Renormalized Solutions for Nonlinear Degenerate Elliptic Problems with L^1 Data

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ABSTRACT

We are interested in a class of nonlinear degenerate diffusion problems with a diffusion function $a(x, u, \nabla u)$ which is not controlled with respect to u and which is not uniformly coercive on the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$. Existence of a renormalized solution is proved in the L^1-setting.

Key words: Renormalized solutions, nonlinear degenerate elliptic equations, weighted Sobolev spaces.

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Introduction

In this paper we investigate the problem of existence of renormalized solutions for a class of nonlinear degenerate elliptic equations of the type

$$-\operatorname{div}(a(x, u, \nabla u)) = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, and the data f is in $L^1(\Omega)$. The operator $-\operatorname{div}(a(x, u, \nabla u))$ is a Leray-Lions operator defined on the weighted

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Sobolev spaces $W_0^{1,p}(\Omega, w)$, but which is not controlled with respect to u (see assumptions (5), (7), and (8) of section 2). For almost any x in Ω and for any $\xi \in \mathbb{R}^N$, the function $a(x, s, \xi)$ is strongly degenerate when $|s|$ grows to $+\infty$ (see (6) and (7)). So, proving existence of a weak solution (i.e., in the distribution meaning) seems to be an arduous task. To overcome this difficulty we use in this paper the framework of renormalized solutions. This notion was introduced by P.-L. Lions and Di Perna [12] for the study of the Boltzmann equation (see also P.-L. Lions [15] for a few applications to fluid mechanics models). This notion was then adapted to the elliptic version of (1) and (2) in Boccardo, J.-L. Diaz, D. Giachetti, F. Murat [11], in P.-L. Lions and F. Murat [16], and F. Murat [16, 17] (see also [8, 9] for nonlinear parabolic problems). At the same time the equivalent notion of entropy solutions have been developed independently by Bénilan and al. [5] for the study of nonlinear elliptic problems.

In the case where $a(x, u, \nabla u)$ is replaced by $A(x, u)\nabla u$ (problems with diffusion matrix which are not uniformly coercive with respect to u) and $f \in L^1(\Omega)$, existence and a partial uniqueness result have been established on the Sobolev spaces $H_0^1(\Omega)$ in D. Blanchard and O. Guibé [6] (see also D. Blanchard, O. Guibé, and H. Redwane [7], K. Ammar [4]).

Note that in the non weighted case, the existence and regularity results for the nonlinear elliptic problem (1), (2) has been proved in A. Alvino, L. Boccardo, V. Ferone, L. Orsina, and G. Trombetti [2] under the condition $a(x, s, \xi)\xi \geq \frac{\alpha}{(1+|s|)^{\theta(p-1)}}|\xi|^p$ and under various assumptions on the function f and on θ, (see also the results of A. Alvino, V. Ferone, and G. Trombetti [3], L. Boccardo, A. Dall'Aglio, and L. Orsina [10]).

In our paper we propose a formulation which takes into account the possible values $+\infty$ or $-\infty$ for the solutions and the operator $-\operatorname{div}(a(x, u, Du))$ is a weighted Leray-Lions operator from the weighted Sobolev space on $W_0^{1,p}(\Omega, \omega)$ into $W^{-1,p'}(\Omega, \omega^*)$.

The paper is organized as follows: In section 1, we precise some basic properties of weighted Sobolev spaces. In section 2, we specify the assumptions on $a(x, s, \xi)$, $b(s)$ and f needed in the present study and we give the definition of a renormalized solution of (1), (2). In section 3, we prove the main result of this paper (Theorem 3.1) which is the existence of a renormalized solution for any data f in $L^1(\Omega)$.

1. Preliminaries

Throughout the paper, we assume that the following assumptions hold true: Ω is a bounded open subset on $\mathbb{R}^N$, $N \geq 1$, $1 < p < \infty$, and $\omega(x) = \{\omega_i(x)\}_{\{0 \leq i \leq N\}}$ is a vector of weight functions. Further, we suppose that every component $\omega_i(x)$ is a measurable function which is strictly positive and satisfies

$$\omega_i \in L^1_{\text{loc}}(\Omega) \qquad \text{and} \qquad {\omega_i}^{-\frac{1}{p-1}} \in L^1_{\text{loc}}(\Omega).$$

We define the weighted Lebesgue space $L^p(\Omega, \omega_0)$ with weight ω_0, as the space of all real-valued measurable functions u for which

$$\|u\|_{p,\omega_0} = \Big(\int_\Omega |u(x)|^p \omega_0(x)\, dx\Big)^{\frac{1}{p}} < +\infty.$$

Similarly, we define the weighted Sobolev space of $W^{1,p}(\Omega, \omega)$, as the space of all real-valued functions $u \in L^p(\Omega, \omega_0)$ such that the derivatives in the sense of distributions satisfy $\frac{\partial u}{\partial x_i} \in L^p(\Omega, \omega_i)$ for all $i = 1, \dots, N$. Equipped with the norm

$$\|u\|_{1,p,\omega} = \Big(\int_\Omega |u(x)|^p \omega_0(x)\, dx + \sum_{i=1}^{N} \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^p \omega_i(x)\, dx\Big)^{\frac{1}{p}}, \tag{3}$$

