumption of $\zeta_i \simeq \zeta$ $(i=1, 2)$ is reasonable, for at a fixed point $u_i{}^*=0$, $\zeta_i{}^*=\infty$ and $(\zeta u_i)^*$ is of $O(\varepsilon)$.

Bare 4-point vertex functions are defined as

$$\Gamma_{\zeta i}(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3\boldsymbol{p}_4)=\frac{\langle\!\langle \zeta\phi_i(\boldsymbol{p}_1)\phi_i(\boldsymbol{p}_2)\phi_i(\boldsymbol{p}_3)\phi_i(\boldsymbol{p}_4)\rangle\!\rangle}{\prod_{j=1}^4\langle\!\langle|\phi_i(\boldsymbol{p}_j)|^2\rangle\!\rangle\delta(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3+\boldsymbol{p}_4)} \quad (i=1, 2) \qquad (2\cdot7\text{a})$$

and

$$\Gamma_{\zeta 3}(\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3\boldsymbol{p}_4)=\frac{\langle\!\langle \zeta\phi_1(\boldsymbol{p}_1)\phi_1(\boldsymbol{p}_2)\phi_2(\boldsymbol{p}_3)\phi_2(\boldsymbol{p}_4)\rangle\!\rangle}{\prod_{j=1}^2\langle\!\langle|\phi_1(\boldsymbol{p}_j)|^2\rangle\!\rangle\prod_{j=3}^4\langle\!\langle|\phi_2(\boldsymbol{p}_j)|^2\rangle\!\rangle\delta(\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3+\boldsymbol{p}_4)}, \qquad (2\cdot7\text{b})$$

where statistical-mechanical average is shown by $\langle\!\langle\cdots\rangle\!\rangle$. Renormalized coupling constants are given by

$$v_i=\zeta Z_{si}^2 Z_i^{-1} u_i \qquad (i=1, 2) \qquad (2\cdot8\text{a})$$

and

$$v_3=\zeta Z_{s1} Z_{s2} Z_3^{-1} u_3 \qquad (2\cdot8\text{b})$$

by the use of renormalization constants Z_i $(i=1, 2, 3)$. Here we set $\tilde{\zeta}_i=\zeta$ for

$$Z_{si}=1+O(\varepsilon^2), \qquad (i=1, 2) \qquad (2\cdot9)$$

as is shown in what follows.

Renormalization constants at a multicritical point $(\bar{m}_1=\bar{m}_2=0)$ are uniquely determined up to $O(\varepsilon)$ by imposing the following conditions:

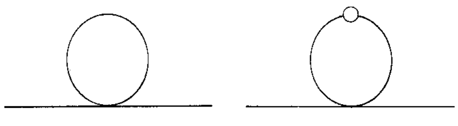

Fig. 1. Typical examples of diagrams which are irrelevant (a) and relevant (b) under Gaussian random fields.

$$\tilde{\Gamma}_i^{(2)}|_{p=0}=0\,, \qquad (i=1,2) \tag{2·10}$$

$$\partial\tilde{\Gamma}_i^{(2)}/\partial p^2|_{\mathrm{sp}}=1 \qquad (i=1,2) \tag{2·11}$$

and

$$\tilde{\Gamma}_{\xi i}|_{\mathrm{sp}}=v_i=\mu^{\varepsilon}g_i\,, \qquad (i=1,2,3) \tag{2·12}$$

where sp stands for renormalization point ($p^2=\mu^2$). From (2·6) together with (2·11), Eq. (2·9) is self-consistently derived.

If we drop out terms contributing only to free energy and canceling out in the statistical-mechanical average, terms smaller than $O(\zeta)$ and terms disappeared in the random average, we finally obtain reduced renormalized Hamiltonian density:

