Squared White Noise and other Non-Gaussian Noises as Lévy Processes on Real Lie Algebras

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Abstract

It is shown how the relations of the renormalized squared white noise defined by Accardi, Lu, and Volovich [ALV99] can be realized as factorizable current representations or Lévy processes on the real Lie algebra sl_2. This allows to obtain its Itô table, which turns out to be infinite-dimensional. The linear white noise without or with number operator is shown to be a Lévy process on the Heisenberg-Weyl Lie algebra or the oscillator Lie algebra. Furthermore, a joint realization of the linear and quadratic white noise relations is constructed, but it is proved that no such realizations exist with a vacuum that is an eigenvector of the central element and the annihilator. Classical Lévy processes are shown to arise as components of Lévy process on real Lie algebras and their distributions are characterized.

1 Introduction

Motivated by physical models with non-linear interaction, Accardi, Lu, and Volovich [ALV99] tried to define a quantum stochastic calculus for the squares of the creation and annihilation processes on the symmetric Fock space. They showed that this requires a renormalization and postulated the algebraic relations a square of white noise process should satisfy. Recently several authors constructed a realization of the square of white noise relations (see below), using different approaches , see [ALV99, AS99a, Śni00].

After the renormalization procedure (which we shall not discuss here, we will simply take its output as our starting point) the problem of defining a squared white noise (SWN) calculus can be formulated as follows. First, one asks for realization of the SWN relations on some (pre-) Hilbert space D, i.e. for maps b, b^+, n from $\Sigma(\mathbb{R}_+)$ to $\mathcal{L}(D)$, such that b^+ and n are linear, b is anti-linear, and the following relations are satisfied for all $\phi, \psi \in \Sigma(\mathbb{R}_+)$,

$$b_\phi b_\psi^+ - b_\psi^+ b_\phi = \gamma\langle\phi,\psi\rangle + n_{\overline{\phi}\psi}, \tag{1a}$$

$$n_\phi b_\psi - b_\psi n_\phi = -2b_{\overline{\phi}\psi}, \tag{1b}$$

$$n_\phi b_\psi^+ - b_\psi^+ n_\phi = 2b_{\phi\psi}^+, \tag{1c}$$

$$(b_\phi)^* = b_\phi^+, \qquad (n_\phi)^* = n_{\overline{\phi}}, \tag{1d}$$

where $\Sigma(\mathbb{R}_+) = \{\phi = \sum_{i=1}^n \phi_i 1_{[s_i,t_i[}; \phi_i \in \mathbb{C}, s_i < t_i \in \mathbb{R}_+, n \in \mathbb{N}\}$ is the algebra of step functions on $\mathbb{R}_+$ with bounded support and finitely many values, $\mathcal{L}(D)$ is the algebra of adjointable linear operators on D, and γ is a fixed real parameter (coming from the renormalization). Furthermore, operators corresponding to functions with disjoint supports should always commute.

The second part of this problem consists in defining quantum stochastic integrals with respect to these three operator processes and finding their Itô table, i.e. their mutual quadratic variations.

The goal of this paper is to show that the theory of factorizable representations of current algebras developped in the early seventies by Araki, Streater, etc. (see, e.g., [PS72, Gui72] and the references therein) solves the first part of the problem.

The simple current algebra $g^{\mathbb{T}}$ of a real Lie algebra g over a measure space $(\mathbb{T}, \mathcal{T}, \mu)$ is defined as the space of simple functions on $\mathbb{T}$ with values in g,

$$g^{\mathbb{T}} = \left\{ X = \sum_{i=1}^{n} X_i 1_{M_i}; X_i \in g, M_i \in \mathcal{T}, n \in \mathbb{N} \right\}.$$

This is a real Lie algebra with the Lie bracket and the involution defined pointwise. The SWN relations (1) imply that any realization of SWN on a pre-Hilbert space D defines a representation π of the current algebra $sl_2^{\mathbb{R}_+}$ of the real Lie algebra sl_2 over $\mathbb{R}_+$ (with the Borel σ-algebra and the Lebesgue measure) on D by

$$B^- 1_{[s,t[} \mapsto b_{1_{[s,t[}}, \quad B^+ 1_{[s,t[} \mapsto b^+_{1_{[s,t[}}, \quad M 1_{[s,t[} \mapsto \gamma(t-s) + n_{1_{[s,t[}},$$

where sl_2 is the three-dimensional real Lie algebra spanned by $\{B^+, B^-, M\}$, with the commutation relations

$$[B^-, B^+] = M, \quad [M, B^\pm] = \pm 2 B^\pm,$$

and the involution $(B^-)^* = B^+$, $M^* = M$. The converse is obviously also true, every representation of the current algebra $sl_2^{\mathbb{R}_+}$ defines a realization of the SWN relations (1). Looking only at indicator functions of intervals we get a family of $*$-representations $(j_{st})_{0 \le s \le t}$ on D of the Lie algebra sl_2,

$$j_{st}(X) = \pi(X 1_{[s,t[}), \qquad \text{for all } X \in sl_2.$$

By the universal property these $*$-representations extend to $*$-representations of the universal enveloping algebra $\mathcal{U}(sl_2)$ of sl_2. If there exists a vector Ω

in $\mathcal{L}(D)$ such that the representations corresponding to disjoint intervals are independent (in the sense of Definition 2.1, Condition 2), i.e. if they commute and their expectations in the state $\Phi(\cdot) = \langle \Omega, \cdot \, \Omega \rangle$ factorize, then $(j_{st})_{0\leq s\leq t}$ is a Lévy process on sl_2 (in the sense of Definition 2.1). This condition is satisfied in the constructions in [ALV99, AS99a, Śni00]. They are of 'Fock type' and have a fixed special vector, the so-called vacuum, and the corresponding vector state has the desired factorization property.

On the other hand, given a Lévy process on sl_2 on a pre-Hilbert space D, we can construct a realization of the SWN relations (1) on D. Simply set

$$b_\phi = \sum_{i=1}^n \overline{\phi_i} j_{s_i,t_i}(B^-),\ b^+_\phi = \sum_{i=1}^n \phi_i j_{s_i,t_i}(B^+),\ n_\phi = \sum_{i=1}^n \phi_i \big(j_{s_i,t_i}(M) - \gamma(t_i - s_i)\mathrm{id}_D \big),$$

for $\phi = \sum_{i=1}^n \phi_i 1_{[s_i,t_i[} \in \Sigma(\mathbb{R}_+)$.

We see that in order to construct realizations of the SWN relations we can construct Lévy processes on sl_2. Furthermore, all realizations that have a vacuum vector in which the expectations factorize, will arise in this way.

In this paper we show how to classify the Lévy processes on sl_2 and how to construct realizations of these Lévy processes acting on (a subspace of) the symmetric Fock space over $L^2(\mathbb{R}_+, H)$ for some Hilbert space H. Given the *generator* L of a Lévy process, we immediately can write down a realization of the process; see Equation (2). It is a linear combination of the four fundamental integrators in Hudson-Parthasarathy quantum stochastic calculus: conservation, creation, annihilation and time. This also allows to write down their Itô tables (see Equation (3)) and reduces the problem of defining quantum stochastic integrals w.r.t. to these operator processes to the Hudson-Parthasarathy calculus.

Even though the theory of Lévy processes has been developed for arbitrary involutive bialgebras, cf. [ASW88, Sch93], we will only consider (enveloping algebras of) real Lie algebras here. This allows some simplification, in particular we do not need to make explicit use of the coproduct. Instead of "Lévy process on the real Lie algebra g" we could also say "factorizable unitary representation of the simple current algebra $g^{\mathbb{R}_+}$ of the real Lie algebra g over $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \mathrm{d}t)$" in this paper. Besides the real Lie algebra sl_2, we shall also consider several other real Lie algebras, including the Heisenberg-Weyl Lie algebra hw, the oscillator Lie algebra osc, and the finite-difference Lie algebra fd.

This paper is organized as follows.

In Section 2, we recall the definitition of Lévy processes on real Lie algebras and present their fundamental properties. We also outline how the Lévy processes on a given real Lie algebra can be characterized and constructed as a linear combination of the four fundamental processes of Hudson-Parthasarathy quantum stochastic calculus.

In Section 3, we list all Gaussian Lévy processes or Lévy processes associated to integrable unitary irreducible representations for several real Lie algebras in terms of their generators or Schürmann triples (see Definition 2.2). We also give explicit realizations on a boson Fock space for several examples. These examples include the processes on the finite-difference Lie algebra defined by Boukas [Bou88, Bou91] and by Parthasarathy and Sinha [PS91] as well as a process on sl_2 that has been considered previously by Feinsilver and Schott [FS93, Section 5.IV]. See also [VGG73] for factorizable current representations of current groups over $SL(2,\mathbb{R})$.

Finally, in Section 4, we show that the restriction of a Lévy process to one single element of the real Lie algebra always gives rise to a classical Lévy process. We give a characterization this process in terms of its Fourier transform. For several examples we also explicitly compute its Lévy measure or its marginal distribution. It turns out that the densities of self-adjoint linear combinations of the SWN operators $b_{1_{[s,t[}}$, $b^+_{1_{[s,t[}}$, $n_{1_{[s,t[}}$ in the realization considered in [ALV99, AS99a, Śni00] are the measures of orthogonality of the Laguerre, Meixner, and Meixner-Pollaczek polynomials.

2 Lévy processes on real Lie algebras

In this section we give the basic definitions and properties of Lévy processes on real Lie algebras. This is a special case of the theory of Lévy processes on involutive bialgebras, for more detailed accounts on these processes see [Sch93], [Mey95, Chapter VII],[FS99]. For a list of references on factorizable representations of current groups and algebras and a historical survey, we refer to [Str00, Section 5].

Let $g_\mathbb{R}$ be a real Lie algebra, g its complexification, and $\mathcal{U}(g)$ its universal enveloping algebra. We denote by $\mathcal{U}_0(g)$ the (non-unital!) subalgebra of $\mathcal{U}$ generated by g. If $X_1,\ldots,X_d$ is a basis of g, then

$$\{X_i^{n_1}\cdots X_d^{n_d}|n_1,\ldots,n_d\in\mathbb{N}, n_1+\cdots+n_d\geq 1\}$$

is a basis of $\mathcal{U}_0(g)$. Furthermore, we equip $\mathcal{U}(g)$ and $\mathcal{U}_0(g)$ with the (unique!) involution for which the elements of $g_\mathbb{R}$ are anti-hermitian.

Definition 2.1 *Let D be a pre-Hilbert space and $\Omega \in D$ a unit vector. We call a family $\big(j_{st} : \mathcal{U}(g) \to \mathcal{L}(D)\big)_{0\le s\le t}$ of unital $*$-representations of $\mathcal{U}(g)$ a* Lévy process *on $g_\mathbb{R}$ over D (with respect to Ω), if the following conditions are satisfied.*

1. *(Increment property) We have*
$$j_{st}(X) + j_{tu}(X) = j_{su}(X)$$
for all $0 \le s \le t \le u$ and all $X \in g$.

2. *(Independence) We have $[j_{st}(X), j_{s't'}(Y)] = 0$ for all $X, Y \in g$, $0 \le s \le t \le s' \le t'$ and*
$$\langle \Omega, j_{s_1t_1}(u_1)\cdots j_{s_nt_n}(u_n)\Omega\rangle = \langle \Omega, j_{s_1t_1}(u_1)\Omega\rangle \cdots \langle \Omega, j_{s_nt_n}(u_n)\Omega\rangle$$
for all $n \in \mathbb{N}$, $0 \le s_1 \le t_1 \le s_2 \le \cdots \le t_n$, $u_1, \ldots, u_n \in \mathcal{U}(g)$.

3. *(Stationarity) The functional $\varphi_{st} : \mathcal{U}(g) \to \mathbb{C}$ defined by*
$$\varphi(u) = \langle \Omega, j_{st}(u)\Omega\rangle, \qquad u \in \mathcal{U}_0(g),$$
depends only on the difference $t - s$.

4. *(Weak continuity) We have $\lim_{t\searrow s}\langle \Omega, j_{st}(u)\Omega\rangle = 0$ for all $u \in \mathcal{U}_0(g)$.*

If $(j_{st})_{0\le s\le t}$ is a Lévy process on $g_\mathbb{R}$, then the functionals $\varphi_t = \langle \Omega, j_{0t}(\cdot)\Omega\rangle : \mathcal{U}(g) \to \mathbb{C}$ are actually states. Furthermore, they are differentiable w.r.t. t and
$$L(u) = \lim_{t\searrow 0} \frac{1}{t}\varphi_t(u), \qquad u \in \mathcal{U}_0(g),$$
defines a positive hermitian linear functional on $\mathcal{U}_0(g)$. Such a functional is called a *generator*.