$X = W^{1,p}(\Omega, \omega)$ is a Banach space. As we are concerned with a Dirichlet problem, we work in the space $X = W_0^{1,p}(\Omega, \omega)$ defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p,\omega}$. Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, \omega)$ and $(W_0^{1,p}(\Omega, \omega), \|\cdot\|_{1,p,\omega})$ is a reflexive Banach space. Note that the expression

$$\|u\|_X = \Big(\sum_{i=1}^{N} \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^p \omega_i(x)\, dx\Big)^{\frac{1}{p}}$$

is a norm defined on X and is equivalent to the norm (3). Moreover $(X, \|\cdot\|_X)$ is a reflexive Banach space, and there exist a weight function σ on Ω and a parameter $1 < q < \infty$ such that the Hardy inequality

$$\Big(\int_\Omega |u|^q \sigma(x)\, dx\Big)^{\frac{1}{q}} \le C\Big(\sum_{i=1}^{N} \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^p \omega_i(x)\, dx\Big)^{\frac{1}{p}} \tag{4}$$

holds for every $u \in X$ with a constant $C > 0$ independent of u. Moreover, the imbedding $X \hookrightarrow L^q(\Omega, \sigma)$ is compact.

We recall that the dual of the weighted Sobolev spaces $W_0^{1,p}(\Omega, \omega)$ is equivalent to $W^{-1,p'}(\Omega, \omega^*)$, where $\omega^* = \{\, \omega_i^* = \omega_i^{1-p'};\ i = 1\dots, N \,\}$ and $p' = \frac{p}{p-1}$ is the conjugate of p. For more details we refer the reader to [13].

2. Assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true: The functional $-\operatorname{div}(a(x, u, \nabla u))$ is a Leray-Lions operator defined on $W_0^{1,p}(\Omega, \omega)$ into $W^{-1,p'}(\Omega, \omega^*)$ and where

$$a\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N \quad \text{is a Carathéodory function,}$$

which is monotone with respect to ξ:

$$[a(x,s,\xi) - a(x,s,\xi')][\xi - \xi'] \geq 0; \tag{6}$$

for any $\xi, \xi' \in \mathbb{R}^N$, for any $s \in \mathbb{R}$ and for almost every $x \in \Omega$, and which satisfies the following weak coercivity condition: there exists a positive function $b \in C^0(\mathbb{R})$ such that for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$,

$$a(x,s,\xi)\cdot\xi \geq b(s)^{p-1}\sum_{i=1}^{N}\omega_i(x)|\xi_i|^p \quad \text{and} \quad \int_{-\infty}^{+\infty} b(s)\,ds < +\infty. \tag{7}$$

Moreover, a satisfies a growth condition of this type: for any $i = 1, \dots, N$

$$|a_i(x,s,\xi)| \leq \omega_i(x)^{\frac{1}{p}}\left[L(x) + \sigma(x)^{\frac{1}{p'}}\left(\int_0^s b(r)\,dr\right)^{\frac{q}{p'}} + b(s)^{p-1}\sum_{j=1}^{N}\omega_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}\right] \tag{8}$$

and the normalization condition $a(x,s,0) = 0$ for almost every $x \in \Omega$, for every s and ξ, and where $L(x)$ is a positive function in $L^{p'}(\Omega)$ and $\sigma(x)$ is defined in (4).

We will study the problem in the general framework, i.e.,

$$f \quad \text{is an element of} \quad L^1(\Omega). \tag{9}$$

Remark 2.1. As already mentioned in the introduction Problem (1), (2) does not admit a weak solution under assumptions (5)–(9). Indeed, as the growth of $a(x,u,\nabla u)$ is not controlled with respect to u, the field $a(x,u,\nabla u)$ is not, in general, defined as a distribution.

The following notations will be used throughout the paper: for any $K \geq 0$, the truncation at height K is defined by $T_K(r) = \max(-K, \min(r,K))$. Moreover, for $n \geq 1$ fixed,

$$\theta_n(r) = T_1(r - T_n(r)) = \begin{cases} 0 & \text{if} \quad |r| \leq n, \\ r - n\ sg(r) & \text{if} \quad n \leq |r| \leq n+1, \\ sg(r) & \text{if} \quad |r| \geq n+1 \end{cases}$$

and $S_n(r) = 1 - |\theta_n(r)|$, $\forall r \in \mathbb{R}$.

We define a renormalized solution for Problem (1), (2) as follows.

Definition 2.2. A measurable function u defined on Ω with values in $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is a renormalized solution of Problem (1), (2) if

$$T_K(u) \in W_0^{1,p}(\Omega,\omega) \quad \forall K \geq 0, \tag{10}$$

$$\int_{\{n\leq|u(x)|\leq n+1\}} a(x,u,\nabla u)\nabla u\,dx \longrightarrow \int_{\{u=+\infty\}} f(x)\,dx - \int_{\{u=-\infty\}} f(x)\,dx, \tag{11}$$

as $n \to +\infty$, and if, for every function S in $W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp}(S)$ is compact, u satisfies

$$\int_\Omega a(x,u,\nabla u)\nabla\Big(S(u)\varphi\Big)\,dx = \int_\Omega fS(u)\varphi\,dx, \quad \forall \varphi \in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega). \tag{12}$$

The following remarks are concerned with a few comments on Definition 2.2.