$$\begin{aligned}\mathscr{H}\simeq&\frac{1}{2}(\nabla\tilde{\phi}_1)^2+\frac{1}{2}\bar{m}_1{}^2\tilde{\phi}_1{}^2+\frac{1}{2}(\nabla\tilde{\phi}_2)+\frac{1}{2}\bar{m}_2\tilde{\phi}_2{}^2+\frac{v_1}{4!\,\zeta}\tilde{\phi}_1{}^4+\frac{v_2}{4!\,\zeta}\tilde{\phi}_2{}^4+\frac{2v_3}{4!\,\zeta}\tilde{\phi}_1{}^2\tilde{\phi}_2{}^2\\&+\frac{Z_1v_1}{6\zeta}\left[\tilde{\phi}_1{}^2(\tilde{\phi}_1\psi_1)+(\tilde{\phi}_1\psi_1)^2+\frac{1}{2}\tilde{\phi}_1{}^2\psi_1{}^2\right]+\frac{Z_2v_2}{6\zeta}\left[\tilde{\phi}_2{}^3\psi_2+\frac{3}{2}\tilde{\phi}_2{}^2\psi_2{}^2\right]\\&+\frac{Z_3v_3}{6\zeta}\left[\tilde{\phi}_1{}^2(\tilde{\phi}_2\psi_2)+\frac{1}{2}\tilde{\phi}_1{}^2\psi_2{}^2+(\tilde{\phi}_1\psi_1)\tilde{\phi}_2{}^2+\frac{1}{2}\psi_1{}^2\tilde{\phi}_2{}^2\right]\\&+\frac{1}{2}(Z_{s1}-1)(\nabla\tilde{\phi}_1)^2+\frac{1}{2}\delta m_1{}^2\tilde{\phi}_1{}^2+\frac{1}{2}(Z_{s2}-1)(\nabla\tilde{\phi}_2)^2+\frac{1}{2}\delta m_2{}^2\tilde{\phi}_2{}^2\\&+\frac{(Z_1-1)v_1}{4!\,\zeta}\tilde{\phi}_1{}^4+\frac{(Z_2-1)v_2}{4!\,\zeta}\tilde{\phi}_2{}^4+\frac{2(Z_3-1)v_3}{4!\,\zeta}\tilde{\phi}_1{}^2\tilde{\phi}_2{}^2\,.\end{aligned} \tag{2·13}$$

Renormalization constants are self-consistently determined up to $O(\varepsilon)$ by imposing the conditions (2·12). It follows that $Z_i=1+O(\varepsilon)$ $(i=1,2,3)$, so that we can set $Z_i=1$ in $Z_iv_i/6\zeta$ $(i=1,2,3)$.

Diagrammatic equations of renormalized 2-point vertices $\tilde{\Gamma}_i^{(2)}$ $(i=1,2)$ and 4-point vertices $\tilde{\Gamma}_{\xi i}$ $(i=1,2,3)$ are sketched in Figs. 2 and 3 respectively. From diagrams of $\tilde{\Gamma}_i^{(2)}$ $(i=1,2)$, results (2·9) are confirmed. These results are unaltered even when the relations $\zeta_i=Z_{si}\zeta$ $(i=1,2)$ are preserved in (2·13). Polarization parts at a multicritical point ($\bar{m}_1=\bar{m}_2=0$) are given by[2),4)]

$$\bar{\Pi}_i(\mu/\Lambda)=-\frac{1}{64\pi^3}\ln(\mu/\Lambda), \qquad (i=1,2,3) \tag{2·14}$$

where Λ is the upper cutoff of momentum. Renormalization constants Z_i $(i=1,2,3)$ are determined via diagrammatic equations (Fig. 3) to give the following results:

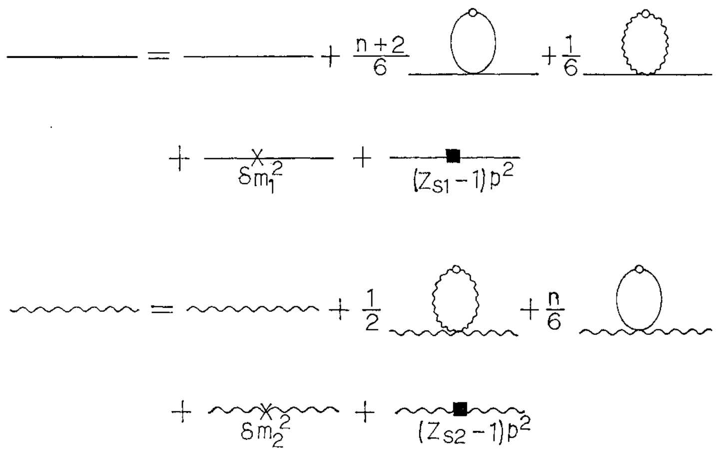

Fig. 2. Diagrammatic equations of renormalized 2-point vertex functions $\tilde{\Gamma}_i^{(2)}$ ($i=1$, 2) up to $O(\varepsilon)$ and $O(\zeta)$. Four lines ——, ~~~, —○— and ~○~ represent $\tilde{\phi}_1$, $\tilde{\phi}_2$, $\langle|\psi_{1q}|^2\rangle$ and $\langle|\psi_{2q}|^2\rangle$, respectively.

