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RENORMING OF $C(K)$ SPACES

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ABSTRACT. If K is a scattered Eberlein compact space, then $C(K)^*$ admits an equivalent dual norm that is uniformly rotund in every direction. The same is shown for the dual to the Johnson-Lindenstrauss space JL_2.

1. Introduction

We will find classes of Banach spaces whose duals admit equivalent dual norms that are uniformly rotund in every direction (URED) or pointwise uniformly rotund (p-UR) (definitions are given below). The notion of p-UR covers both the weak and weak* uniform rotundity (W*UR). It can be shown from the Šmulyan theorem (see, e.g., [5, p. 63]), that the existence of a dual p-UR norm on X^* implies the existence of a "big" set in X^{**} on which the bidual norm is uniformly Gâteaux differentiable.

In Section 2 we will prove a three-space-like theorem for the following properties of a Banach space X: *X^* admits an equivalent dual URED (p-UR) norm.* This result enables us to renorm duals to spaces, such as the Johnson-Lindenstrauss space or $C(K)$ for K scattered with $K^{(\omega)} = \emptyset$, by dual norms that are simultaneously locally uniformly rotund (LUR) and p-UR. On the example of $C(K)$, where K is the so-called "two arrow" compact space, it is shown that properties of the duals to be equivalently renormed by dual URED norm (or p-UR norm) are not three space properties.

In Section 3, we will apply previous results, use a result from [1] and a method from [9], [11]. It will be proved that if K is an Eberlein and scattered compact space, then $C(K)^*$ admits an equivalent dual LUR and p-UR norm.

Recently it was shown in [6] that if X^* admits weak* uniformly rotund norm, then X is a subspace of weakly compactly generated space. However, in [12, Th. 1] it is shown that if X has an unconditional Schauder basis and X^* admits an equivalent (not necessarily dual) URED norm, then X^* admits an equivalent dual weak* uniformly rotund norm. Hence the space JL_2 from Section 2 shows that Theorem 1 in [12] does not hold without the assumption of unconditional Schauder

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basis. From the result in Section 3 we can deduce that if K is scattered Eberlein compact, that is not uniform Eberlein compact, then $C(K)^*$ is a dual to the weakly compactly generated space and admits an equivalent dual p-UR norm, but no equivalent dual weak* uniformly rotund norm, i.e., there is weakly compactly generated Banach space X, such that its dual X^* admits an equivalent dual URED and LUR norm and no W*UR norm.

Let $(X, \|.\|)$ be a Banach space. Let S_X and B_X denote the unit sphere and the unit ball respectively, i.e., $S_X = \{x \in X; \|x\| = 1\}$ and $B_X = \{x \in X; \|x\| \leq 1\}$. The norm $\|.\|$ on a Banach space X is said to be *uniformly rotund in every direction* (*URED* for short), if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ whenever $x_n, y_n \in S_X$ are such that $x_n - y_n = \lambda_n z$ for some $z \in X, \lambda_n \in \mathbb{R}$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$. We will say that the norm $\|.\|$ on X is *pointwise uniformly rotund* (*p-UR*), if there exists a w^*-dense set $\mathbb{F} \subset X^*$ such that $\lim_{n\to\infty} f(x_n - y_n) = 0$ whenever $x_n, y_n \in S_{(X,\|.\|)}$, $\lim_{n\to\infty} \|x_n + y_n\| = 2$, and $f \in \mathbb{F}$. More precisely, we say that the norm is *p-UR with* $\mathbb{F}$. Clearly, if the norm is p-UR, then it is URED. In the case of a dual Banach space $X = Y^*$ we say that the norm is *weak** *uniformly rotund* (*W***UR*), if it is p-UR with $\mathbb{F} = Y \subset Y^{**}$. The norm $\|.\|$ is said to be *locally uniformly rotund* (*LUR*), if $\lim_{n\to\infty} \|x - x_n\| = 0$ whenever $x, x_n \in S_X$ are such that $\lim_{n\to\infty} \|x + x_n\| = 2$.

A compact space K is an *Eberlein compact* if K is homeomorphic to a weakly compact subset of a Banach space in its weak topology. A compact space K is a *uniform Eberlein compact* if K is homeomorphic to a weakly compact subset of a Hilbert space. A compact space is called *scattered* if every closed subset $L \subset K$ has an isolated point in L. For scattered compact spaces the Cantor derivative sets are defined as follows: $K^{(0)} = K, K^{(1)} = K'$ is the set of all limit points of K. If α is an ordinal and $K^{(\beta)}$ are defined for all $\beta < \alpha$, then we put $K^{(\alpha)} = (K^{(\beta)})'$ for $\alpha = \beta + 1$ and $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}$ for α a limit ordinal.