Remark 2.3. Notice that, thanks to our regularity assumptions (10) and the choice of S, all terms in (12) are well defined.

The following identifications are made in (12).

- $a(x,u,\nabla u)\nabla(S(u)\varphi)$ identifies with $a\big(x,T_K(u),\nabla T_K(u)\big)\nabla\big(S(T_K(u))\varphi\big)$ a.e. in Ω, where $K>0$ and $\operatorname{supp}(S)\subset[-K,K]$. As a consequence of (8), (10) and of $S\in W^{1,\infty}(\mathbb{R})$, $\varphi\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$, it follows that

$$a\big(x,T_K(u),\nabla T_K(u)\big)\nabla\big(S(T_K(u))\varphi\big)\in L^1(\Omega).$$

Indeed,

$$\nabla(S(u)\varphi)=\nabla\big(S(T_K(u))\varphi\big)\in\prod_{i=1}^N L^p(\Omega,\omega_i)$$

and, by Holder inequality, we have, for $i=1,\dots,N$,

$$|a_i(x,T_K(u),\nabla T_K(u))| \le \omega_i(x)^{\frac{1}{p}}\Big[L(x)+\sigma(x)^{\frac{1}{p'}}C_1^{\frac{q}{p'}}+C_2^{p-1}\sum_{j=1}^N\omega_j^{\frac{1}{p'}}(x)\Big|\frac{\partial T_K(u)}{\partial x_j}\Big|^{p-1}\Big]$$

where $C_1=\int_{-\infty}^{+\infty}b(s)\,ds$ and $C_2=\max_{|s|\le K}|b(s)|$.

- $fS(u)\varphi\in L^1(\Omega)$, because $f\in L^1(\Omega)$ and $S(u)\varphi\in L^\infty(\Omega)$.

3. Existence result

This section is devoted to establish the existence theorem.

Theorem 3.1. *Under the assumptions* (5)–(9) *there exists at least a renormalized solution u of Problem* (1), (2).

Proof. The proof is divided into 6 steps. In step 1, we introduce an approximate problem. Step 2 is devoted to establish a few *a priori* estimates on the approximate solutions u^ε and on the limit solution u. In particular, we prove that u satisfies (10). In step 3, we prove the monotonicity estimate. In step 4, we identify the weak limit X_K of $a^\varepsilon(x,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon))$ and we prove the weak L^1 convergence of the energy $a^\varepsilon(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon))\nabla T_K(u^\varepsilon)$ as ε tends to zero. Step 5 is devoted to prove that u satisfies (11). Finally, in step 6, we prove that u satisfies (12) of Definition 2.2.

- *Step 1.* We proceed by approximation: for $\varepsilon > 0$, define the regularized functions

$$b^\varepsilon : \mathbb{R} \longrightarrow \mathbb{R},\ r \longmapsto b^\varepsilon(r) = b(T_{\frac{1}{\varepsilon}}(r)), \tag{13}$$

$$a^\varepsilon(x, s, \xi) = a(x, T_{\frac{1}{\varepsilon}}(s), \xi) \quad \text{a.e. in } \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N, \tag{14}$$

$$f^\varepsilon \in L^{p'}(\Omega), \quad \|f^\varepsilon\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)},$$

$$f^\varepsilon \longrightarrow f \quad \text{strongly in } L^1(\Omega) \quad \text{as} \quad \varepsilon \text{ tends to } 0, \tag{15}$$

Let us now consider the following regularized problem:

$$-\operatorname{div}(a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)) = f^\varepsilon \quad \text{in } \Omega, \tag{16}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega. \tag{17}$$

In view of (7), (8), (13), and (14), b^ε and a^ε satisfy

$$0 < \alpha_\varepsilon \equiv \min_{\{|r| \le \frac{1}{\varepsilon}\}} (b(r)) \le b^\varepsilon(s) \le \max_{\{|r| \le \frac{1}{\varepsilon}\}} (b(r)) \equiv C_\varepsilon \quad \forall s \in \mathbb{R}.$$

$$a^\varepsilon(x, s, \xi) \cdot \xi \ge b^\varepsilon(s)^{p-1} \sum_{i=1}^N \omega_i(x) |\xi_i|^p \ge \alpha_\varepsilon^{p-1} \sum_{i=1}^N \omega_i(x) |\xi_i|^p, \tag{18}$$

and, for $i = 1, \ldots, N$,

$$|a_i^\varepsilon(x, s, \xi)| \le \omega_i(x)^{\frac{1}{p}} \Big[L(x) + C_\varepsilon^{\frac{q}{p'}} \sigma(x)^{\frac{1}{p'}} |s|^{\frac{q}{p'}} + C_\varepsilon^{p-1} \sum_{j=1}^N \omega_j^{\frac{1}{p'}}(x) |\xi_j|^{p-1} \Big]$$

a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$.