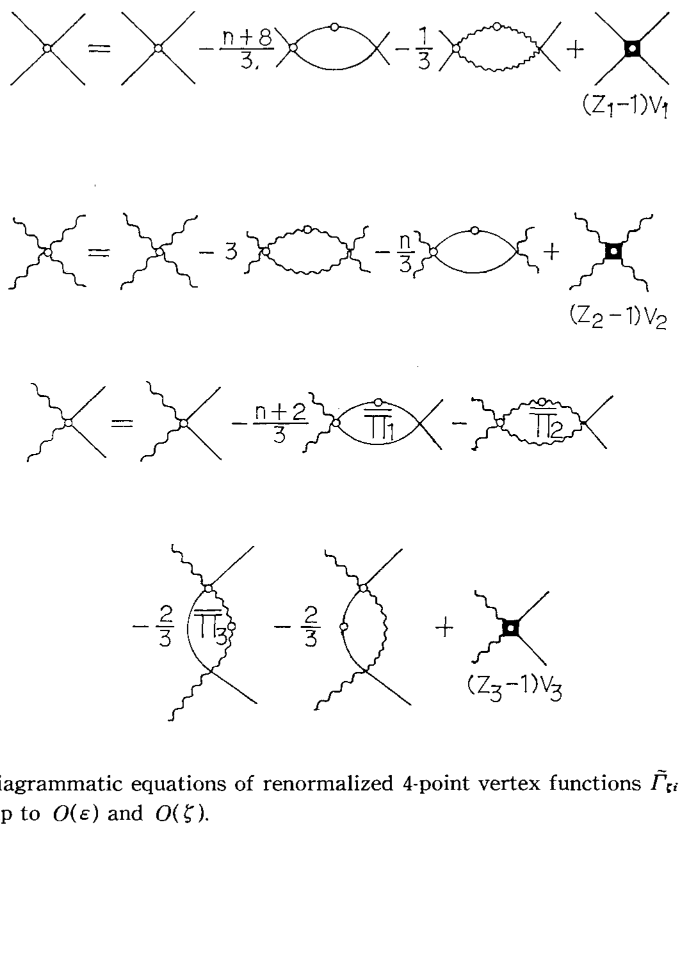

Fig. 3. Diagrammatic equations of renormalized 4-point vertex functions $\tilde{\Gamma}_{\zeta i}$ ($i=1$, 2, 3) up to $O(\varepsilon)$ and $O(\zeta)$.

$$Z_1 = 1 - \frac{1}{192\pi^3}[(n+8)g_1 + g_3^2/g_1]\ln(\mu/\Lambda), \tag{2·15a}$$

$$Z_2 = 1 - \frac{1}{192\pi^3}[9g_2 + ng_3^2/g_2]\ln(\mu/\Lambda) \tag{2·15b}$$

and

$$Z_3 = 1 - \frac{1}{192\pi^3}[(n+2)g_1 + 3g_2 + 4g_3]\ln(\mu/\Lambda). \tag{2·15c}$$

Note that (2·15a)~(2·15c) differ from (A·1a)~(A·1c) in the Appendix only by multiplicative constant $1/192\pi^3$.

§ 3. β functions and the stability of fixed points

Dimensionless renormalized coupling constants g_i ($i=1, 2, 3$) are given by (2·9). On the other hand, dimensionless bare coupling constants g_{i0} ($i=1, 2, 3$) are defined as

$$v_{i0} = \zeta u_i = \Lambda^\varepsilon g_{i0}\,. \qquad (i=1, 2, 3) \tag{3·1}$$

From (2·8a), (2·8b), (2·12), (2·15a)~(2·15c) and (3·1), we derive β functions given by

$$\beta_i = g_i \mu \frac{\partial}{\partial \mu}\bigg|_{g_{10}, g_{20}, g_{30}, \Lambda} \ln g_i\,. \qquad (i=1, 2, 3) \tag{3·2}$$

It follows that

$$\beta_1 = -\varepsilon g_1 + \frac{n+8}{192\pi^3}g_1^2 + \frac{1}{192\pi^3}g_3^2\,, \tag{3·3a}$$

$$\beta_2 = -\varepsilon g_2 + \frac{3}{64\pi^3}g_2^2 + \frac{n}{192\pi^3}g_3^2 \tag{3·3b}$$

and

$$\beta_3 = g_3\left[-\varepsilon + \frac{n+2}{192\pi^3}g_1 + \frac{1}{64\pi^3}g_2 + \frac{1}{48\pi^3}g_3\right]. \tag{3·3c}$$