Let $\big(j^{(1)}_{st} : \mathcal{U}(g) \to \mathcal{L}(D^{(1)})\big)_{0\le s\le t}$ and $\big(j^{(2)} : \mathcal{U}(g) \to \mathcal{L}(D^{(2)})\big)_{0\le s\le t}$ be two Lévy processes on $g_\mathbb{R}$ with respect to the state vectors $\Omega^{(1)}$ and $\Omega^{(2)}$, resp. We call them *equivalent*, if all their moments agree, i.e. if
$$\langle \Omega^{(1)}, j^{(1)}_{s_1t_1}(u_1)\cdots j^{(1)}_{s_nt_n}(u_n)\Omega^{(1)}\rangle = \langle \Omega^{(2)}, j^{(2)}_{s_1t_1}(u_1)\cdots j^{(2)}_{s_nt_n}(u_n)\Omega^{(2)}\rangle,$$
for all $n \in \mathbb{N}$, $0 \le s_1 \le t_1 \le s_2 \le \cdots \le t_n$, $u_1, \ldots, u_n \in \mathcal{U}(g)$.

By a GNS-type construction, one can associate to every generator a so-called Schürmann triple.

Definition 2.2 *A* Schürmann triple *on $g_{\mathbb{R}}$ is a triple (ρ, η, L), where ρ is a $*$-representation of $\mathcal{U}_0(g)$ on some pre-Hilbert space D, $\eta : \mathcal{U}_0(g) \to D$ is a surjective ρ-1-cocycle, i.e. it satisfies*

$$\eta(uv) = \rho(u)\eta(v),$$

for all $u, v \in \mathcal{U}_0(g)$, and $L : \mathcal{U}_0(g) \to \mathbb{C}$ is a hermitian linear functional such that the linear map $(u, v) \mapsto -\langle \eta(u^*), \eta(v)\rangle$ is the 2-coboundary of L (w.r.t. the trivial representation), i.e.

$$L(uv) = \langle \eta(u^*), \eta(v)\rangle$$

for all $u, v \in \mathcal{U}_0(g)(\rho, \eta, L)$.

Let (ρ, η, L) be a Schürmann triple on $g_{\mathbb{R}}$, acting on a pre-Hilbert space D. We can define a Lévy process on the symmetric Fock space $\Gamma\big(L^2(\mathbb{R}_+, D)\big) = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+, D)^{\odot n}$ by setting

$$j_{st}(X) = \Lambda_{st}\big(\rho(X)\big) + A^*_{st}\big(\eta(X)\big) + A_{st}\big(\eta(X^*)\big) + L(X)(t-s)\mathrm{id}, \quad (2)$$

for $X \in g$, where Λ_{st}, A^*_{st}, A_{st} denote the conservation, creation, and annihilation processes on $\Gamma\big(L^2(\mathbb{R}_+, D)\big)$, cf. [Par92, Mey95]. It is straightforward to check that we have

$$\big[j_{st}(X), j_{st}(Y)\big] = j_{st}\big([X,Y]\big), \quad \text{and} \quad j_{st}(X)^* = j_{st}(X^*)$$

for all $0 \leq s \leq t$, $X, Y \in g$. By the universal property, the family

$$\Big(j_{st} : g \to \mathcal{L}\Big(\Gamma\big(L^2(\mathbb{R}_+, D)\big)\Big)\Big)_{0\leq s\leq t}$$

extends to a unique family $(j_{st})_{0\leq s\leq t}$ of unital $*$-representations of $\mathcal{U}(g)$, and it is not difficult to verify that this family is a Lévy process with generator L on g over $\Gamma\big(L^2(\mathbb{R}_+, D)\big)$ with respect to the Fock vacuum Ω.

We summarize the relation between Lévy processes, convolution semi-groups of states, generators and Schürmann triples in the following theorem.

Theorem 2.3 *[Sch93] The Lévy processes on $g_{\mathbb{R}}$ (modulo equivalence), convolution semi-groups of states on $\mathcal{U}(g)$, generators on $\mathcal{U}_0(g)$, and Schürmann triples on $g_{\mathbb{R}}$ (modulo unitary equivalence) are in one-to-one correspondence. If $(k_{st})_{0\leq s\leq t}$ is a Lévy process with generator L and (ρ, η, L) a Schürmann triple, then $(k_{st})_{0\leq s\leq t}$ is equivalent to the Lévy process $(j_{st})_{0\leq s\leq t}$ associated to (ρ, η, L) defined in Equation (2).*

Remark 2.4 *Since we know the Itô table for the four H-P integrators,*

$\bullet$	$\mathrm{d}A^*(u)$	$\mathrm{d}\Lambda(F)$	$\mathrm{d}A(u)$	$\mathrm{d}t$
$\mathrm{d}A^*(v)$	0	0	0	0
$\mathrm{d}\Lambda(G)$	$\mathrm{d}A^*(Gu)$	$\mathrm{d}\Lambda(GF)$	0	0
$\mathrm{d}A(v)$	$\langle v,u\rangle\mathrm{d}t$	$\mathrm{d}A(F^*v)$	0	0
$\mathrm{d}t$	0	0	0	0

for all $F,G\in\mathcal{L}(D)$, $u,v\in D$, we can deduce the Itô tables for the Lévy processes on g. The map d_L associating elements u of the universal enveloping algebra to the corresponding quantum stochastic differentials $\mathrm{d}_L u$ defined by

$$\mathrm{d}_L u = \mathrm{d}\Lambda\big(\rho(u)\big) + \mathrm{d}A^*\big(\eta(u)\big) + \mathrm{d}A\big(\eta(u^*)\big) + L(u)\mathrm{d}t, \tag{3}$$

is a (non-unital!) $*$-homomorphism from $\mathcal{U}_0(g)$ to the Itô algebra over D, see [FS99, Proposition 4.4.2]. It follows that the dimension of the Itô algebra generated by $\{\mathrm{d}_L X; X\in g\}$ is at least the dimension of D (since η is supposed surjective) and not bigger than $(\dim D+1)^2$. If D is infinite-dimensional, then its dimension is also infinite. Note that it depends on the choice of the Lévy process.

Due to Theorem 2.3, the problem of characterizing and constructing all Lévy processes on a given real Lie algebra can be decomposed into the following steps. First, classify all $*$-representations of $\mathcal{U}(g)$ (modulo unitary equivalence), this will give the possible choices for the representation ρ in the Schürmann triple. Next determine all surjective ρ-1-cocycles. We distinguish between trivial cocycles, i.e. cocycles which are of the form

$$\eta(u) = \rho(u)\omega, \qquad u\in\mathcal{U}_0(g)$$

for some vector $\omega\in D$ in the representation space of ρ, and non-trivial cocycles, i.e. cocycles, which can not be written in this form. We will denote the space of all cocycles of a given $*$-representation ρ on some pre-Hilbert space D be $Z^1(\mathcal{U}_0(g),\rho,D)$, that of trivial ones by $B^1(\mathcal{U}_0(g),\rho,D)$. The quotient $H^1(\mathcal{U}_0(g),\rho,D) = Z^1(\mathcal{U}_0(g),\rho,D)/B^1(\mathcal{U}_0(g),\rho,D)$ is called the fi

rst cohomology group of ρ. In the last step we determine all generators L that turn a pair (ρ,η) into a Schürmann triple (ρ,η,L). This can again also be viewed as a cohomological problem. If η is a ρ-1-cocycle, then the linear map $(u,v)\mapsto -\langle\eta(u^*),\eta(v)\rangle$ is a 2-cocycle for the trivial representation, i.e.

it satisfies $-\langle\eta\big((uv)^*\big),\eta(w)\rangle+\langle\eta(u^*),\eta(vw)\rangle=0$ for all $u,v,w\in\mathcal{U}_0(g)$. For L we can take any hermitian functional that has the map $(u,v)\mapsto -\langle\eta(u^*),\eta(v)\rangle$ as coboundary, i.e. L has to satisfy $L(u^*)=\overline{L(u)}$ and $L(uv)=\langle\eta(u^*),\eta(v)\rangle$ for all $u,v\in\mathcal{U}_0(g)$. If η is trivial, then such a functional always exists, we can take $L(u)=\langle\omega,\rho(u)\omega\rangle$. For a given pair (ρ,η), L is determined only up to a hermitian 0-1-cocycle, i.e. a hermitian functional ℓ that satisfies $\ell(uv)=0$ for all $u,v\in\mathcal{U}_0(g)$.

Remark 2.5 *Let us call a linear $*$-map $\pi: g\to\mathcal{L}(D)$ an projective $*$-representation of g, if there exists a linear map $\alpha: g\times g\to\mathbb{C}$, such that*

$$\big[\pi(X),\pi(Y)\big]=\pi\big([X,Y]\big)+\alpha(X,Y)\mathrm{id},$$

for all $X,Y\in g$. Every projective $*$-representation defines a $*$-representation of a central extension $\tilde{g}$ of g. As a vector space $\tilde{g}$ is defined as $\tilde{g}=g\oplus\mathbb{C}$. The Lie bracket and the involution are defined by

$$\big[(X,\lambda),(Y,\mu)\big]=\big([X,Y],\alpha(X,Y)\big),\qquad (X,\lambda)^*=(X^*,\overline{\lambda})$$

for $(X,\lambda),(Y,\mu)\in\tilde{g}$. It is not hard to check that

$$\tilde{\pi}\big((X,\lambda)\big)=\pi(X)+\lambda\mathrm{id}$$

defines a $*$-representation of $\tilde{g}$. If the cocycle α is trivial, i.e. if there exists a (hermitian) linear functional β such that $\alpha(X,Y)=\beta([X,Y])$ for all $X,Y\in g$, then the central extension is trivial, i.e. $\tilde{g}$ is isomorphic to the direct sum of g with the (abelian) one-dimensional Lie algebra $\mathbb{C}$. Such an isomorphism is given by $g\oplus\mathbb{C}\ni(X,\mu)\mapsto(X,\beta(X)+\mu)\in\tilde{g}$. This implies that in this case

$$\pi_\beta(X)=\tilde{\pi}\big((X,\beta(X))\big)=\pi(X)+\beta(X)\mathrm{id}$$

defines a $*$-representation of g.

For a pair (ρ,η) consisting of a $*$-representation ρ and a ρ-1-cocycle η we can always define a family of projective $*$-representations $(k_{st})_{0\le s\le t}$ of g by setting

$$k_{st}\big((X,\lambda)\big)=\Lambda_{st}\big(\rho(X)\big)+A^*_{st}\big(\eta(X)\big)+A_{st}\big(\eta(X^*)\big),$$

for $(X,\lambda)\in\tilde{g}$, $0\le s\le t$. Using the commutation relations of the creation, annihilation, and conservation operators, one finds that the 2-cocycle α is given by $(X,Y)\mapsto\alpha(X,Y)=\langle\eta(X^*),\eta(Y)\rangle-\langle\eta(Y^*),\eta(X)\rangle$. If it is trivial, *then $(k_{st})_{0\leq s\leq t}$ can be used to define a Lévy process on g. More precisely, if there exists a hermitian functional ψ on $\mathcal{U}_0(g)$ such that $\psi(uv)=\langle\eta(u^*),\eta(v)\rangle$ holds for all $u,v\in\mathcal{U}_0(g)$, then (ρ,η,ψ) is a Schürmann triple on g and therefore defines a Lévy process on g. But even if such a hermitian functional ψ does not exist, we can define a Lévy process on $\tilde{g}$ by setting*

$$\tilde{k}_{st}\big((X,\lambda)\big)=\Lambda_{st}\big(\rho(X)\big)+A^*\big(\eta(X)\big)+A\big(\eta(X^*)\big)+(t-s)\lambda\mathrm{id},$$

for $(X,\lambda)\in\tilde{g}$, $0\leq s\leq t$.

We close this section with a two useful lemmas on the relevent cohomology groups.

Schürmann triples (ρ,η,L), where the $*$-representation ρ is equal to the trivial representation defined by $0:\mathcal{U}_0(g)\ni u\mapsto 0\in\mathcal{L}(D)$ are called *Gaussian*, as well as the corresponding processes, cocycles, and generators. The following lemma completely classifies all Gaussian cocycles of a given Lie algebra.