As a consequence, proving the existence of a weak solution $u^\varepsilon \in W_0^{1,p}(\Omega, \omega)$ of (16) and (17) is an easy task (see, e.g., Theorem 2.1 and Remark 2.1 in chapter 2 of [14] and see also [1]).

- *Step 2.* A priori estimates and pointwise convergence of u^ε.
  Using $T_K(u^\varepsilon)$ as a test function in (16) leads to

$$\int_\Omega a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla T_K(u^\varepsilon)\, dx = \int_\Omega f^\varepsilon T_K(u^\varepsilon)\, dx \le K \|f\|_{L^1(\Omega)}. \tag{19}$$

Since a^ε satisfies (18), we deduce from (19) that

$$\int_\Omega b^\varepsilon(u^\varepsilon)^{p-1} \sum_{i=1}^N \Big| \frac{\partial T_K(u^\varepsilon)}{\partial x_i} \Big|^p \omega_i(x)\, dx \le K\, \|f\|_{L^1(\Omega)}.$$

and

$$\alpha_{\frac{1}{K}}^{p-1} \int_\Omega \sum_{i=1}^{N} \Big|\frac{\partial T_K(u^\varepsilon)}{\partial x_i}\Big|^p \omega_i(x)\,dx \le K\,\|f\|_{L^1(\Omega)}, \tag{20}$$

where $\alpha_{\frac{1}{K}} \equiv \min_{\{|s|\le K\}}(b(s))$. From (20) and (4), we deduce with a classical argument (see, e.g., [1]) that, for a subsequence still indexed by ε,

$$u^\varepsilon \longrightarrow u \quad \text{a.e. in } \Omega, \tag{21}$$

$$T_K(u^\varepsilon) \longrightarrow T_K(u) \quad \text{weakly in } W_0^{1,p}(\Omega,\omega) \quad \text{and strongly in } L^q(\Omega,\sigma), \tag{22}$$

as ε tends to 0, where u is a measurable function defined on Ω with values in $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.

Taking now $Z^\varepsilon = \int_0^{T_K(u^\varepsilon)} b^\varepsilon(s)\,ds$ as a test function in (16) gives

$$\int_\Omega a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla Z^\varepsilon\,dx = \int_\Omega f^\varepsilon Z^\varepsilon\,dx. \tag{23}$$

Since a^ε satisfies (18), (23) leads to

$$\int_\Omega \sum_{i=1}^{N} \Big|\frac{\partial Z^\varepsilon}{\partial x_i}\Big|^p \omega_i(x)\,dx \le C_K\,\|f\|_{L^1(\Omega)}. \tag{24}$$

where $|Z^\varepsilon| \le C_K = 2K\max_{|s|\le K} b(s)$ is a constant independent of ε.

Now, for fixed $K>0$, assumption (8) gives, for $i=1,\dots,N$,

$$|a_i^\varepsilon(x,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon))| \le \omega_i(x)^{\frac{1}{p}}\Big[L(x) + \sigma(x)^{\frac{1}{p'}} C_K^{\frac{q}{p'}} + \sum_{j=1}^{N} \omega_j^{\frac{1}{p'}}(x)\Big|\frac{\partial Z^\varepsilon}{\partial x_j}\Big|^{p-1}\Big] \tag{25}$$

In view of (24) and (25), we deduce that

$$a^\varepsilon(x,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon)) \quad \text{is bounded in} \quad \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}), \tag{26}$$

then there exists a function $X_K \in \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$ such that

$$a^\varepsilon(x,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon)) \rightharpoonup X_K \quad \text{weakly in} \quad \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}) \quad \text{as } \varepsilon \to 0. \tag{27}$$

Let us now take $T_K(v^\varepsilon)$ as a test function in (16), where $v^\varepsilon = \int_0^{u^\varepsilon} b^\varepsilon(s)\,ds$. We obtain

$$\int_\Omega a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla T_K(v^\varepsilon)\,dx = \int_\Omega f^\varepsilon T_K(v^\varepsilon)\,dx \le K\,\|f\|_{L^1(\Omega)}.$$

Then (18) yields

$$\int_\Omega \sum_{i=1}^N \Big|\frac{\partial T_K(v^\varepsilon)}{\partial x_i}\Big|^p \omega_i(x)\,dx \le K\,\|f\|_{L^1(\Omega)}. \tag{28}$$

We deduce with a classical argument that, for a subsequence still indexed by ε,

$$\begin{aligned} v^\varepsilon &\longrightarrow v && \text{a.e. in } \Omega,\\ T_K(v^\varepsilon) &\longrightarrow T_K(v) && \text{weakly in } W_0^{1,p}(\Omega,\omega), \end{aligned} \tag{29}$$

as ε tends to 0, where v is a measurable function defined on Ω which is finite a.e. in Ω.