In order to examine the stability of fixed points of β functions, we should evaluate the following matrix here:

$$\mathscr{B} = \left(\frac{\partial \beta_i}{\partial g_j}\right)$$

$$=\begin{bmatrix} -\varepsilon+\dfrac{n+8}{96\pi^3}g_1 & 0 & \dfrac{1}{96\pi^3}g_3 \\ 0 & -\varepsilon+\dfrac{3}{32\pi^3}g_2 & \dfrac{n}{96\pi^3}g_3 \\ \dfrac{n+2}{192\pi^3}g_3 & \dfrac{1}{64\pi^3}g_3 & -\varepsilon+\dfrac{n+2}{192\pi^3}g_1+\dfrac{1}{64\pi^3}g_3+\dfrac{1}{24\pi^3}g_3 \end{bmatrix}. \quad (3\cdot4)$$

Setting $\beta_i=0$ ($i=1, 2, 3$) in (3·3a)~(3·3c), we find the following six fixed points. Let us inquire the stability of them in turn.

[A] $g_1{}^*=g_2{}^*=g_3{}^*=192\pi^3\varepsilon/(n+9)$

Obviously Hamiltonian density (1·2) restores $O(n+1)$ symmetry in this case. To examine the stability, we seek eigenvalues of $\mathcal{B}$:

$$\mathcal{B}=\frac{\varepsilon}{n+9}\begin{pmatrix} n+7 & 0 & 2 \\ 0 & 9-n & 2n \\ n+2 & 3 & 4 \end{pmatrix}. \quad (3\cdot5)$$

Apart from the coefficient $\varepsilon/(n+9)$, solutions of secular equation of (3·5) are $\lambda=n+9, 8, 3-n$. Thus all eigenvalues are positive for $3>n$ and one eigenvalue is negative for $n>3$. Judging from Liu and Fisher's criterion,[8] an inequality $g_3^{*2}\geqq g_1{}^*g_2{}^*$ of bicriticality is satisfied. Therefore bicritical point is stable for $n<3$.[1]

The following three cases are all infrared unstable.

[B] $g_1{}^*=g_2{}^*=g_3{}^*=0$ [C] $g_1{}^*=g_3{}^*=0$ $g_2{}^*=64\pi^3\varepsilon/3$

[D] $g_2{}^*=g_3{}^*=0$ $g_1{}^*=192\pi^3\varepsilon/(n+8)$

[E] $g_1{}^*=192\pi^3\varepsilon/(n+8)$ $g_2{}^*=64\pi^3\varepsilon/3$ $g_3{}^*=0$.

An inequality of tetracriticality $g_1{}^*g_2{}^*>g_3^{*2}$ [8] holds in this case. $\mathcal{B}$ matrix becomes

$$\mathcal{B}=\begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & (n-10)\varepsilon/3(n+8) \end{pmatrix}. \quad (3\cdot6)$$

Thus the decoupled tetracritical point is stable for $n>10$.

[F] $\beta_1=\beta_2=0$ $(n+2)g_1{}^*+3g_2{}^*+4g_3{}^*=192\pi^3\varepsilon$

Coefficients of these three algebraic equations differ from those of Nelson-Kosterlitz-Fisher by multiplicative constants arising from the difference of the definitions of g_i. Solutions are found to be in agreement with (12) and (14) in Ref. 1), when we replace n and $\bar{\varepsilon}$ in (12) and (14) by $n+1$ and $768\pi^3\varepsilon$ respectively. This case is known to satisfy the tetracritical inequality $g_1{}^*g_2{}^*>g_2^{*2}$ for $n>-1$.[1]

To sum up the above results, for $n<3$ bicritical point is dominant, for $3<n<10$ coupled tetracritical behavior is observed and for $n>10$ the decoupled tetracritical point of doubly ordered phase consisting of $O(n)$ Heisenberg and Ising systems is stable. This conclusion is consistent with the one of Nelson-Kosterlitz-Fisher,[1] although these transitions exist only for $4<d<6$.