If we consider spaces such as $c_0(\Gamma), \ell_1(\Gamma), l_\infty(\Gamma)$, by the symbol e_γ we mean the standard unit vector.

For more information in this area we refer to [3], [5], [7, Ch. 12], [10] and [14].

2. The three space problem

Theorem 1. *Let X be a Banach space such that $c_0(\Gamma) \subset X$. Let the dual to $Y = X/c_0(\Gamma)$ admit an equivalent dual p-UR (URED) norm. Then X^* admits an equivalent dual p-UR (URED) norm.*

Proof. Let $i : c_0(\Gamma) \to X$ be the inclusion map and $q : X \to Y$ be the quotient map. Then the dual mappings are $i^* : X^* \to c_0(\Gamma)^*$, which is a quotient map and a restriction, and $q^* : Y^* \to X^*$, which is an inclusion. Because of the lifting property of the space $\ell_1(\Gamma) \cong c_0(\Gamma)^*$ there is a bounded linear map $l : \ell_1(\Gamma) \to X^*$ (the so-called lifting; see, e.g., [4]) such that $i^*(l(e)) = e$ for all $e \in \ell_1(\Gamma)$. Hence we have an isomorphism $X^* \cong \ell_1(\Gamma) \oplus Y^*$, where the duality between $(f, g) \in \ell_1(\Gamma) \oplus Y^*$ and $x \in X$ is given by the formula

$$\langle (f, g), x\rangle = \langle l(f), x\rangle + \langle q^*(g), x\rangle\,.$$

Let $\|.\|_{Y^*}$ be a dual norm on Y^* which is p-UR with $\mathbb{F}$. We will prove that there is an equivalent dual norm $\|.\|_u$ on X^*, that is, p-UR with $\mathbb{G} = \{(e_\gamma, 0); \gamma \in \Gamma\} \cup \{(0, f); f \in \mathbb{F}\}$, where we identify X^{**} with $l_\infty(\Gamma) \oplus Y^{**}$ and where $\{e_\gamma; \gamma \in \Gamma\}$ denote the standard unit vectors in $c_0(\Gamma) \subset l_\infty(\Gamma)$. The proof that the norm $\|.\|_u$ is URED if $\|.\|_{Y^*}$ is URED proceeds in the same way.

Let $\|.\|_{X^*}$ be a dual norm on X^* which extends the norm $\|.\|_{Y^*}$. Let $\|.\|_{\ell_1(\Gamma)}$ be the standard norm on $\ell_1(\Gamma)$. We choose $a > 1$ such that

$$a^{-1}\|(f,g)\|_{X^*} \leq \|f\|_{\ell_1(\Gamma)} + \|g\|_{Y^*} \leq a\|(f,g)\|_{X^*}.$$

Put

$$\|(f,g)\|_w = \left(\|f\|^2_{\ell_1(\Gamma)} + \|f\|^2_{l_2(\Gamma)} + \|g\|^2_{Y^*}\right)^{\frac{1}{2}}.$$

This is an equivalent norm on $X^* \cong \ell_1(\Gamma) \oplus Y^*$. The norm $\|.\|_w$ need not be a dual norm, but it is p-UR with $\mathbb{G}$. This convexity property will be used at the end of this proof. To have a dual norm, let us define

$$\|(f,g)\| = \|(f,g)\|_w + a\|f\|_{\ell_1(\Gamma)}.$$

Observation. The norm $\|.\|$ is a dual norm on X^*.