Using the admissible test function $\theta_n(v^\varepsilon)$ in (16) leads to

$$\int_\Omega a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla\theta_n(v^\varepsilon)\,dx = \int_\Omega f^\varepsilon\theta_n(v^\varepsilon)\,dx \tag{30}$$

As a consequence of the above convergence results, we are in a position to pass to the limit as ε tends to 0 in (30). Indeed, the pointwise convergence of $\theta_n(v^\varepsilon)$ to $\theta_n(v)$ as ε tends to zero and $|\theta_n(v^\varepsilon)| \le 1$ a.e. in Ω (independently of ε and n) leads to

$$\lim_{\varepsilon\to0}\int_\Omega a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla\theta_n(v^\varepsilon)\,dx = \int_\Omega f\theta_n(v)\,dx. \tag{31}$$

The pointwise convergence of $\theta_n(v)$ to zero as n tends to $+\infty$, the bounded character of θ_n ($|\theta_n(v^\varepsilon)| \le 1$ a.e. in Ω, independently of ε and n) and $f \in L^1(\Omega)$, Lebesgue's convergence theorem shows that $\int_\Omega f\theta_n(v)\,dx \to 0$, as n tends to $+\infty$. Passing to the limit in (31) we obtain

$$\lim_{n\to+\infty}\lim_{\varepsilon\to0}\int_{\{n\le|v^\varepsilon|\le n+1\}} a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla v^\varepsilon\,dx = 0. \tag{32}$$

- *Step 3.* In this step we prove the following monotonicity estimate:

LEMMA 3.2. *The subsequence of u^ε defined in step 1 satisfies for any $K \ge 0$:*

$$\lim_{\varepsilon\to0}\int_\Omega \big[a^\varepsilon(x,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon)) - a^\varepsilon(x,T_K(u^\varepsilon),\nabla T_K(u))\big] \times \big[\nabla T_K(u^\varepsilon) - \nabla T_K(u)\big]\,dx = 0 \tag{33}$$

Proof of Lemma 3.2. Let $K \ge 0$ be fixed. The left hand side of equality (33) is split into

$$\int_\Omega \big[a^\varepsilon(x,T_K(u^\varepsilon),\nabla T_K(u^\varepsilon)) - a^\varepsilon(x,T_K(u^\varepsilon),\nabla T_K(u))\big] \times \big[\nabla T_K(u^\varepsilon) - \nabla T_K(u)\big]\,dx = A_1^\varepsilon + A_2^\varepsilon + A_3^\varepsilon, \tag{34}$$

where

$$A_1^\varepsilon = \int_\Omega a^\varepsilon(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon))\nabla T_K(u^\varepsilon)\, dx,$$
$$A_2^\varepsilon = -\int_\Omega a^\varepsilon(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon))\nabla T_K(u)\, dx,$$

and

$$A_3^\varepsilon = -\int_\Omega a^\varepsilon(x, T_K(u^\varepsilon), \nabla T_K(u))\big[\nabla T_K(u^\varepsilon) - \nabla T_K(u)\big]\, dx.$$

In the sequel we pass to the limit in (34) when ε tends to 0.

– *Limit of* A_1^ε. Using the admissible test function $S_n(v^\varepsilon)T_K(u)$ in (16) leads to

$$\int_\Omega S_n(v^\varepsilon)a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)\nabla T_K(u)\, dx + \int_\Omega a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)\nabla S_n(v^\varepsilon)\cdot T_K(u)\, dx$$
$$= \int_\Omega f^\varepsilon S_n(v^\varepsilon)T_K(u)\, dx, \quad (35)$$

where $v^\varepsilon = \int_0^{u^\varepsilon} b^\varepsilon(s)\, ds$.

Passing to the limit as ε tends to 0 in (35), since $\operatorname{supp}(S_n) \subset [-(n+1), n+1]$, we have for $i = 1, \ldots, N$ that

$$|a_i^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)S_n(v^\varepsilon)|$$
$$\leq \|S_n\|_{L^\infty(\mathbb{R})}\omega_i(x)^{\frac{1}{p}}\Big[L(x) + \sigma(x)^{\frac{1}{p'}}(n+1)^{\frac{q}{p'}} + \sum_{j=1}^N \omega_j^{\frac{1}{p'}}(x)\Big|\frac{\partial T_{n+1}(v^\varepsilon)}{\partial x_j}\Big|^{p-1}\Big] \quad (36)$$

In view of (28) and (36), we deduce that, for fixed $n \geq 1$,

$$a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)S_n(v^\varepsilon) \quad \text{is bounded in} \quad \prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'}),$$

independently of ε. Then there exists a function $Y_n \in \prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$ such that, for fixed $n \geq 1$,

$$S_n(v^\varepsilon)a^\varepsilon\Big(x, u^\varepsilon, \nabla u^\varepsilon\Big) \rightharpoonup Y_n \quad \text{weakly in} \quad \prod_{i=1}^N L^{p'}(\Omega, w_i^{1-p'}) \quad \text{as } \varepsilon \to 0. \quad (37)$$