§ 4. Discussion

In the previous sections, we have verified that, even if Gaussian random fields are applied to antiferromagnets with weak spin anisotropy, they experience just the same multicritical behaviors for $4<d<6$ as those for $2<d<4$ without random fields. As for the validity of $d \to d-2$ rule, it is not necessary that we estimate crossover exponents explicitly. Comparing renormalization constants Z_i $(i=1,2,3)$ in (2·15a)~(2·15c) with (A·1a)~(A·1c), we conclude that the well-known $d \to d-2$ rule[7),*)] holds in the case of multicritical points. We showed one example supporting this conclusion in this paper.

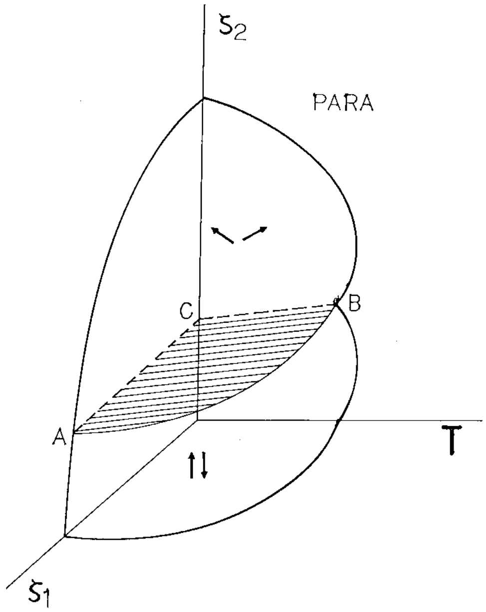

Fig. 4. Anticipated phase diagram of weakly anisotropic antiferromagnets under Gaussian random fields. Line AB is a 2nd-order bicritical line and a plane ABC is a 1st-order spin flop plane.

Experiments of $GdAlO_3$,[9)] which is a good sample of weakly anisotropic antiferromagnets, show the existence of a bicritical line. On the other hand, La-doped $GdAlO_3$,[10)] which is a sample realizing weakly anisotropic antiferromagnets with uniaxially random fields,[11)] is theoretically[6)] and experimentally[10)] known to have a bicritical point at $\zeta_1=0$ plane. Our results extend these results and predict that, if transverse random fields as well as longitudinal ones are applied, second-order (although smeared) bicritical line will be observed around the 1st-order spin flop plane (Fig. 4).

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*) Aharony-Imry-Ma[7)] derived a general relation [Eq. (5)] between most divergent diagrams with weak Gaussian random fields and those for pure system: $\tilde{I}_l{}^d(\boldsymbol{q}_i, r_i)=(\lambda/4\pi)I_l^{d-2}(\boldsymbol{q}_i, r_i)$. A coefficient $4\pi\lambda$ in their results may be misprinted and a correct coefficient is $\lambda/4\pi$. This is also pointed out by Young.[7)] When we compare the results in §§ 2 and 3 with those in the Appendix, our results satisfy the above general relation, i.e., $\zeta/192\pi^3=(\zeta/4\pi)/48\pi^2$.

Acknowledgements

The author is indebted to Professor R. Abe for continuous encouragement. He also expresses his sincere thanks to Professor T. Izuyama for critical reading of this manuscript and encouragement.

Appendix

—— *Antiferromagnets with Weak Spin Anisotropy* ——

The Nelson-Kosterlitz-Fisher model[1),*)] (1·1) is analyzed via renormalized field theory.[3)] With the application of the usual renormalized field theory to (1·1), we obtain diagrammatic equations up to $O(\varepsilon)$ $[\varepsilon=4-d]$ of renormalized 2-point and 4-point vertex functions $\tilde{\Gamma}_i^{(2)}$ $(i=1,2)$ and $\tilde{\Gamma}_i$ $(i=1,2,3)$ pictured in Figs. 5 and 6, respectively. These equations yield the results:

$$Z_1=1-\frac{1}{48\pi^2}[(n+8)g_1+g_3{}^2/g_1]\ln(\mu/\Lambda), \qquad (\mathrm{A\cdot 1a})$$

$$Z_2=1-\frac{1}{48\pi^2}[9g_2+ng_3{}^2/g_2]\ln(\mu/\Lambda) \qquad (\mathrm{A\cdot 1b})$$

and

$$Z_3=1-\frac{1}{48\pi^2}[(n+2)g_1+3g_2+4g_3]\ln(\mu/\Lambda), \qquad (\mathrm{A\cdot 1c})$$

where g_i $(i=1,2,3)$ are dimensionless renormalized coupling constants defined as $\tilde{\Gamma}_i|_{\mathrm{SP}}=\tilde{u}_i=\mu^\varepsilon g_i$ $(i=1,2,3)$.