Proof of the Observation. We will follow the proof published in [8] and show that the unit ball is closed in the weak* topology. To prove this, let $\{(f_\alpha, g_\alpha)\}_{\alpha\in A}$ be a net in the unit ball in $(X^*, \|.\|)$, which weak* converges to (f,g). Because $c_0(\Gamma) \subset X$ and $\ell_1(\Gamma) \cong c_0(\Gamma)^*$, $\{f_\alpha\}_{\alpha\in A}$ converges coordinatewise to f. To see this, choose $x \in c_0(\Gamma)$. We have

$$\begin{aligned}\langle (f_\alpha, g_\alpha), i(x)\rangle &= \langle l(f_\alpha), i(x)\rangle + \langle q^*(g_\alpha), i(x)\rangle \\ &= \langle i^*(l(f_\alpha)), x\rangle + \langle g_\alpha, q(i(x))\rangle = \langle f_\alpha, x\rangle .\end{aligned}$$

To estimate the norm of (f,g) we will decompose f_α in a special way. For each $\alpha \in A$, we can find elements $f^1_\alpha, f^2_\alpha \in \ell_1(\Gamma)$ such that $f_\alpha = f^1_\alpha + f^2_\alpha$, the supports of f^1_α, f^2_α are disjoint and $\lim_{\alpha\in A} \|f_\alpha - f^1_\alpha\|_{\ell_1(\Gamma)} = 0$. By passing to a subnet, we can assume that $\{(f^2_\alpha, 0)\}_{\alpha\in A}$ weak* converges to some $(0, g^1)$ and $\{(0, g_\alpha)\}_{\alpha\in A}$ weak* converges to $(0, g_2) = (0, g - g_1)$. Then

$$\begin{aligned}\|f\|_{\ell_1(\Gamma)} &\leq \liminf_{\alpha\in A} \|f^1_\alpha\|_{\ell_1(\Gamma)}, \\ \|g_1\|_{Y^*} = \|(0,g_1)\|_{X^*} &\leq \liminf_{\alpha\in A} \|(f^2_\alpha, 0)\|_{X^*} \leq a \liminf_{\alpha\in A} \|f^2_\alpha\|_{\ell_1(\Gamma)}, \\ \|g_2\|_{Y^*} = \|(0,g_2)\|_{X^*} &\leq \liminf_{\alpha\in A} \|(0,g_\alpha)\|_{X^*} = \liminf_{\alpha\in A} \|g_\alpha\|_{Y^*},\end{aligned}$$

where we used that $\|.\|_{X^*}$ is the dual norm. It follows from previous estimates that

$$\begin{aligned}\|(f,g)\| &= a\|f\|_{\ell_1(\Gamma)} + \left(\|f\|^2_{\ell_1(\Gamma)} + \|f\|^2_{\ell_2(\Gamma)} + \|g_1 + g_2\|^2_{Y^*}\right)^{\frac{1}{2}} \\ &\leq a\|f\|_{\ell_1(\Gamma)} + \|g_1\|_{Y^*} + (\|f\|^2_{\ell_1(\Gamma)} + \|f\|^2_{\ell_2(\Gamma)} + \|g_2\|^2_{Y^*})^{\frac{1}{2}} \\ &\leq \liminf_{\alpha\in A}\Big(a\|f^1_\alpha\|_{\ell_1(\Gamma)} + a\|f^2_\alpha\|_{\ell_1(\Gamma)} + \left(\|f^1_\alpha\|^2_{\ell_1(\Gamma)} + \|f^1_\alpha\|^2_{\ell_2(\Gamma)} + \|g_\alpha\|^2_{Y^*}\right)^{\frac{1}{2}}\Big) \\ &\leq \limsup_{\alpha\in A} \|(f_\alpha, g_\alpha)\| \leq 1.\end{aligned}$$

Thus dual unit ball is w^*-closed and the Observation is proved. □

We will continue with the proof of Theorem 1. Let us define the norm $\|.\|_u$ on X^* by the formula

$$\|(f,g)\|^2_u = \|(f,g)\|^2 + \|f\|^2_{\ell_1(\Gamma)}.$$

It is a dual norm, because it is w^*-lower semicontinuous as is the seminorm $\|.\|_{\ell_1(\Gamma)}$ on X^*. We prove that it is p-UR with $\mathbb{G}$. To do this, we use the following fact, which can be found with the proof in [5, Ch. II].

Fact. Let a_n, b_n be bounded elements of a Banach space $(E, \|.\|)$ such that

$$\lim_{n\to\infty}\big(2\|a_n\|^2 + 2\|b_n\|^2 - \|a_n + b_n\|^2\big) = 0.$$

Then $\lim_{n\to\infty}\big(\|a_n\| - \|b_n\|\big) = 0$ and $\lim_{n\to\infty}\big(\|a_n\| + \|b_n\| - \|a_n + b_n\|\big) = 0$.