Now, for $K \leq n$ we have

$$S_n(v^\varepsilon)a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)\chi_{\{|u^\varepsilon| \leq K\}} = S_n(v^\varepsilon)a^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\big)$$

a.e. in Ω, which implies that, through the use of (27), (29), and (37), and passing to the limit as ε tends to 0,

$$Y_n \chi_{\{|u|<K\}} = S_n(v) X_K \tag{38}$$

a.e. in $\Omega \backslash \{|u| = K\}$. As a consequence of (37) we have for $K \leq n$ that

$$Y_n \nabla T_K(u) = S_n(v) X_K \nabla T_K(u) \quad \text{a.e. in } \Omega. \tag{39}$$

We are now in a position to exploit (35), which together with (37) and (39), gives

$$\lim_{\varepsilon\to 0} \int_\Omega S_n(v^\varepsilon) a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla T_K(u)\, dx = \int_\Omega Y_n \nabla T_K(u)\, dx = \int_\Omega S_n(v) X_K \nabla T_K(u)\, dx \tag{40}$$

Passing to the limit as n tends to $+\infty$ in (40) leads to

$$\lim_{n\to+\infty} \lim_{\varepsilon\to 0} \int_\Omega S_n(v^\varepsilon) a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla T_K(u)\, dx = \int_\Omega X_K \nabla T_K(u)\, dx \tag{41}$$

Now, we estimate the second term of (35):

$$\left| \int_\Omega a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla S_n(v^\varepsilon) \cdot T_K(u)\, dx \right| \leq K \int_{\{n \leq |v^\varepsilon| \leq n+1\}} a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla v^\varepsilon\, dx.$$

Then (32) implies that

$$\lim_{n\to+\infty} \lim_{\varepsilon\to 0} \int_\Omega a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla S_n(v^\varepsilon) \cdot T_K(u)\, dx = 0. \tag{42}$$

In view of (41) and (42), passing to the limit as ε tends to 0 and n tends to $+\infty$ in (35) is an easy task and leads to

$$\int_\Omega X_K \nabla T_K(u)\, dx = \int_\Omega f T_K(u)\, dx \tag{43}$$

We are now in a position to exploit (43). Using the test function $T_K(u^\varepsilon)$ in (16), we obtain

$$\int_\Omega a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla T_K(u^\varepsilon)\, dx = \int_\Omega f^\varepsilon T_K(u^\varepsilon)\, dx \tag{44}$$

Passing to the limit as ε tends to 0 in (44). In view (43), we have

$$\lim_{\varepsilon\to 0} A_1^\varepsilon = \lim_{\varepsilon\to 0} \int_\Omega a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) \nabla T_K(u^\varepsilon)\, dx = \int_\Omega X_K \nabla T_K(u)\, dx \tag{45}$$

– *Limit of* A_2^ε. In view of (21), (26), we have

$$\lim_{\varepsilon\to 0} A_2^\varepsilon = -\int_\Omega X_K \nabla T_K(u)\, dx \tag{46}$$

– *Limit of* A_3^ε. Let us remark that (5), (14), and (21) imply that

$$a^\varepsilon(x, T_K(u^\varepsilon), \nabla T_K(u)) \to a(x, T_K(u), \nabla T_K(u)) \quad \text{a.e. in } \Omega,$$

as ε tends to 0, and that, for $i = 1, \dots, N$,

$$\begin{aligned}&\big|a_i^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u)\big)\big|\\ &\quad\le w_i^{\frac{1}{p}}(x)\Big[L(x) + \sigma^{\frac{1}{p'}}(x)\Big(\int_{-K}^{K} b(s)\, ds\Big)^{\frac{q}{p'}} + C_K{}^{p-1}\sum_{j=1}^{N} w_j^{\frac{1}{p'}}\Big|\frac{\partial T_K(u)}{\partial x_j}\Big|^{p-1}\Big]\end{aligned}$$

a.e. in Ω, uniformly with respect to ε and where $C_K = \max_{|s|\le K}(b(s))$.

It follows that

$$a^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u)\big) \longrightarrow a\big(x, T_K(u), \nabla T_K(u)\big) \quad \text{strongly in } \prod_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}), \tag{47}$$

as ε tends to 0. In view of (21), we conclude that

$$\big(\nabla T_K(u^\varepsilon) - \nabla T_K(u)\big) \rightharpoonup 0 \quad \text{weakly in } \prod_{i=1}^{N} L^p(\Omega, w_i), \text{ as } \varepsilon \text{ goes to } 0. \tag{48}$$

As a consequence of (47) and (48) we have for all $K \ge 0$ that

$$\lim_{\varepsilon\to 0} A_3^\varepsilon = 0. \tag{49}$$

In view of (45), (46), and (49), we can pass to the limit as ε tends to zero in (34) and obtain (33) of Lemma 3.2. ∎

• *Step 4.* In this step we identify the weak limit X_K and we prove the weak L^1 convergence of the "truncated" energy $a^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\big)\nabla T_K(u^\varepsilon)$ as ε tends to 0.