Lemma 2.6 *Let D be an arbitrary complex vector space, and 0 the trivial representation of g on D. We have*

$$Z^1(\mathcal{U}_0(g),0,D)\cong\big(g/[g,g]\big)^*,\qquad B^1(\mathcal{U}_0(g),0,D)=\{0\},$$

and therefore $\dim H^1(\mathcal{U}_0(g),0,D)=\dim g/[g,g]$.

Proof 1 *Let ϕ be a linear functional on $g/[g,g]$, then we can extend it to a (unique) 0-1-cocycle on the algebra $\mathcal{U}_0(g/[g,g])$ (this is the free abelian algebra over $g/[g,g]$), which we denote by $\tilde{\phi}$. Denote by π the canonical projection from g to $g/[g,g]$, by the universal property of the enveloping algebra it has a unique extension $\tilde{\pi}:\mathcal{U}_0(g)\to\mathcal{U}_0(g/[g,g])$. We can define a cocycle η_ϕ on $\mathcal{U}_0(g)$ by $\eta_\phi=\tilde{\phi}\circ\tilde{\pi}$. Furthermore, since any 0-1-cocycle on $\mathcal{U}_0(g)$ has to vanish on $[g,g]$ (because $Y=[X_1,X_2]$ implies $\eta(Y)=0\eta(X_2)-0\eta(X_1)=0$), the map $\phi\mapsto\eta_\phi$ is bijective.*

The following lemma shows that a representation of $\mathcal{U}(g)$ can only have non-trivial cocycles, if the center of $\mathcal{U}_0(g)$ acts trivially.

Lemma 2.7 *Let ρ be a representation of g on some vector space D and let $C\in\mathcal{U}_0(g)$ be central. If $\rho(C)$ is invertible, then*

$$H^1(\mathcal{U}_0(g),\rho,D)=\{0\}.$$

Proof 2 *Let η be a ρ-cocycle on $\mathcal{U}_0(g)$ and $C \in \mathcal{U}_0(g)$ such that $\rho(C)$ is invertible. Then we get*

$$\rho(C)\eta(u) = \eta(Cu) = \eta(uC) = \rho(u)\eta(C)$$

and therefore $\eta(u) = \rho(u)\rho(C)^{-1}\eta(C)$ for all $u \in \mathcal{U}_0(g)$, i.e. $\eta(u) = \rho(u)\omega$, where $\omega = \rho(C)^{-1}\eta(C)$. This shows that all ρ-cocycles are trivial.

3 Examples

In this section we completely classify the Gaussian generators for several real Lie algebras and determine the non-trivial cocycles for some or all of their integrable unitary irreducible representations, i.e. those representations that arise by differentiating unitary irreducible representations of the corresponding Lie group. These are $*$-representations of the enveloping algebra $\mathcal{U}(g)$ on some pre-Hilbert space D for which the Lie algebra elements are mapped to essentially self-adjoint operators. For some of the processes we give explicit realizations on the boson Fock space.

3.1 White noise or Lévy processes on *hw* and *osc*

The Heisenberg-Weyl Lie algebra *hw* is the three-dimensional Lie algebra with basis $\{A^+, A^-, E\}$, commutation relations

$$[A^-, A^+] = E, \quad [A^\pm, E] = 0,$$

and involution $(A^-)^* = A^+$, $E^* = E$. Adding a hermitian element N with commutation relations

$$[N, A^\pm] = \pm A^\pm, \quad [N, E] = 0,$$

we obtain the four-dimensional oscillator Lie algebra *osc*.

We begin with the classification of all Gaussian generators on these two Lie algebras.

Proposition 3.1 a *Let $v_1, v_2 \in \mathbb{C}^2$ be two vectors and $z \in \mathbb{C}$ an arbitrary complex number. Then*

$$\rho(A^+) = \rho(A^-) = \rho(E) = 0,$$
$$\eta(A^+) = v_1, \quad \eta(A^-) = v_2, \quad \eta(E) = 0,$$
$$L(A^+) = z, \quad L(A^-) = \overline{z}, \quad L(E) = ||v_1||^2 - ||v_2||^2,$$

defines the Schürmann triple on $D = \mathrm{span}\,\{v_1, v_2\}$ *of a Gaussian generator on* $\mathcal{U}_0(hw)$*. Furthermore, all Gaussian generators on* $\mathcal{U}_0(hw)$ *arise in this way.*

b *The Schürmann triples of Gaussian generators on* $\mathcal{U}_0(osc)$ *are all of the form*

$$\begin{aligned} \rho(N) = \rho(A^+) = \rho(A^-) = \rho(E) = 0, \\ \eta(N) = v, \quad \eta(A^+) = \eta(A^-) = \eta(E) = 0, \\ L(N) = b, \quad L(A^+) = L(A^-) = L(E) = 0, \end{aligned}$$

with $v \in \mathbb{C}$*,* $b \in \mathbb{R}$*.*

Proof 3 *The form of the Gaussian cocycles on* $\mathcal{U}_0(hw)$ *and* $\mathcal{U}_0(osc)$ *follows from Lemma 2.6. Then one checks that for all these cocycles there do indeed exist generators and computes their general form.*

For an arbitrary Gaussian Lévy process on hw we therefore get

$$\begin{aligned} \mathrm{d}_L A^+ &= \mathrm{d}A^*(v_1) + \mathrm{d}A(v_2) + z\mathrm{d}t, \\ \mathrm{d}_L A^- &= \mathrm{d}A^*(v_2) + \mathrm{d}A(v_1) + \overline{z}\mathrm{d}t, \\ \mathrm{d}_L E &= \big(||v_1||^2 - ||v_2||^2\big)\mathrm{d}t, \end{aligned}$$

and the Itô table

$\bullet$	$\mathrm{d}_L A^+$	$\mathrm{d}_L A^-$	$\mathrm{d}_L E$
$\mathrm{d}_L A^+$	$\langle v_2, v_1\rangle \mathrm{d}t$	$\langle v_2, v_2\rangle \mathrm{d}t$	0
$\mathrm{d}_L A^-$	$\langle v_1, v_1\rangle \mathrm{d}t$	$\langle v_1, v_2\rangle \mathrm{d}t$	0
$\mathrm{d}_L E$	0	0	0

.

For $||v_1||^2 = 1$ and $v_2 = 0$, this is the usual Itô table for the creation and annihilation process in Hudson-Parthasarathy calculus.

Any integrable unitary irreducible representation of hw is equivalent to either one of the one-dimensional representations defined by

$$\pi_z(A^+) = z, \quad \pi_z(A^-) = \overline{z}, \quad \pi_z(E) = 0,$$

for some $z \in \mathbb{C}$, or one of the infinite-dimensional representations defined by

$$\rho_h(A^+)e_n = \sqrt{(n+1)h}\, e_{n+1}, \quad \rho_h(A^-) = \sqrt{nh}\, e_{n-1}, \quad \rho_h(E)e_n = he_n, \quad (4)$$

and

$$\rho_{-h}(A^-)e_n = \sqrt{(n+1)h}\, e_{n+1}, \quad \rho_{-h}(A^+) = \sqrt{nh}\, e_{n-1}, \quad \rho_{-h}(E)e_n = -he_n,$$

where $h > 0$, and $\{e_0, e_1, \ldots\}$ is a orthonormal basis of ℓ^2. By Lemma 2.7, the representations ρ_h have no non-trivial cocycles. But by a simple computation using the defining relations of hw we see that, for $z \neq 0$, the representations of the form $\pi_z \, \mathrm{id}_D$ also have only one trivial cocycle. From $A^+E = EA^+$ we get

$$z\eta(E) = \eta(A^*E) - \eta(EA^+) = \pi_z(E)\eta(A^+) = 0$$

and therefore $\eta(E) = 0$. But $E = A^-A^+ - A^+A^-$ implies

$$0 = \eta(E) = \pi_z(A^-)\eta(A^+) - \pi_z(A^+)\eta(A^-) = \overline{z}\eta(A^+) - z\eta(A^-),$$

and we see that η is the coboundary of $\omega = z^{-1}\eta(A^+)$. Thus the integrable unitary irreducible representations (except the trivial one) of hw have no non-trivial cocycles.

Let us now consider the oscillator Lie algebra osc. The elements E and $NE - A^+A^-$ generate the center of $\mathcal{U}_0(osc)$. If we want an irreducible representation of $\mathcal{U}(osc)$, which has non-trivial cocycles, they have to be represented by zero. But this implies that we have also $\rho(A^+) = \rho(A^-) = 0$ (since we are only interested in $*$-representations). Thus we are lead to study the representations ρ_ν defined by

$$\rho_\nu(N) = \nu \, \mathrm{id_D}, \quad \rho_\nu(A^+) = \rho_\nu(A^-) = \rho_\nu(E) = 0,$$

with $\nu \in \mathbb{R}\backslash\{0\}$. It is straightforward to determine all their cocycles and generators.

Proposition 3.2 *For $\nu \in \mathbb{R}$, $\nu \notin \{-1, 0, 1\}$, all cocycles of ρ_ν are of the form*

$$\eta(N) = v, \quad \eta(A^+) = \eta(A^-) = \eta(E) = 0,$$

for some $v \in D$ and thus trivial (coboundaries of $\omega = \nu^{-1}v$).

For $\nu = 1$ they are of the form

$$\eta(N) = v_1, \quad \eta(A^+) = v_2, \quad \eta(A^-) = \eta(E) = 0,$$

and for $\nu = -1$ of the form

$$\eta(N) = v_1, \quad \eta(A^-) = v_2, \quad \eta(A^+) = \eta(E) = 0,$$

with some vectors $v_1, v_2 \in D$*. Therefore we get*

$$\dim H^1(\mathcal{U}_0(osc), \rho_{\pm 1}, D) = 1, \quad \dim B^1(\mathcal{U}_0(osc), \rho_{\pm 1}, D) = 1$$

and

$$\dim H^1(\mathcal{U}_0(osc), \rho_\nu, D) = 0, \quad \dim B^1(\mathcal{U}_0(osc), \rho_\nu, D) = 1$$

for $\nu \in \mathbb{R}\backslash\{-1,0,1\}$*.*

Let now $\nu = 1$, the case $\nu = -1$ is similar, since ρ_1 and ρ_{-1} are related by the automorphism $N \mapsto -N$, $A^+ \mapsto A^-$, $A^- \mapsto A^+$, $E \mapsto -E$. It turns out that for all the cocycles given in the preceding proposition there exists a generator, and we obtain the following result.

Proposition 3.3 *Let* $v_1, v_2 \in \mathbb{C}^2$ *and* $b \in \mathbb{R}$*. Then* $\rho = \rho_1$*,*

$$\begin{gathered} \eta(N) = v_1, \quad \eta(A^+) = v_2, \quad \eta(A^-) = \eta(E) = 0, \\ L(N) = b, \quad L(E) = ||v_2||^2, \quad L(A^+) = L(A^-) = 0, \end{gathered}$$

defines a Schürmann triple on osc acting on $D = \mathrm{span}\,\{v_1, v_2\}$*. The corresponding quantum stochastic differentials are*

$$\begin{aligned} \mathrm{d}_L N &= \mathrm{d}\Lambda(\mathrm{id}) + \mathrm{d}A^*(v_1) + \mathrm{d}A(v_1) + b\mathrm{d}t, \\ \mathrm{d}_L A^+ &= \mathrm{d}A^*(v_2), \\ \mathrm{d}_L A^- &= \mathrm{d}A(v_2), \\ \mathrm{d}_L E &= ||v_2||^2\mathrm{d}t, \end{aligned}$$

and they satisfy the following Itô table

$\bullet$	$\mathrm{d}_L A^+$	$\mathrm{d}_L N$	$\mathrm{d}_L A^-$	$\mathrm{d}_L E$
$\mathrm{d}_L A^+$	0	0	0	0
$\mathrm{d}_L N$	$\mathrm{d}_L A^+ + \langle v_1, v_2\rangle \mathrm{d}t$	$\mathrm{d}_L N + \big(\|\|v_1\|\|^2 - b\big)\mathrm{d}t$	0	0
$\mathrm{d}_L A^-$	$\mathrm{d}_L E = \|\|v_2\|\|^2\mathrm{d}t$	$\mathrm{d}_L A^- + \langle v_2, v_1\rangle \mathrm{d}t$	0	0
$\mathrm{d}_L E$	0	0	0	0

.

Note that for $||v_2||^2 = 1$, $||v_1||^2 = b$, and $v_1 \perp v_2$, this is the usual Itô table of the four fundamental noises of Hudson-Parthasarathy calculus.