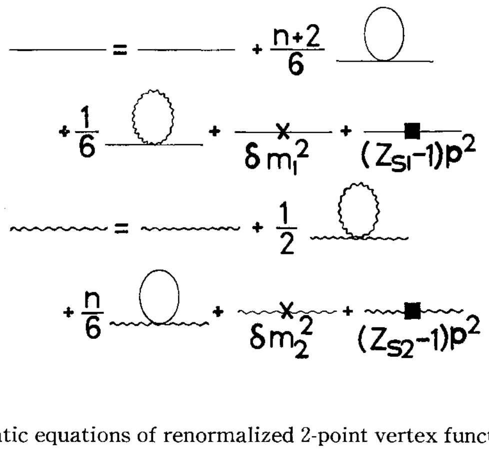

Fig. 5. Diagrammatic equations of renormalized 2-point vertex functions up to $O(\varepsilon)$.

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*) More general case that ϕ_2 field has $m(\geq 1)$ components is argued independently by Lyuksyutov-Pokrovskii-Khmel'nitskii[12)] and Kosterlitz-Nelson-Fisher.[12)]

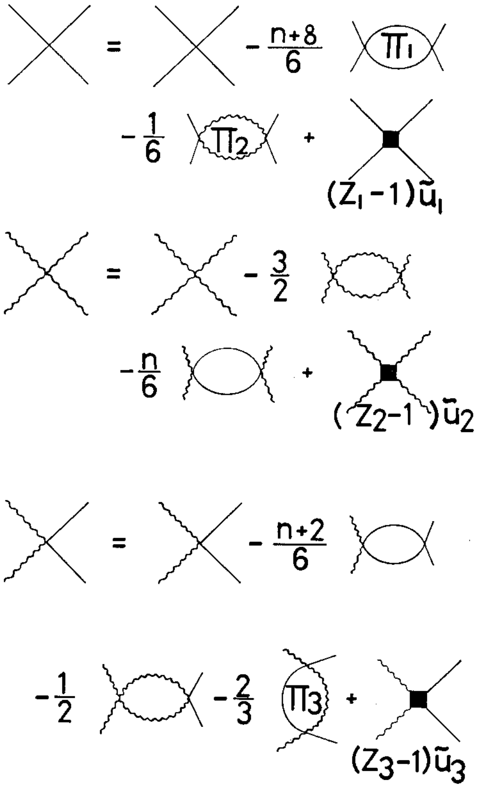

Fig. 6. Diagrammatic equations of renormalized 4-point vertex functions up to $O(\varepsilon)$.

β functions and $\mathscr{B}$ matrix are derived from (A·1a)~(A·1c) in the same manners as in § 3. Results are summarized as follows:

$$\beta_1 = -\varepsilon g_1 + \frac{n+8}{48\pi^2} g_1{}^2 + \frac{1}{48\pi^2} g_3{}^2 \,, \tag{A·2a}$$

$$\beta_2 = -\varepsilon g_2 + \frac{3}{16\pi^2} g_2{}^2 + \frac{n}{48\pi^2} g_3{}^2 \,, \tag{A·2b}$$

$$\beta_3 = g_3 \left[-\varepsilon + \frac{n+2}{48\pi^2} g_1 + \frac{1}{16\pi^2} g_2 + \frac{1}{12\pi^2} g_3 \right] \tag{A·2c}$$

and

$$\mathscr{B} = \begin{pmatrix} -\varepsilon + \frac{n+8}{48\pi^2} g_1 & 0 & \frac{1}{24\pi^2} g_3 \\ 0 & -\varepsilon + \frac{3}{8\pi^2} g_2 & \frac{n}{24\pi^2} g_3 \\ \frac{n+2}{48\pi^2} g_3 & \frac{1}{16\pi^2} g_2 & -\varepsilon + \frac{n+2}{48\pi^2} g_1 + \frac{1}{16\pi^2} g_2 + \frac{1}{6\pi^2} g_3 \end{pmatrix} . \tag{A·3}$$

Through (A·2a)~(A·3), we see just the same multicritical phenomena for $2<d<4$ as [A]~[F] in § 3, by replacing $192\pi^3$ with $48\pi^2$. Thus we confirm the results of Nelson-Kosterlitz-Fisher[1)] simply and explicitly in the framework of renormalized field theory.

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