LEMMA 3.3. *For fixed* $K \ge 0$, *we have*

$$X_K = a\big(x, T_K(u), \nabla T_K(u)\big) \quad \textit{a.e. in } \Omega. \tag{50}$$

And, as ε *tends to* 0,

$$a^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\big)\nabla T_K(u^\varepsilon) \rightharpoonup a\big(x, T_K(u), \nabla T_K(u)\big)\nabla T_K(u) \quad \textit{weakly in } L^1(\Omega). \tag{51}$$

Proof of Lemma 3.3. Let $K \geq 0$ be fixed. In view of (22) and (26) it is possible to obtain from (33) of Lemma 3.2 that

$$\lim_{\varepsilon \to 0} \int_\Omega a^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\big) \nabla T_K(u^\varepsilon)\, dx = \int_\Omega X_K \nabla T_K(u)\, dx. \tag{52}$$

We use the monotone character a (with respect to ξ) and for all $\psi \in \prod_{i=1}^N L^p(\Omega, w_i)$ we have

$$\begin{aligned}
0 &\leq \lim_{\varepsilon \to 0} \int_\Omega \big[a^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\big) - a^\varepsilon\big(x, T_K(u^\varepsilon), \psi\big)\big][\nabla T_K(u^\varepsilon) - \psi]\, dx \\
&= \lim_{\varepsilon \to 0} \int_\Omega a^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\big)[\nabla T_K(u^\varepsilon) - \psi]\, dx \\
&\quad - \lim_{\varepsilon \to 0} \int_\Omega a^\varepsilon(x, T_K(u^\varepsilon), \psi)[\nabla T_K(u^\varepsilon) - \psi]\, dx \\
&= \int_\Omega X_K [\nabla T_K(u) - \psi]\, dx - \int_\Omega a(x, T_K(u), \psi)[\nabla T_K(u) - \psi]\, dx \\
&= \int_\Omega [X_K - a(x, T_K(u), \psi)][\nabla T_K(u) - \psi]\, dx
\end{aligned}$$

The usual Minty's argument applies in view of (52). It follows that (50) of Lemma 3.3 holds true.

In order to prove (51), we use the monotone character of a (with respect to ξ) and (33) to have for any $K \geq 0$ that

$$\big(a^\varepsilon(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)) - a^\varepsilon(x, T_K(u^\varepsilon), \nabla T_K(u))\big)[\nabla T_K(u^\varepsilon) - \nabla T_K(u)] \tag{53}$$

converges to zero, strongly in $L^1(\Omega)$ as ε tends to 0. Moreover (22), (25), (47), and (50) imply that

$$a^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\big) \nabla T_K(u) \rightharpoonup a\big(x, T_K(u), \nabla T_K(u)\big) \nabla T_K(u) \tag{54}$$

weakly in $L^1(\Omega)$ as tends to 0,

$$a^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u)\big) \nabla T_K(u^\varepsilon) \rightharpoonup a\big(x, T_K(u), \nabla T_K(u)\big) \nabla T_K(u) \tag{55}$$

weakly in $L^1(\Omega)$ as tends to 0, and

$$a^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u)\big) \nabla T_K(u) \rightharpoonup a\big(x, T_K(u), \nabla T_K(u)\big) \nabla T_K(u) \tag{56}$$

strongly in $L^1(\Omega)$ as tends to 0.

Using the above convergence results (54), (55), and (56) in (53) we get, for any $K \geq 0$,

$$a^\varepsilon\big(x, T_K(u^\varepsilon), \nabla T_K(u^\varepsilon)\big) \nabla T_K(u^\varepsilon) \rightharpoonup a\big(x, T_K(u), \nabla T_K(u)\big) \nabla T_K(u)$$

weakly in $L^1(\Omega)$ as tends to 0. ∎

- *Step 5.* In this step we prove that u satisfies (11).
  Using $\big(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon)\big)S_p(v^\varepsilon)$ as a test function in (16) leads to

$$\int_\Omega S_p(v^\varepsilon)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla\big(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon)\big)\,dx + \int_\Omega \big(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon)\big)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla S_p(v^\varepsilon)\,dx = \int_\Omega f^\varepsilon S_p(v^\varepsilon)\big(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon)\big)\,dx.$$

Remark that for any fixed $n\geq 0$ and $p\geq 0$ one has

$$\int_{\{n\leq|u^\varepsilon(x)|\leq n+1\}} S_p(v^\varepsilon)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon\,dx = \int_\Omega S_p(v^\varepsilon)a^\varepsilon\big(x,T_{n+1}(u^\varepsilon),\nabla T_{n+1}(u^\varepsilon)\big)\nabla T_{n+1}(u^\varepsilon)\,dx - \int_\Omega S_p(v^\varepsilon)a^\varepsilon\big(x,T_n(u^\varepsilon),\nabla T_n(u^\varepsilon)\big)\nabla T_n(u^\varepsilon)\,dx.$$

According to (51), one can pass to the limit as ε tends to zero for fixed n and p to obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\{n\leq|u^\varepsilon(x)|\leq n+1\}} S_p(v^\varepsilon)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla u^\varepsilon\,dx &= \int_\Omega S_p(v)a\big(x,T_{n+1}(u),\nabla T_{n+1}(u)\big)\nabla T_{n+1}(u)\,dx \\ &\quad - \int_\Omega S_p(v)a\big(x,T_n(u),\nabla T_n(u)\big)\nabla T_n(u)\,dx \\ &= \int_{\{n\leq|u(x)|\leq n+1\}} S_p(v)a(x,u,\nabla u)\nabla u\,dx. \end{aligned} \tag{57}$$