3.2 SWN or Lévy processes on sl_2

The Lie algebra sl_2 is the three-dimensional simple Lie algebra with basis $\{B^+, B^-, M\}$, commutation relations

$$[B^-, B^+] = M, \quad [M, B^\pm] = \pm 2B^\pm,$$

and involution $(B^-)^* = B^+$, $M^* = M$. Its center is generated by the Casimir element

$$C = M(M-2) - 4B^+B^- = M(M+2) - 4B^-B^+.$$

We have $[sl_2, sl_2] = sl_2$, and so $\mathcal{U}_0(sl_2)$ has no Gaussian cocycles, cf. Lemma 2.6, and therefore no Gaussian generators either. Let us now determine all the non-trivial cocycles for the integrable unitary irreducible representations of sl_2.

We have to consider two kinds of representations of $\mathcal{U}(sl_2)$. The first are the lowest and highest weight representations, i.e. the representations induced by $\rho(M)\Omega = m_0\Omega$, $\rho(B^-)\Omega = 0$, and $\rho(M)\Omega = -m_0\Omega$, $\rho(B^+)\Omega = 0$, respectively. The lowest weight representations are spanned by the vectors $v_n = \rho(B^+)^n\Omega$, with $n \in \mathbb{N}$. We get

$$\begin{aligned}
\rho(B^+)v_n &= v_{n+1}, \\
\rho(B^-)v_n &= \rho\left(B^-(B^+)^n\right)\Omega = \rho\left(\frac{1}{4}\big(M(M+2) - C\big)(B^+)^{n-1}\right)\Omega \\
&= n(n+m_0-1)\rho(B^-)^{n-1}\Omega = n(n+m_0-1)v_{n-1}, \\
\rho(M)v_n &= (2n+m_0)v_n).
\end{aligned}$$

If we want to define an inner product on span $\{v_n; n \in \mathbb{N}\}$ such that $M^* = M$ and $(B^-)^* = B^+$, then the v_n have to be orthogonal and their norms have to satisfy the recurrence relation

$$||v_{n+1}||^2 = \langle \rho(B^+)v_n, v_{n+1}\rangle = \langle v_n, \rho(B^-)v_{n+1}\rangle = n(n+m_0-1)||v_n||^2. \quad (5)$$

It follows there exists an inner product on span $\{v_n; n \in \mathbb{N}\}$ such that the lowest weight representation with $\rho(M)\Omega = m_0\Omega$, $\rho(B^-)\Omega = 0$ is a $*$-representation, if and only if the coefficients $n(n+m_0-1)$ in Equation (5) are non-negative for all $n = 0, 1, \ldots$, i.e. if and only if $m_0 \geq 0$. For $m_0 = 0$ we get the trivial one-dimensional representation $\rho_0(B^+)\Omega = \rho_0(B^-)\Omega = \rho_0(M)\Omega = 0$ (since $||v_1||^2 = 0$), for $m_0 > 0$ we get

$$\rho^+_{m_0}(B^+)e_n = \sqrt{(n+1)(n+m_0)}\, e_{n+1}, \quad (6a)$$

$$\rho^+_{m_0}(M)e_n = (2n+m_0)e_n, \tag{6b}$$

$$\rho^+_{m_0}(B^-)e_n = \sqrt{n(n+m_0-1)}\, e_{n-1}, \tag{6c}$$

where $\{e_0, e_1, \ldots\}$ is an orthonormal basis of ℓ^2. Similarly we see that there exists a $*$-representation ρ containing a vector Ω such that $\rho(B^+)\Omega = 0$, $\rho(M)\Omega = -m_0\Omega$, if and only if $m_0 \geq 0$. For $m_0 = 0$ this is the trivial representation, for $m_0 > 0$ it is of the form

$$\rho^-_{m_0}(B^-)e_n = \sqrt{(n+1)(n+m_0)}\, e_{n+1}, \tag{7a}$$

$$\rho^-_{m_0}(M)e_n = -(2n+m_0)e_n, \tag{7b}$$

$$\rho^-_{m_0}(B^+)e_n = \sqrt{n(n+m_0-1)}\, e_{n-1}. \tag{7c}$$

But there exists also another kind of integrable unitary irreducible representations of $\mathcal{U}(sl_2)$, those without a highest or lowest weight vector. To construct these, we consider the representation induced by $\rho(M)\Omega = m_0\Omega$, $\rho(C)\Omega = c\Omega$. The vectors $\{v_{\pm n} = \rho(B^\pm)^n\Omega; n \in \mathbb{N}\}$ form a basis for the induced representation,

$$\begin{aligned}
\rho(M) &= (2n+m_0)v_n \\
\rho(B^+)v_n &= \begin{cases} v_{n+1} & \text{if} \quad n \geq 0, \\ \frac{(m_0+2n+2)(m_0+2n)-c}{4} v_{n+1} & \text{if} \quad n < 0, \end{cases} \\
\rho(B^-)v_n &= \begin{cases} \frac{(m_0+2n-2)(m_0+2n)-c}{4} v_{n-1} & \text{if} \quad n > 0, \\ v_{n-1} & \text{if} \quad n \leq 0, \end{cases}
\end{aligned}$$

We look again for an inner product that turns this representation into a $*$-representation. The v_n have to be orthogonal for such an inner product and their norms have to satisfy the recurrence relations

$$\begin{aligned}
||v_{n+1}||^2 &= \frac{(m_0+2n+2)(m_0+2n)-c}{4}||v_n||^2, \qquad \text{for } n \geq 0, \\
||v_{n-1}||^2 &= \frac{(m_0+2n-2)(m_0+2n)-c}{4}||v_n||^2, \qquad \text{for } n \leq 0.
\end{aligned}$$

Therefore we can define a positive definite inner product on span $\{v_n; n \in \mathbb{Z}\}$, if and only if $\lambda(\lambda+2) > c$ for all $\lambda \in m_0 + 2\mathbb{Z}$. We can restrict ourselves to $m_0 \in [0, 2[$, because the representations induced by (c, m_0) and $(c, m_0 + 2k)$,

$k \in \mathbb{Z}$ turn out to be unitarily equivalent. We get the following family of integrable unitary irreducible representations of $\mathcal{U}(sl_2)$,

$$\rho_{cm_0}(B^+)e_n = \frac{1}{2}\sqrt{(m_0+2n+2)(m_0+2n)-c}\, e_{n+1}, \tag{8a}$$

$$\rho_{cm_0}(M)e_n = (2n+m_0)e_n, \tag{8b}$$

$$\rho_{cm_0}(B^-)e_n = \frac{1}{2}\sqrt{(m_0+2n-2)(m_0+2n)-c}\, e_{n-1}, \tag{8c}$$

where $\{e_n; n\in\mathbb{Z}\}$ is an orthonormal basis of $\ell^2(\mathbb{Z})$, $m_0 \in [0,2[$, $c < m_0(m_0-2)$.

Any integrable unitary irreducible representation representation of sl_2 is equivalent to either the trivial representation ρ_0 or one of the representations given in Equations (6), (7), and (8).

Due to Lemma 2.7, we are interested in representations in which C is mapped to zero. There are, up to unitary equivalence, only three such representations, the trivial or zero representation (which has no cocycles at all, by Lemma 2.6), and the two representations $\rho^\pm = \rho_2^\pm$ on ℓ^2 defined by

$$\begin{array}{rcl} \rho^\pm(M)e_n &=& \pm(2n+2)e_n, \\ \rho^+(B^+)e_n &=& \sqrt{(n+1)(n+2)}\, e_{n+1}, \\ \rho^+(B^-)e_n &=& \sqrt{n(n+1)}\, e_{n-1}, \\ \rho^-(B^+)e_n &=& \sqrt{n(n+1)}\, e_{n-1}, \\ \rho^-(B^-)e_n &=& \sqrt{(n+1)(n+2)}\, e_{n+1}, \end{array} \tag{9}$$

for $n\in\mathbb{N}$, where $\{e_0, e_1, \ldots\}$ is an orthonormal basis of ℓ^2. The representations ρ^+ and ρ^- are not unitarily equivalent, but they are related by the automorphism $M\mapsto -M$, $B^+\mapsto B^-$, $B^-\mapsto B^+$. Therefore it is sufficient to study ρ^+. Let η be a ρ^+-1-cocycle. Since $\rho^+(B^+)$ is injective, we see that η is already uniquely determined by $\eta(B^+)$, since the relations $[M,B^+]=2B^+$ and $[B^-,B^+]=M$ imply

$$\begin{array}{rcl} \eta(M) &=& \rho^+(B^+)^{-1}\big(\rho^+(M)-2\big)\eta(B^+), \\ \eta(B^-) &=& \rho^+(B^+)^{-1}\big(\rho^+(B^-)\eta(B^+)-\eta(M)\big). \end{array}$$

In fact, we can choose any vector for $\eta(B^+)$, the definitions above and the formula $\eta(uv)=\rho^+(u)\eta(v)$ for $u,v\in\mathcal{U}_0(sl_2)$ will extend it to a unique ρ^+-1-cocycle. This cocycle is a coboundary, if and only if the coeffient v_0

in the expansion $\eta(B^+) = \sum_{n=0}^\infty v_n e_n$ of $\eta(B^+)$ vanishes, and an arbitrary ρ^+-1-cocycle is a linear combination of the non-trivial cocyle η_1 defined by

$$\eta_1\left((B^+)^n M^m (B^-)^r\right) = \begin{cases} 0 & \text{if} \quad n = 0, \\ \delta_{r,0}\delta_{m,0}\rho(B^+)^{n-1}e_0 & \text{if} \quad n \geq 1, \end{cases} \tag{10}$$

and a coboundary. In particular, for η with $\eta(B^+) = \sum_{n=0}^\infty v_n e_n$, we get $\eta = v_0\eta_1 + \partial\omega$ with $\omega = \sum_{n=0}^\infty \frac{v_{n+1}}{\sqrt{(n+1)(n+2)}} e_n$. Thus we have shown the following.

Proposition 3.4 *We have*

$$\dim H^1(\mathcal{U}_0(sl_2), \rho^\pm, \ell^2) = 1$$

and $\dim H^1(\mathcal{U}_0(sl_2), \rho, \ell^2) = 0$ *for all other integrable unitary irreducible representations of* sl_2*.*

Since $[sl_2, sl_2] = sl_2$, all elements of $\mathcal{U}_0(sl_2)$ can be expressed as linear combinations of products of elements of $\mathcal{U}_0(sl_2)$. Furthermore one checks that

$$L(u) = \langle \eta(u_1^*), \eta(u_2)\rangle, \qquad \text{for } u = u_1u_2, \quad u_1, u_2 \in \mathcal{U}_0(sl_2)$$

is independent of the decomposition of u into a product and defines a hermitian linear functional. Thus there exists a unique generator for every cocycle on sl_2.

Example 3.5 *We will now construct the Lévy process for the cocycle* η_1 *defined in Equation (10) and the corresponding generator. We get*

$$\begin{aligned} L(M) &= \langle \eta_1(B^+), \eta_1(B^+)\rangle - \langle \eta_1(B^-), \eta_1(B^-)\rangle = 1, \\ L(B^+) &= L(B^-) = 0, \end{aligned}$$

and therefore

$$\begin{aligned} \mathrm{d}_L M &= \mathrm{d}\Lambda\big(\rho^+(M)\big) + \mathrm{d}t, \\ \mathrm{d}_L B^+ &= \mathrm{d}\Lambda\big(\rho^+(B^+)\big) + \mathrm{d}A^*(e_0), \\ \mathrm{d}_L B^- &= \mathrm{d}\Lambda\big(\rho^+(B^-)\big) + \mathrm{d}A(e_0). \end{aligned} \tag{11}$$

The Itô table is infinite-dimensional. This is the process that leads to the realization of SWN that was constructed in the previous works [ALV99, AS99a, Śni00].

For the Casimir element we get

$$\mathrm{d}_L C = \mathrm{d}t.$$

For this process we have $j_{st}(B^-)\Omega = 0$ *and* $j_{st}(M)\Omega = (t-s)\Omega$ *for all* $0 \le s \le t$*. From our previous considerations about the lowest weight representation of* sl_2 *we can now deduce that for fixed* s *and* t *the representation* j_{st} *of* sl_2 *restricted to the subspace* $j_{st}\big(\mathcal{U}(sl_2)\big)\Omega$ *is equivalent to the representation* ρ^+_{t-s} *defined in Equation (6).*

Example 3.6 *Let now* ρ *be one of the lowest weight representations defined in (6) with* $m_0 > 0$*, and let* η *be the trivial cocycle defined by*

$$\eta(u) = \rho^+_{m_0}(u)e_0,$$

for $u \in \mathcal{U}_0(sl_2)$*. There exists a unique generator for this cocycle, and the corresponding Lévy process is defined by*

$$\begin{array}{rcl} \mathrm{d}_L M &=& \mathrm{d}\Lambda\big(\rho^+_{m_0}(M)\big) + m_0 \mathrm{d}A^*(e_0) + m_0 \mathrm{d}A(e_0) + m_0 \mathrm{d}t, \\ \mathrm{d}_L B^+ &=& \mathrm{d}\Lambda\big(\rho^+_{m_0}(B^+)\big) + \sqrt{m_0}\mathrm{d}A^*(e_1), \\ \mathrm{d}_L B^- &=& \mathrm{d}\Lambda\big(\rho^+_{m_0}(B^-)\big) + \sqrt{m_0}\mathrm{d}A(e_1). \end{array} \tag{12}$$

For the Casimir element we get

$$\mathrm{d}_L C = m_0(m_0 - 2)\big(\mathrm{d}\Lambda(\mathrm{id}) + \mathrm{d}A^*(e_0) + \mathrm{d}A(e_0) + \mathrm{d}t\big).$$

3.3 White noise and its square or Lévy processes on $sl_2 \oplus_\alpha hw$

We can define an action α of the Lie algebra sl_2 on hw by

$$\alpha(M): \begin{array}{lcl} A^+ &\mapsto& A^+, \\ E &\mapsto& 0, \\ A^- &\mapsto& -A^-, \end{array} \qquad \alpha(B^+): \begin{array}{lcl} A^+ &\mapsto& 0, \\ E &\mapsto& 0, \\ A^- &\mapsto& -A^+, \end{array} \qquad \alpha(B^-): \begin{array}{lcl} A^+ &\mapsto& A^-, \\ E &\mapsto& 0, \\ A^- &\mapsto& 0. \end{array}$$

The $\alpha(X)$ are derivations and satisfy $\big(\alpha(X)Y\big)^* = -\alpha(X^*)Y^*$ for all $X \in sl_2$, $Y \in hw$. Therefore we can define a new Lie algebra $sl_2 \oplus_\alpha hw$ as the semidirect sum of sl_2 and hw, it has the commutation relations $\big[(X_1, Y_1), (X_2, Y_2)\big] =$

$\big([X_1, X_2], [Y_1, Y_2]{+}\alpha(X_1)Y_2{-}\alpha(X_2)Y_1\big)$ and the involution $(X, Y)^* = (X^*, Y^*)$. In terms of the basis $\{B^\pm, M, A^\pm, E\}$ the commutation relations are

$$\begin{gathered}
[B^-, B^+] = M \quad [M, B^\pm] = \pm 2B^\pm, \\
[A^-, A^+] = E, \quad [E, A^\pm] = 0, \\
[B^\pm, A^\mp] = \mp A^\pm, \quad [B^\pm, A^\pm] = 0, \\
[M, A^\pm] = \pm A^\pm, \quad [E, B^\pm] = 0, \quad [M, E] = 0.
\end{gathered}$$

In the following we identify $\mathcal{U}(hw)$ and $\mathcal{U}(sl_2)$ with the corresponding subalgebras in $\mathcal{U}(sl_2 \oplus_\alpha hw)$.

Note that for any $c \in \mathbb{R}$, $\mathrm{span}\{N = M + cE, A^+, A^-, E\}$ forms a Lie subalgebra of $sl_2 \oplus_\alpha hw$ that is isomorphic to osc.

There exist no Gaussian Lévy processes on $sl_2 \oplus_\alpha hw$, since $[sl_2 \oplus_\alpha hw, sl_2 \oplus_\alpha hw] = sl_2 \oplus_\alpha hw$. But, like for every real Lie algebra, there exist non-trivial $*$-representations of $sl_2 \oplus_\alpha hw$, and thus also Lévy processes, it is sufficient to take, e.g., a trivial cocycle.

The following result shows that the usual creation and annihilation calculus can not be extended to a joint calculus of creation and annihilation and their squares.

Proposition 3.7 *Let (ρ, η, L) be the Schürmann triple on hw defined in Proposition 3.1 a), and denote the corresponding Lévy process by $(j_{st})_{0\le s\le t}$. There exists no Lévy process $(\tilde{\jmath}_{st})_{0\le s\le t}$ on $sl_2 \oplus_\alpha hw$ such that*

$$\big(\tilde{\jmath}_{st}|_{\mathcal{U}(hw)}\big) \cong (j_{st}),$$

unless $(j_{st})_{0\le s\le t}$ is trivial, i.e. $j_{st}(u) = 0$ for all $u \in \mathcal{U}_0(hw)$.

Proof 4 *We will assume that $(\tilde{\jmath}_{st})$ exists and show that this implies $||v_1||^2 = ||v_2||^2 = |z|^2 = 0$, i.e. $L = 0$.*

Let $(\tilde{\rho}, \tilde{\eta}, \tilde{L})$ be the Schürmann triple of $(\tilde{\jmath}_{st})$. If $\big(\tilde{\jmath}_{st}|_{\mathcal{U}(hw)}\big) \cong (j_{st})$, then we have $\tilde{L}|_{\mathcal{U}_0(hw)} = L$, and therefore the triple on hw obtained by restriction of $(\tilde{\rho}, \tilde{\eta}, \tilde{L})$ is equivalent to (ρ, η, L) and there exists an isometry from $D = \eta\big(\mathcal{U}_0(hw)\big)$ into $\tilde{D} = \tilde{\eta}\big(\mathcal{U}_0(sl_2 \oplus_\alpha hw)\big)$, such that we have

$$\tilde{\rho}|_{\mathcal{U}(hw)\times D} = \rho, \qquad \text{and } \tilde{\eta}|_{\mathcal{U}_0(hw)} = \eta,$$

if we identify D with its image in $\tilde{D}$.

From $[B^+, A^-] = -A^+$ *and* $[B^-, A^+] = A^-$*, we get*

$$-\tilde{\eta}(A^+) = \tilde{\rho}(B^+)\eta(A^-) - \tilde{\rho}(A^-)\tilde{\eta}(B^+),$$
$$\tilde{\eta}(A^-) = \tilde{\rho}(B^-)\eta(A^+) - \tilde{\rho}(A^+)\tilde{\eta}(B^-).$$

Taking the inner product with $\tilde{\eta}(A^+) = \eta(A^+) = v_1$ *and* $\tilde{\eta}(A^-) = \eta(A^-) = v_2$*, resp., we get*

$$\begin{aligned} -||v_1||^2 &= \langle v_1, \rho(B^+)v_2\rangle - \langle v_1, \tilde{\rho}(A^-)\tilde{\eta}(B^+)\rangle \\ &= \langle v_1, \rho(B^+)v_2\rangle - \langle \rho(A^+)v_1, \tilde{\eta}(B^+)\rangle = \langle v_1, \rho(B^+)v_2\rangle, \\ ||v_2||^2 &= \langle v_2, \rho(B^-)v_1\rangle, \end{aligned}$$

since $\tilde{\rho}(A^\pm)|_D = \rho(A^\pm)$*. Therefore*

$$-||v_1||^2 = \langle v_1, \rho(B^+)v_2\rangle = \overline{\langle v_2, \rho(B^-)v_1\rangle} = ||v_2||^2,$$

and thus $||v_1||^2 = ||v_2||^2 = 0$*. But* $A^+ = -[B^+, A^-]$ *and*

$$L(A^+) = \tilde{L}(A^+) = \langle \tilde{\eta}(A^+), \tilde{\eta}(B^+)\rangle - \langle \tilde{\eta}(B^-), \tilde{\eta}(A^-)\rangle = \langle v_1, \tilde{\eta}(B^+)\rangle - \langle \tilde{\eta}(B^-), v_2\rangle$$

now implies $z = L(A^+) = 0$*.*

Śniady [Śni00] has posed the question, if it is possible to define a joint calculus for the linear white noise and the square of white noise. Formulated in our context, his answer to this question is that there exists no Lévy process on $sl_2 \oplus_\alpha hw$ such that

$$j_{st}(E) = (t-s)\mathrm{id}, \qquad \text{and} \qquad j_{st}(A^-)\Omega = j_{st}(B^-)\Omega = 0,$$

for all $0 \leq s \leq t$. We are now able to show the same under apparently much weaker hypotheses.

Corollary 3.8 *Every Lévy process on* $sl_2 \oplus_\alpha hw$ *such that the state vector* Ω *is an eigenvector for* $j_{st}(E)$ *and* $j_{st}(A^-)$ *for some pair* s *and* t *with* $0 \leq s < t$ *is trivial on* hw*, i.e. it has to satisfy* $j_{st}|_{\mathcal{U}_0(hw)} = 0$ *for all* $0 \leq s \leq t$*.*

Proof 5 *Assume that such a Lévy process exists. Then it would be equivalent to its realization on a boson Fock space defined by Equation (2). Therefore we see that the state vector is an eigenvector of* $j_{st}(E)$ *and* $j_{st}(A^-)$*, if and only if the Schürmann triple of* $(j_{st})_{0\leq s\leq t}$ *satisfies* $\eta(E) = \eta(A^-) = 0$*. If we* *show that the only Schürmann triples on hw satisfying this condition are the Gaussian Schürmann triples, then our result follows from Proposition 3.7.*

Let (ρ, η, L) be a Schürmann triple on hw such that $\eta(E) = \eta(A^-) = 0$. Then the vector $\eta(A^+)$ has to be cyclic for ρ. We get

$$\rho(E)\eta(A^+) = \rho(A^+)\eta(E) = 0,$$

since E and A^+ commute. From $[A^-, A^-] = E$, we get

$$\rho(A^-)\eta(A^+) = \rho(A^+)\eta(A^-) + \eta(E) = 0.$$

But

$$\begin{aligned} ||\rho(A^+)\eta(A^+)||^2 &= \langle \eta(A^+), \rho(A^-)\rho(A^+)\eta(A^+)\rangle \\ &= \langle \eta(A^-), \rho(A^+)\rho(A^-)\eta(A^+)\rangle + \langle \eta(A^+), \rho(E)\eta(A^+)\rangle \\ &= 0 \end{aligned}$$

shows that $\rho(A^+)$ also acts trivially on $\eta(A^+)$ and therefore the restriction of the triple (ρ, η, L) to $\mathcal{U}(hw)$ is Gaussian.

The SWN calculus defined in Example 3.5 can only be extended in the trivial way, i.e. by setting it equal to zero on hw, $\tilde{\jmath}_{st}|_{hw} = 0$.

Proposition 3.9 *Let $(j_{st})_{0 \le s \le t}$ be the Lévy process on sl_2 defined in (11). The only Lévy process $(\tilde{\jmath}_{st})_{0 \le s \le t}$ on $sl_2 \oplus_\alpha hw$ such that*

$$(\tilde{\jmath}_{st}|_{\mathcal{U}(sl_2)}) \cong (j_{st})$$

is the process defined by $\tilde{\jmath}_{st} = j_{st} \circ \pi$ for $0 \le s \le t$, where π is the canonical homomorphism $\pi : \mathcal{U}(sl_2 \oplus_\alpha hw) \to \mathcal{U}\big((sl_2 \oplus_\alpha hw)/hw\big) \cong \mathcal{U}(sl_2)$.