Taking the limit as p tends to $+\infty$ and as n tends to $+\infty$ in (57) we obtain

$$\lim_{n\to+\infty}\lim_{p\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega S_p(v^\varepsilon)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla\big(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon)\big)\,dx = \lim_{n\to+\infty}\int_{\{n\leq|u(x)|\leq n+1\}} a(x,u,\nabla u)\nabla u\,dx$$

Since $\operatorname{supp}(S'_p)\subset[-(p+1),-p]\cup[p,p+1]$, we have

$$\left|\int_\Omega \big(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon)\big)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla S_p(v^\varepsilon)\,dx\right| \leq \int_{\{p\leq|v^\varepsilon(x)|\leq p+1\}} |a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla v^\varepsilon|\,dx$$

In view of (21), (29), and (32), we obtain

$$\lim_{n\to+\infty}\lim_{p\to+\infty}\lim_{\varepsilon\to 0}\left|\int_\Omega \big(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon)\big)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla S_p(v^\varepsilon)\,dx\right| = 0$$

$$\lim_{n\to+\infty}\lim_{p\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega f^\varepsilon S_p(v^\varepsilon)\big(T_{n+1}(u^\varepsilon)-T_n(u^\varepsilon)\big)\,dx = \int_{\{u=+\infty\}} f(x)\,dx - \int_{\{u=-\infty\}} f(x)\,dx.$$

- *Step 6.* In this step, u is shown to satisfy (12). Let $\varphi \in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$ and let S be a function in $W^{1,\infty}(\mathbb{R})$ such that S has a compact support. Let K be a positive real number such that $\operatorname{supp} S \subset [-K,K]$ and $v^\varepsilon = \int_0^{u^\varepsilon} b^\varepsilon(s)\,ds$. Using $S(u)S_n(v^\varepsilon)\varphi$ as a test function in (16) leads to

$$\int_\Omega S_n(v^\varepsilon)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla(S(u)\varphi)\,dx + \int_\Omega S(u)\varphi a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla S_n(v^\varepsilon)\,dx = \int_\Omega f^\varepsilon S_n(v^\varepsilon)S(u)\varphi\,dx. \quad (58)$$

In the following, we pass to the limit as ε tends to 0 and n tends to $+\infty$ in each term of (58).

– *Limit of the first term in* (58).
In view of (37), (38), and (50), passing to the limit as ε tends to 0, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega S_n(v^\varepsilon)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla(S(u)\varphi)\,dx &= \int_\Omega Y_n\nabla(S(u)\varphi)\,dx = \int_\Omega Y_n\chi_{\{|u|\le K\}}\nabla(S(u)\varphi)\,dx\\ &= \int_\Omega S_n(v)X_K\nabla(S(u)\varphi)\,dx\\ &= \int_\Omega S_n(v)a\big(x,T_K(u),\nabla T_K(u)\big)\nabla(S(u)\varphi)\,dx\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega S_n(v^\varepsilon)a^\varepsilon(x,u^\varepsilon,\nabla u^\varepsilon)\nabla(S(u)\varphi)\,dx &= \lim_{n\to+\infty}\int_\Omega S_n(v)a\big(x,T_K(u),\nabla T_K(u)\big)\nabla(S(u)\varphi)\,dx\\ &= \int_\Omega a\big(x,T_K(u),\nabla T_K(u)\big)\nabla(S(u)\varphi)\,dx\\ &= \int_\Omega a(x,u,\nabla u)\nabla(S(u)\varphi)\,dx.\end{aligned}$$

– *Limit of the second term in* (58). Since $\operatorname{supp}(S_n') \subset [-(n+1), -n] \cup [n+1, n]$ for any $n \geq 1$, we have, as a consequence, that

$$\left| \int_\Omega S(u)\varphi a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)\nabla S_n(v^\varepsilon)\, dx \right| \\ \leq \|S\|_{L^\infty(\mathbb{R})} \|\varphi\|_{L^\infty(\Omega)} \int_{\{n \leq |v^\varepsilon| \leq n+1\}} |a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)\nabla v^\varepsilon|\, dx. \quad (59)$$

Taking the limit as ε tends to 0 and n tends to $+\infty$ in (59) and using the estimate (32) yields

$$\lim_{n\to+\infty} \lim_{\varepsilon\to 0} \left| \int_\Omega S(u)\varphi a^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)\nabla S_n(v^\varepsilon)\, dx \right| = 0.$$

– *Limit of the right-hand side of* (58). Due to (15) and (29), we have

$$\lim_{n\to+\infty} \lim_{\varepsilon\to 0} \int_\Omega f^\varepsilon S_n(v^\varepsilon) S(u)\varphi\, dx = \int_\Omega f S(u)\varphi\, dx.$$

Thanks to the above convergence results, we are in a position to pass to the limit as ε tends to 0 in (58) and to conclude that u satisfies (12). The proof of Theorem 3.1 is achieved. □

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