Proof 6 *We proceed as in the proof of Proposition 3.7, we assume that $(\tilde{\jmath}_{st})_{0 \le s \le t}$ is such an extension, and then we show that this necessarily implies $\tilde{\rho}|_{\mathcal{U}_0(hw)} = 0$, $\tilde{\eta}|_{\mathcal{U}_0(hw)} = 0$, and $\tilde{L}|_{\mathcal{U}_0(hw)} = 0$ for its Schürmann triple $(\tilde{\rho}, \tilde{\eta}, \tilde{L})$. We know that the restriction of the Schürmann triple $(\tilde{\rho}, \tilde{\eta}, \tilde{L})$ to the subalgebra sl_2 and the representation space $D = \tilde{\eta}\big(\mathcal{U}_0(sl_2)\big)$ has to be equivalent to the Schürmann triple (ρ, η, L) defined in Example 3.5.*

Our main tool are the following two facts, which can be deduced from our construction of the irreducible $*$-representations of sl_2 in Subsection 3.2. There exists no $*$-representation π of sl_2 containing non-zero vector v that *satisfies* $\pi(B^-)v = 0$ *and* $\pi(M)v = \lambda v$*, with* $\lambda < 0$*. And if we have a vector* $v \neq 0$ *in the representation space of a* $*$*-representation* π *that satisfies* $\pi(B^-)v = 0$ *and* $\pi(M)v = \lambda v$ *with* $\lambda \geq 0$*, then* π *restricted to* $\pi\big(\mathcal{U}(sl_2)\big)v$ *is equivalent to the lowest weight representation* $\rho^+_{m_0}$ *with* $m_0 = \lambda$*.*

First, we get $\tilde{\eta}(A^-) = 0$*, since the relations* $[B^-, A^-] = 0$ *and* $[M, A^-] = -A^-$ *imply* $\tilde{\rho}(B^-)\tilde{\eta}(A^-) = \tilde{\rho}(A^+)\eta(B^+) = 0$ *and* $-\tilde{\eta}(A^-) = \tilde{\rho}(M)\tilde{\eta}(A^-) - \tilde{\rho}(A^-)\eta(M) = \tilde{\rho}(M)\tilde{\eta}(A^-)$*.*

Therefore we have $\tilde{\eta}(A^+) = \tilde{\rho}(A^-)\eta(B^+) - \tilde{\rho}(B^+)\tilde{\eta}(A^-) = \tilde{\rho}(A^-)\eta(B^+)$*. Using the defining relations of* $sl_2 \oplus_\alpha hw$*, we also get* $\tilde{\rho}(B^-)\tilde{\eta}(A^+) = \tilde{\rho}(B^-)\tilde{\eta}(E) = 0$*,* $\tilde{\rho}(M)\tilde{\eta}(A^+) = \tilde{\eta}(A^+)$*. This implies that* $u_0 = \tilde{\eta}(A^+)$ *generates a representation that is equivalent to* ρ^+_1*.*

Similarly, we see that the restriction of $\tilde{\rho}$ *to the subspace generated by* sl_2 *from* $v_0 = \tilde{\eta}(E) = \tilde{\rho}(A^-)\tilde{\eta}(A^+)$ *is equivalent to the trivial representation, i.e.* $\tilde{\rho}(B^+)\tilde{\eta}(E) = \tilde{\rho}(M)\tilde{\eta}(E) = \tilde{\rho}(B^-)\tilde{\eta}(E) = 0$*. Furthermore, on the subspace generated by* sl_2 *from* $w_0 = \tilde{\rho}(A^+)\tilde{\eta}(A^+)$*, we have copy of the lowest weight representation* ρ^+_2*.*

Thus we have three lowest weight representations in $\tilde{D}$*. Set* $f_k = \eta\big((B^+)^{k+1}\big)$*,* $u_k = \tilde{\rho}(B^+)^k\eta(A^+)$*, and* $w_k = \tilde{\rho}(B^+)^k\tilde{\rho}(A^+)\tilde{\eta}(A^+)$*. From Equation (6), we get*

$$\begin{aligned} \tilde{\rho}(M)f_k &= (2k+2)f_k, \qquad & \tilde{\rho}(B^-)f_k &= k(k+1)f_{k-1}, \\ \tilde{\rho}(M)u_k &= (2k+1)u_k, \qquad & \tilde{\rho}(B^-)u_k &= k^2u_{k-1}, \\ \tilde{\rho}(M)w_k &= (2k+2)w_k, \qquad & \tilde{\rho}(B^-)u_k &= k(k+1)u_{k-1}, \end{aligned}$$

for $k = 1, 2, \ldots$*. Since* A^+ *and* B^+ *commute, we also get* $\tilde{\rho}(A^+)f_k = u_{k+1}$ *and* $\tilde{\rho}(A^+)u_k = w_k$*. From the relations* $[B^-, A^+] = A^-$ *and* $[A^+, A^+] = E$ *we can compute the action of* A^- *and* E *on the* e_k*,*

$$\tilde{\rho}(A^-)f_k = (k+1)u_k, \quad \tilde{\rho}(A^-)u_k = kw_k, \quad \tilde{\rho}(E)f_k = 0,$$

for $k = 0, 1, \ldots$*. This implies*

$$||u_{k+1}||^2 = \langle\tilde{\rho}(A^+)e_k, \tilde{\rho}(A^+)e_k\rangle = \langle\tilde{\rho}(A^-)e_k, \tilde{\rho}(A^-)e_k\rangle = (k+1)^2||u_k||^2$$

for $k = 1, 2, \ldots$*, for the norms of the* u_k*. But on the other hand, we have*

$$\begin{aligned} ||u_{k+1}||^2 &= \langle\tilde{\rho}(B^+)u_k, \tilde{\rho}(B^+)u_k\rangle = \langle\tilde{\rho}(B^-)u_k, \tilde{\rho}(B^-)u_k\rangle + \langle u_k, \tilde{\rho}(M)u_k\rangle \\ &= k^4||u_{k-1}||^2 + (2k+1)||u_k||^2, \end{aligned}$$

for all $k = 0, 1, \ldots$. This is only possible, if $||u_k|| = 0$ for all $k = 0, 1, \ldots$, and so we have showed $\tilde{\eta}|_{\mathcal{U}_0(hw)} = 0$. This immediately implies $\tilde{L}|_{\mathcal{U}_0(hw)} = 0$.

Finally, we also have

$$\tilde{\rho}(A^+)\eta(B^+) = u_1 = 0, \qquad \tilde{\rho}(A^-)\eta(B^+) = u_0 = 0,$$

and therefore $\tilde{\rho}|_{\mathcal{U}_0(hw)} = 0$, because $\eta(B^+)$ has to be cyclic for $\tilde{\rho}$.

But there do exist Lévy processes such that $j_{st}(A^-)\Omega = j_{st}(B^-)\Omega = 0$ for all $0 \leq s \leq t$, as the following example shows.

Example 3.10 *Let $h > 0$ and let ρ_h be the Fock representation of $\mathcal{U}(hw)$ defined in (4). This extends to a representation of $\mathcal{U}(sl_2 \oplus_\alpha hw)$, if we set*

$$\rho_h(B^+) = \frac{\rho_h(A^+)^2}{2h}, \quad \rho_h(M) = \frac{\rho_h(A^+A^- + A^-A^+)}{2h}, \quad \rho_h(B^+) = \frac{\rho_h(A^-)^2}{2h}.$$

For the cocycle we take the coboundary of the 'lowest weight vector' $e_0 \in \ell^2$, i.e. we set

$$\eta(u) = \rho_h(u)e_0$$

for $u \in \mathcal{U}_0(sl_2 \oplus_\alpha hw)$, and for the generator

$$L(u) = \langle \Omega, \rho_h(u)\Omega \rangle$$

for $u \in \mathcal{U}_0(sl_2 \oplus_\alpha hw)$. This defines a Schürmann triple on $sl_2 \oplus_\alpha hw$ over ℓ^2 and therefore

$$\begin{aligned}
\mathrm{d}_L B^+ &= \frac{1}{2h}\mathrm{d}\Lambda\left(\rho_h(A^+)^2\right) + \frac{1}{\sqrt{2}}\mathrm{d}A^*(e_2), \\
\mathrm{d}_L A^+ &= \mathrm{d}\Lambda\big(\rho_h(A^+)\big) + \sqrt{h}\mathrm{d}A^*(e_1), \\
\mathrm{d}_L M &= \frac{1}{2h}\mathrm{d}\Lambda\big(\rho_h(A^+A^- + A^-A^+)\big) + \frac{1}{2}\mathrm{d}A^*(e_0) + \frac{1}{2}\mathrm{d}A(e_0) + \frac{1}{2}\mathrm{d}t, \\
\mathrm{d}_L E &= h\mathrm{d}\Lambda(\mathrm{id}) + h\mathrm{d}A^*(e_0) + h\mathrm{d}(e_0) + h\mathrm{d}t, \\
\mathrm{d}_L A^- &= \mathrm{d}\Lambda\big(\rho_h(A^-)\big) + \sqrt{h}\mathrm{d}A(e_1), \\
\mathrm{d}_L B^- &= \frac{1}{2h}\mathrm{d}\Lambda\left(\rho_h(A^-)^2\right) + \frac{1}{\sqrt{2}}\mathrm{d}A(e_2),
\end{aligned}$$

defines a Lévy process $sl_2 \oplus_\alpha hw$, acting on the Fock space over $L^2(\mathbb{R}_+, \ell^2)$. The Itô table of this process is infinite-dimensional. The restriction of this process to sl_2 is equivalent to the process defined in Example 3.6 with $m_0 = \frac{1}{2}$.

One can easily verify that $j_{st}(A^-)$ and $j_{st}(B^-)$ annihilate the vacuum vector of $\Gamma\big(L^2(\mathbb{R}_+,\ell^2)\big)$.

We have $\rho_h(C)=\frac{5}{4}\mathbb{P}_0+3\mathbb{P}_1$, where $\mathbb{P}_0$ is the projection onto $\overline{\mathrm{span}\,\{e_0,e_2,e_4,\ldots\}}$ and $\mathbb{P}_1=\mathrm{id}-\mathbb{P}_0$, and therefore

$$\mathrm{d}_L C=\frac{5}{4}\mathrm{d}\Lambda(\mathbb{P}_0)+3\mathrm{d}\Lambda(\mathbb{P}_1)+\frac{5}{4}\mathrm{d}A^*(e_0)+\frac{5}{4}\mathrm{d}A(e_0)+\frac{5}{4}\mathrm{d}t.$$

3.4 Higher order noises

Let us now consider the infinite-dimensional real Lie algebra wn that is spanned by $\{B_{n,m};n,m\in\mathbb{N}\}$ with the commutation relations

$$\begin{aligned}[B_{n_1,m_1},B_{n_2,m_2}] &= \sum_{k=1}^{n_2\wedge m_1}\frac{m_1!n_2!}{(m_1-k)!(n_2-k)!k!}c^kB_{n_1+n_2-k,m_1+m_2-k}\\ &\quad-\sum_{k=1}^{n_1\wedge m_2}\frac{m_2!n_1!}{(m_2-k)!(n_1-k)!k!}c^kB_{n_1+n_2-k,m_1+m_2-k}\end{aligned}$$

for $n_1,n_2,m_1,m_2\in\mathbb{N}$, and involution $(B_{n,m})^*=B_{m,n}$, where $c\geq 0$ is some fixed positive parameter. These relations can be obtained by taking the quotient of the universal enveloping algebra $\mathcal{U}(hw)$ of hw with respect to the ideal generated by $E=c1$. The basis elements $B_{n,m}$ are the images of $(A^+)^n(A^-)^m$.

We can embed hw and $sl_2\otimes_\alpha hw$ into wn by

$$A^+\mapsto\frac{B_{1,0}}{\sqrt{c}},\quad A^-\mapsto\frac{B_{0,1}}{\sqrt{c}},\quad E\mapsto B_{0,0},$$
$$B^+\mapsto\frac{1}{2c}B_{2,0},\quad B^-\mapsto\frac{1}{2c}B_{0,2},\quad M\mapsto\frac{1}{c}B_{1,1}+\frac{1}{2}B_{0,0}.$$

There exist no Gaussian Lévy processes on hw, since $[hw,hw]=hw$.

Let ρ_c be the Fock representation defined in Equation (4). Setting

$$\rho(B_{n,m})=\rho_c\left((A^+)^n(A^-)^m\right),\qquad n,m\in\mathbb{N},$$

we get a $*$-representation $\mathcal{U}(wn)$. If we set $\eta(u)=\rho(u)e_0$ and $L(u)=\langle e_0,\rho(u)e_0\rangle$ for $u\in\mathcal{U}_0(wn)$, then we obtain a Schürmann triple on wn. For this triple we get

$$\mathrm{d}_LB_{n,m}=\mathrm{d}\Lambda\left(\rho_c(A^+)^n\rho_c(A^-)^m\right)+\delta_{m0}\sqrt{c^nn!}\,\mathrm{d}A^*(e_n)+\delta_{n0}\sqrt{c^mm!}\,\mathrm{d}A(e_m)+\delta_{n0}\delta_{m0}\mathrm{d}t,$$

for the differentials. Note that we have $j_{st}(B_{nm})\Omega=0$ for all $m\geq 1$ and $0\leq s\leq t$ for the associated Lévy process.

3.5 Other examples: Lévy processes on fd and gl_2

The goal of this subsection is to explain the relation of the present paper to previous works by Boukas [Bou88, Bou91] and Parthasarathy and Sinha [PS91].

We introduce the two real Lie algebras fd and gl_2. The finite-difference Lie algebra fd is the three-dimensional solvable real Lie algebra with basis $\{P, Q, T\}$, commutation relations

$$[P, Q] = [T, Q] = [P, T] = T,$$

and involution $P^* = Q$, $T^* = T$, cf. [Fei87]. This Lie algebra is actually the direct sum of the (unique!) non-abelian two-dimensional real Lie algebra and the one-dimensional abelian Lie algebra, its center is spanned by $T - P - Q$.

The Lie algebra gl_2 of the general linear group $GL(2; \mathbb{R})$ is the direct sum of sl_2 with the one-dimensional abelian Lie algebra. As a basis of gl_2 we will choose $\{B^+, B^-, M, I\}$, where B^+, B^-, and M are a basis of the Lie subalgebra sl_2, and I is hermitian and central. Note that $T \mapsto M + B^+ + B^-$, $P \mapsto (M - I)/2 + B^-$, $Q \mapsto (M - I)/2 + B^+$ defines an injective Lie algebra homomorphism from fd into gl_2, i.e. we can regard fd as a Lie subalgebra of gl_2.

Following ideas by Feinsilver [Fei89], Boukas [Bou88, Bou91] constructed a calculus for fd, i.e. he constructed a Lévy process on it and defined stochastic integrals with respect to it. He also derived the Itô formula for these processes and showed that their Itô table is infinite-dimensional. His realization is not defined on the boson Fock space, but on the so-called finite-difference Fock space especially constructed for his fd calculus. Parthasarathy and Sinha constructed another Lévy process on fd, acting on a boson Fock space, in [PS91]. They gave an explicite decomposition of the operators into conservation, creation, annihilation, and time, thereby reducing its calculus to Hudson-Parthasarathy calculus.

Accardi and Skeide [AS99a, AS99b] noted that they were able to recover Boukas' fd calculus from their SWN calculus. In fact, since gl_2 is a direct sum of sl_2 and the one-dimensional abelian Lie algebra, any Lévy process $(j_{st})_{0\le s\le t}$ on sl_2 can be extended (in many different ways) to a Lévy process $(\tilde{\jmath}_{st})_{0\le s\le t}$ on gl_2. We will only consider the extensions defined by

$$\tilde{\jmath}_{st}|_{sl_2} = j_{st}, \quad \text{and} \quad \tilde{\jmath}_{st}(I) = \lambda(t - s)\mathrm{id}, \qquad \text{for } 0 \le s \le t,$$

for $\lambda \in \mathbb{R}$. Since fd is a Lie subalgebra of gl_2, we also get a Lévy process on fd by restricting $(\tilde{\jmath}_{st})_{0\le s\le t}$ to $\mathcal{U}(fd)$.

If we take the Lévy process on sl_2 defined in Example 3.5 and $\lambda = 1$, then we get

$$\begin{aligned}
\mathrm{d}_L P &= \mathrm{d}\Lambda\big(\rho^+(M/2 + B^-)\big) + \mathrm{d}A(e_0),\\
\mathrm{d}_L Q &= \mathrm{d}\Lambda\big(\rho^+(M/2 + B^+)\big) + \mathrm{d}A^*(e_0),\\
\mathrm{d}_L T &= \mathrm{d}\Lambda\big(\rho^+(M + B^+ + B^-)\big) + \mathrm{d}A^*(e_0) + \mathrm{d}A(e_0) + \mathrm{d}t.
\end{aligned}$$

It can be checked that this Lévy process is equivalent to the one defined by Boukas.

If we take instead the Lévy process on sl_2 defined in Example 3.6, then we get

$$\begin{aligned}
\mathrm{d}_L P &= \mathrm{d}\Lambda\big(\rho^+_{m_0}(M/2 + B^-)\big) + \mathrm{d}A^*\left(\frac{m_0}{2}e_0\right) + \mathrm{d}A\left(\frac{m_0}{2}e_0 + \sqrt{m_0}e_1\right) + \frac{m_0-\lambda}{2}\mathrm{d}t,\\
\mathrm{d}_L Q &= \mathrm{d}\Lambda\big(\rho^+_{m_0}(M/2 + B^-)\big) + \mathrm{d}A^*\left(\frac{m_0}{2}e_0 + \sqrt{m_0}e_1\right) + \mathrm{d}A\left(\frac{m_0}{2}e_0\right) + \frac{m_0-\lambda}{2}\mathrm{d}t,\\
\mathrm{d}_L T &= \mathrm{d}\Lambda\big(\rho^+_{m_0}(M + B^+ + B^-)\big) + \mathrm{d}A^*(m_0e_0 + \sqrt{m_0}e_1) + \mathrm{d}A(m_0e_0 + \sqrt{m_0}e_1) + m_0\mathrm{d}t\\
&= \mathrm{d}_L P + \mathrm{d}_L Q + \lambda\mathrm{d}t.
\end{aligned}$$

For $m_0 = \lambda = 2$, this is exactly the Lévy process defined in [PS91]. Note that in that case the representation $\rho^+_2 = \rho^+$ and the Fock space agree with the those of Boukas' process, only the cocycle and the generator are different.

4 Classical processes

Let $(j_{st})_{0\le s\le t}$ be a Lévy process on a real Lie algebra g over $\Gamma = \Gamma\big(L^2(\mathbb{R}_+, D)\big)$, fix a real element $Y \in g_\mathbb{R}$ and define a map $y : \Sigma(\mathbb{R}_+) \to \mathcal{L}(\Gamma)$ by

$$y_\phi = -i\sum_{k=1}^n \phi_k j_{s_k t_k}(Y), \qquad \text{for } \phi = \sum_{k=1}^n \phi_k 1_{[s_k,t_k[} \in \Sigma(\mathbb{R}_+)$$

It is clear that the operators $\{y_\phi; \phi \in \Sigma(\mathbb{R}_+)\}$ commute, since iy is the restriction of $\pi : g^{\mathbb{R}_+} \ni \psi = \sum_{k=1}^n \psi_k 1_{[s_k,t_k[} \mapsto \sum_{k=1}^n j_{s_k t_k}(\psi_k) \in \mathcal{L}(\Gamma)$ to the (abelian!) current algebra $\mathbb{C}Y^{\mathbb{R}_+}$ over $\mathbb{C}Y$. Furthermore, if ϕ is real-valued, then y_ϕ is hermitian, since Y is anti-hermitian. Therefore there exists a classical stochastic process $(\tilde{Y}_t)_{t\ge0}$ whose moments are given by

$$\mathbb{E}(\tilde{Y}_{t_1}\cdots\tilde{Y}_{t_n}) = \langle\Omega, y_{1_{[0,t_1[}}\cdots y_{1_{[0,t_n[}}\Omega\rangle, \qquad \text{for all } t_1,\ldots,t_n \in \mathbb{R}_+.$$

Since the expectations of $(j_{st})_{0\le s\le t}$ factorize, we can choose $(\tilde{Y}_t)_{t\ge 0}$ to be a Lévy process. If $ij_{st}(Y)$ is even essentially self-adjoint, then the marginal distribution of $(\tilde{Y}_t)_{t\ge 0}$ is uniquely determined.

We will now give a characterization of $(\tilde{Y}_t)_{t\ge 0}$. First, we need two lemmas.

Lemma 4.1 *Let $X \in \mathcal{L}(D)$, $u, v \in D$, and suppose furthermore that the series $\sum_{n=0}^{\infty} \frac{(tX)^n}{n!} w$ and $\sum_{n=0}^{\infty} \frac{(tX^*)^n}{n!} w$ converge in D for all $w \in D$. Then we have*

$$\begin{aligned} e^{\Lambda(X)} A(v) &= A\left(e^{-X^*} v\right) e^{\Lambda(X)} \\ e^{A^*(u)} A(v) &= \big(A(v) - \langle v, u\rangle\big) e^{A^*(u)} \\ e^{A^*(u)} \Lambda(X) &= \big(\Lambda(X) - A^*(Xu)\big) e^{A^*(u)} \end{aligned}$$

on the algebraic boson Fock space over D.

Proof 7 *This can be deduced from the formula for the adjoint actions,* $\mathrm{Ad}e^X Y = e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2}\big[X,[X,Y]\big] + \cdots = e^{\mathrm{ad}X} Y$.

Lemma 4.2 *Let $X \in \mathcal{L}(D)$ and $u, v \in D$ and suppose furthermore that the series $\sum_{n=0}^{\infty} \frac{(tX)^n}{n!} w$ and $\sum_{n=0}^{\infty} \frac{(tX^*)^n}{n!} w$ converge in D for all $w \in D$. Then we have*

$$\exp\big(\Lambda(X) + A^*(u) + A(v) + \alpha\big) = \exp\big(A^*(\tilde{u})\big) \exp\big(\Lambda(X)\big) \exp\big(A(\tilde{v})\big) \exp(\tilde{\alpha})$$

on the algebraic boson Fock space over D, where

$$\tilde{u} = \sum_{n=1}^{\infty} \frac{X^{n-1}}{n!} u, \quad \tilde{v} = \sum_{n=1}^{\infty} \frac{(X^*)^{n-1}}{n!} v, \quad \tilde{\alpha} = \alpha + \sum_{n=1}^{\infty} \langle v, \frac{X^{n-2}}{n!} u\rangle.$$

Proof 8 *Let $\omega \in D$ and set $\omega_1(t) = \exp t\big(\Lambda(X) + A^*(u) + A(v) + \alpha\big)\omega$ and*

$$\omega_2(t) = \exp\Big(A^*\big(\tilde{u}(t)\big)\Big) \exp\big(t\Lambda(X)\big) \exp\Big(A\big(\tilde{v}(t)\big)\Big) \exp\big(\tilde{\alpha}(t)\big)\omega$$

for $t \in [0,1]$, where

$$\tilde{u}(t) = \sum_{n=1}^{\infty} \frac{t^n X^{n-1}}{n!} u,\ \tilde{v}(t) = \sum_{n=1}^{\infty} \frac{t^n (X^*)^{n-1}}{n!} v,\ \tilde{\alpha}(t) = t\alpha + \sum_{n=1}^{\infty} \langle v, \frac{t^n X^{n-2}}{n!} u\rangle.$$

Then we have $\omega_1(0) = \omega = \omega_2(0)$. Using Lemma 4.1, we can also check that

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_1(t) = \big(\Lambda(X) + A^*(u) + A(v) + \alpha\big)\omega \exp t\big(\Lambda(X) + A^*(u) + A(v) + \alpha\big)\omega$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\omega_2(t) &= A^*\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t}(t)\right)\exp\Big(A^*\big(\tilde{u}(t)\big)\Big)\exp\big(t\Lambda(X)\big)\exp\Big(A\big(\tilde{v}(t)\big)\Big)\exp\big(\tilde{\alpha}(t)\big)\omega \\ &\quad + \exp\Big(A^*\big(\tilde{u}(t)\big)\Big)\Lambda(X)\exp\big(t\Lambda(X)\big)\exp\Big(A\big(\tilde{v}(t)\big)\Big)\exp\big(\tilde{\alpha}(t)\big)\omega \\ &\quad + \exp\Big(A^*\big(\tilde{u}(t)\big)\Big)\exp\big(t\Lambda(X)\big)A\left(\frac{\mathrm{d}\tilde{v}}{\mathrm{d}t}(t)\right)\exp\Big(A\big(\tilde{v}(t)\big)\Big)\exp\big(\tilde{\alpha}(t)\big)\omega \\ &\quad + \exp\Big(A^*\big(\tilde{u}(t)\big)\Big)\exp\big(t\Lambda(X)\big)\exp\Big(A\big(\tilde{v}(t)\big)\Big)\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t}(t)\exp\big(\tilde{\alpha}(t)\big)\omega\end{aligned}$$

coincide for all $t \in [0,1]$. Therefore we have $\omega_1(1) = \omega_2(1)$.

Theorem 4.3 *Let L be a generator on $\mathcal{U}_0(g)$ with Schürmann triple (ρ, η, L) and Lévy process $(j_{st})_{0\le s\le t}$. Suppose that $\eta(Y)$ is analytic for $\rho(Y)$. Then the associated classical Lévy process $(\tilde{Y}_t)_{t\ge 0}$ has characteristic exponent*

$$\Psi(\lambda) = \lambda L(Y) + \sum_{n=2}^{\infty}\frac{\lambda^n}{n!}\langle \eta(Y^*), \rho(Y)^{n-2}\eta(Y)\rangle,$$

for λ in some neighborhood of zero.

Proof 9 *The characteristic exponent $\Psi(\lambda)$, $\lambda \in \mathbb{R}$, is defined by $\mathbb{E}(e^{i\lambda\tilde{Y}_t}) = e^{t\Psi(\lambda)}$, so we have to compute*

$$\mathbb{E}\left(e^{i\lambda\tilde{Y}_t}\right) = \langle \Omega, e^{\lambda j_{0t}(Y)}\Omega\rangle$$

for $j_{0t}(Y) = \Lambda_{0t}\big(\rho(Y)\big) + A^*_{0t}\big(\eta(Y)\big) + A_{0t}\big(\eta(Y)\big) + tL(Y)$. Using Lemma 4.2, we get

$$\mathbb{E}\left(e^{i\lambda\tilde{Y}_t}\right) = \exp\left(t\lambda L(Y) + t\sum_{n=2}^{\infty}\left\langle \eta(Y^*), \frac{\lambda^n\rho(Y)^{n-2}}{n!}\eta(Y)\right\rangle\right).$$

Remark 4.4 *Note that $\Psi(\lambda)$ is nothing else than $\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}L(Y^n)$. It is also possible to give a more direct proof of the theorem, using the convolution of functionals on $\mathcal{U}(g)$ instead of the boson Fock space realization of $(j_{st})_{0\le s\le t}$.*

We give two corollaries of this result, the first justifies our definition of Gaussian generators.

Corollary 4.5 *Let L be a Gaussian generator on g with corresponding Lévy process $(j_{st})_{0\leq s\leq t}$. Then the classical Lévy process $(\tilde{Y}_t)_{t\geq 0}$ is Gaussian with mean and variance*

$$\mathbb{E}(\tilde{Y}_t) = -itL(Y), \qquad \mathbb{E}(\tilde{Y}_t^2) = ||\eta(Y)||^2 t, \qquad \text{for } t \geq 0.$$

We see that in this case we can take $\big(||\eta(Y)||B_t - iL(Y)t\big)_{t\geq 0}$ for $(\tilde{Y}_t)_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion.

The next corollary deals with the case where L is the restriction to $\mathcal{U}_0(g)$ of a positive functional on $\mathcal{U}(g)$.

Corollary 4.6 *Let (ρ, η, L) be a Schürmann triple on $\mathcal{U}(g)$ whose cocycle is trivial, i.e. there exists a vector $\omega \in D$ such that $\eta(u) = \rho(u)\omega$ for all $u \in \mathcal{U}_0(g)$, and whose generator is of the form $L(u) = \langle \omega, \big(\rho(u) - \varepsilon(u)\,\mathrm{id}_D\big)\omega\rangle$, for all $u \in \mathcal{U}_0(g)$. Suppose furthermore that the vector ω is analytical for $\rho(Y)$, i.e. that $e^{u\rho(Y)}\omega := \sum_{n=1}^{\infty} \frac{u^n \rho(Y)^n}{n!}\omega$ converges for sufficiently small u. Then the classical stochastic process $(\tilde{Y}_t)_{t\geq 0}$ associated to (j_{st}) is composed Poisson process with characteristic exponent*

$$\Psi(u) = \langle \omega, \big(e^{u\rho(Y)} - 1\big)\,\omega\rangle.$$

Remark 4.7 *If the operator $i\rho(Y)$ is even (essentially) self-adjoint, then we get the Lévy measure of $(\tilde{Y}_t)_{t\geq 0}$ by evaluating its spectral measure in the state vector ω,*

$$\mu(\mathrm{d}\lambda) = \langle \omega, \mathrm{d}\mathbb{P}_\lambda \omega\rangle,$$

where $i\rho(Y) = \int \lambda \mathrm{d}\mathbb{P}_\lambda$ is the spectral resolution of (the closure of) $i\rho(Y)$.

Corollary 4.6 suggests to call a Lévy process on g with trivial cocycle $\eta(u) = \rho(u)\omega$ and generator $L(u) = \langle \omega, \rho(u)\omega\rangle$ for $u \in \mathcal{U}_0(g)$ a *Poisson process* on g.

Example 4.8 *Let $(j_{st})_{0\leq s\leq t}$ be the Lévy process on sl_2 defined in Example 3.6 and let $Y = -i(B^+ + B^- + \beta M)$ with $\beta \in \mathbb{R}$. The operator $X = i\rho^+_{m_0}(Y)$ is essentially self-adjoint. We now want to characterize the classical Lévy process $(\tilde{Y}_t)_{t\geq 0}$ associated to Y and $(j_{st})_{0\leq s\leq t}$ in the manner described* *above. Corollary 4.6 tells us that* $(\tilde{Y}_t)_{t\geq 0}$ *is a compound Poisson process with characteristic exponent*

$$\Psi(u) = \langle e_0, \left(e^{iuX} - 1\right) e_0\rangle.$$

We want to determine the Lévy measure of $(\tilde{Y}_t)_{t\geq 0}$*, i.e. we want to determine the measure* μ *on* $\mathbb{R}$*, for which*

$$\Psi(u) = \int \left(e^{iux} - 1\right) \mu(\mathrm{d}x).$$

This is the spectral measure of X *evaluated in the state* $\langle e_0, \cdot\, e_0\rangle$*. Note that the polynomials* $p_n \in \mathbb{R}[x]$ *defined by the condition*

$$e_n = p_n(X)e_0,$$

$n = 0, 1, \ldots,$ *are orthogonal w.r.t.* μ*, since*

$$\int p_n(x)p_m(x)\mu(\mathrm{d}x) = \langle e_0, p_n(X)p_m(X)e_0\rangle = \langle p_n(X)e_0, p_m(X)e_0\rangle = \delta_{nm},$$

for $n, m \in \mathbb{N}$*. Looking at the definition of* X*, we can easily identitfy the three-term-recurrence relation satisfied by the* p_n*. We get*

$$Xe_n = \sqrt{(n+1)(n+m_0)}e_{n+1} + \beta(2n+m_0)e_n + \sqrt{n(n+m_0-1)}e_{n-1},$$

for $n \in \mathbb{N}$*, and therefore*

$$(n+1)P_{n+1} + (2\beta n + \beta m_0 - x)P_n + (n+m_0-1)P_{n-1} = 0,$$

with initial condition $P_{-1} = 0$*,* $P_1 = 1$*, for the rescaled polynomials*

$$P_n = \prod_{k=1}^{n} \sqrt{\frac{n}{n+m_0}}\, p_n.$$

According to the value of β *we have to distinguish three cases.*

1. $|\beta| = 1$*: In this case we have, up to rescaling, Laguerre polynomials, i.e.*
$$P_n(x) = (-\beta)^n L_n^{(m_0-1)}(\beta x)$$

where the Laguerre polynomials $L_n^{(\alpha)}$ are defined as in [KS94, Equation (1.11.1)]. The measure μ can be obtained by normalizing the measure of orthogonality of the Laguerre polynomials, it is equal to

$$\mu(\mathrm{d}x) = \frac{|x|^{m_0-1}}{\Gamma(m_0)} e^{-\beta x} 1_{\beta\mathbb{R}_+}.$$

If $\beta = +1$, then this measure is also known in probability theory as the χ^2-distribution. The operator X is then positive and therefore $(\tilde{Y}_t)_{t\geq 0}$ is a subordinator, i.e. a Lévy process with values in $\mathbb{R}_+$, or, equivalently, a Lévy process with non-decreasing sample paths.

2. $|\beta| < 1$*: In this case we find the Meixner-Pollaczek polynomials after rescaling,*

$$P_n(x) = P_n^{(m_0/2)}\left(\frac{x}{2\sqrt{1-\beta^2}}; -\arccos\beta\right).$$

For the definition of these polynomials see, e.g., [KS94, Equation (1.7.1)]. For the measure μ we get

$$\mu(\mathrm{d}x) = C\exp\left(-\frac{(2\arccos\beta+\pi)x}{2\sqrt{1-\beta^2}}\right)\left|\Gamma\left(\frac{m_0}{2}+\frac{ix}{2\sqrt{1-\beta^2}}\right)\right|^2,$$

where C has to be chosen such that μ is a probability measure.

3. $|\beta| > 1$*: In this case we get Meixner polynomials after rescaling,*

$$P_n(x) = \begin{cases} (-1)^n \prod_{k=1}^n \frac{n+m_0-1}{n} M_n\left(\frac{x}{c-1/c} - \frac{m_0}{2}; m_0; c^2\right) & \text{if} \quad \beta > 0, \\ \prod_{k=1}^n \frac{n+m_0-1}{n} M_n\left(-\frac{x}{c-1/c} + \frac{m_0}{2}; m_0; c^2\right) & \text{if} \quad \beta > 0, \end{cases}$$

where

$$c = \begin{cases} \beta - \sqrt{\beta^2-1} & \text{if} \quad \beta > +1, \\ -\beta - \sqrt{\beta^2-1} & \text{if} \quad \beta < -1. \end{cases}$$

The definition of these polynomials can be found, e.g., in [KS94, Equation (1.9.1)]. The density μ is again the measure of orthogonality of the polynomials P_n (normalized to a probability measure). We therefore get

$$\mu = C\sum_{n=0}^{\infty} \frac{c^{2n}(m_0)_n}{n!}\delta_{sgn\beta((c-1/c)(n+m_0/2))},$$

where $C^{-1} = \sum_{n=0}^{\infty} \frac{c^{2n}(m_0)_n}{n!} = (1-c^2)^{-m_0}$. *Here* $(m_0)_n$ *denotes the Pochhammer symbol,* $(m_0)_n = m_0(m_0+1)\cdots(m_0+n-1)$.

Example 4.9 *Let now* $(j_{st})_{0\le s\le t}$ *be the Lévy process on* sl_2 *defined in Example 3.5 and let again* $Y = -i(B^+ - B^- + \beta M)$ *with* $\beta \in \mathbb{R}$. *We already noted in Example 3.5 that* j_{st} *is equivalent to* ρ^+_{t-s} *for fixed* s *and* t. *Therefore the marginal distributions of the classical Lévy process* $(\tilde{Y}_t)_{t\ge 0}$ *are exactly the distributions of the operator* X *that we computed in the previous example (with* $m_0 = t$*).*

For $\beta = 1$, *we recover [Bou91, Theorem 2.2]. The classical Lévy process associated to* $-iT = -i(B^+ + B^- + M)$ *is an exponential or Gamma process with Fourier transform*

$$\mathbb{E}\left(e^{iu\tilde{Y}_t}\right) = (1-iu)^{-t}$$

and marginal distribution $\nu_t(\mathrm{d}x) = \frac{x^{t-1}}{\Gamma(t)} e^{-x} 1_{\mathbb{R}_+} \mathrm{d}x$. *Its Lévy measure is* $x^{-1}e^{-x}1_{\mathbb{R}_+}\mathrm{d}x$, *see, e.g., [Ber96].*

For $\beta > 1$, *we can write the Fourier transform of the marginal distributions* ν_t *as*

$$\mathbb{E}(e^{iu\tilde{Y}_t}) = \exp t\left(\frac{iu(c-1/c)}{2} + t\sum_{n=1}^{\infty} \frac{c^{2n}}{n}\left(e^{iun(c-1/c)} - 1\right)\right).$$

This shows that we can define $(\tilde{Y}_t)_{t\ge 0}$ *as sum of Poisson processes with a drift, i.e. if* $\left(\left(N_t^{(n)}\right)_{t\ge 0}\right)_{n\ge 1}$ *are independent Poisson processes (with intensity and jump size equal to one), then we can take*

$$\tilde{Y}_t = (c-1/c)\left(\sum_{n=1}^{\infty} nN^{(n)}_{c^{2n}t/n} + \frac{t}{2}\right), \qquad \text{for } t \ge 0.$$

The marginal distributions of these processes for the different values of β *and their relation to orthogonal polynomials are also discussed in [FS93, Chapter 5].*

5 Conclusion

We have shown that the theories of factorizable current representations of Lie algebras and Lévy processes on $*$-bialgebras provide an elegant and efficient formalism for defining and studying quantum stochastic calculi with respect to additive operator processes satisfying Lie algebraic relations. The theory of Lévy processes on $*$-bialgebras can also handle processes whose increments are not simply additive, but are composed by more complicated formulas, the main restriction is that they are independent (in the tensor sense). This allows to answer questions that could not be handled by ad hoc construction methods, such as the computation of the SWN Itô table, the simultaneous realization of linear and squared white noise on the same Hilbert space, or the characterization of the associated classical processes.

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