Now assume that $x_n = (x_n^1, x_n^2), y_n = (y_n^1, y_n^2) \in X^*$ satisfy

$$\lim_{n\to\infty}\big(2\|x_n\|_u^2 + 2\|y_n\|_u^2 - \|x_n + y_n\|_u^2\big) = 0. \tag{1}$$

By the previous Fact we have

$$\lim_{n\to\infty}\big(2\|x_n^1\|_{\ell_1(\Gamma)}^2 + 2\|y_n^1\|_{\ell_1(\Gamma)}^2 - \|x_n^1 + y_n^1\|_{\ell_1(\Gamma)}^2\big) = 0.$$

And again by the Fact

$$\begin{gathered}\lim_{n\to\infty}\big(\|x_n^1\|_{\ell_1(\Gamma)} - \|y_n^1\|_{\ell_1(\Gamma)}\big) = 0,\\ \lim_{n\to\infty}\big(\|x_n^1\|_{\ell_1(\Gamma)} + \|y_n^1\|_{\ell_1(\Gamma)}\big) - \lim_{n\to\infty}\|x_n^1 + y_n^1\|_{\ell_1(\Gamma)} = 0.\end{gathered} \tag{2}$$

By (1) and by the Fact it follows that

$$\lim_{n\to\infty}\big(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2\big) = 0.$$

Hence by the Fact

$$\lim_{n\to\infty}\big(\|x_n\| + \|y_n\|\big) - \lim_{n\to\infty}\|x_n + y_n\| = 0. \tag{3}$$

By (2) and (3) we have

$$\begin{gathered}\lim_{n\to\infty}\big(\|x_n\|_w + \|y_n\|_w\big) = \lim_{n\to\infty}\|x_n + y_n\|_w,\\ \lim_{n\to\infty}\|x_n\|_w = \lim_{n\to\infty}\|y_n\|_w.\end{gathered}$$

Hence

$$\lim_{n\to\infty}\Big\|\frac{x_n}{\|x_n\|_w} + \frac{y_n}{\|y_n\|_w}\Big\|_w = \lim_{n\to\infty}\Big\|\frac{x_n}{\|x_n\|_w} + \frac{y_n}{\|x_n\|_w}\Big\|_w = 2.$$

The norm $\|.\|_w$ on $\ell_1(\Gamma) \oplus Y^*$ is p-UR with $\mathbb{G}$; thus, for all $G \in \mathbb{G}$ we have

$$\lim_{n\to\infty} G\Big(\frac{x_n}{\|x_n\|_w} - \frac{y_n}{\|x_n\|_w}\Big) = 0,$$

which finishes the proof of Theorem 1. □

Remark. Moreover, if the norm $\|.\|_{Y^*}$ on Y^* is LUR, the norm $\|.\|_u$ on X^* is LUR as well because the norm $\|.\|_w$ is.

Corollary 2. *Let K be a scattered compact space with $K^{(\omega)} = \emptyset$. Then $C(K)^*$ admits an equivalent dual norm that is simultaneously LUR and p-UR with $\mathbb{F} = \{e_k; k \in K\} \subset c_0(K) \subset C(K)^{**}$.*

Proof. By a compactness argument, there is some $n \in \mathbb{N}$ such that $K^{(n)} = \emptyset$. We shall prove the corollary by an induction. If $n = 0, 1$, the claim is trivial. Let $n > 1$. It is easy to see that the space $Y = \{f \in C(K); f|_{K'} = 0\}$ is isometric to the space $c_0(K \setminus K')$. Moreover, $C(K)/Y = C(K')$, so we can use Theorem 1. □

In fact, the following theorem holds.

Theorem (Deville)**.** *Let K be a scattered compact space, such that $K^{(\omega_1)} = \emptyset$. Then $C(K)^*$ admits an equivalent dual norm, that is LUR and p-UR.*

Proof. See [5, Theorem 7.4.7]. There is an equivalent dual LUR norm constructed on $C(K)^*$. One can compute that this norm is, moreover, p-UR with $\mathbb{F} = \{e_k; k \in K\}$. □

Note that it is shown in [8] (see also [13]) that there is a Banach space JL_2 with the following properties:

(1) $c_0 \subset \mathrm{JL}_2, \mathrm{JL}_2/c_0 = \ell_2(\Gamma)$, where the cardinality of the set Γ is a continuum,
(2) JL_2 is not a subspace of any WCG space; in particular, JL_2 is not isomorphic to the $c_0 \oplus \ell_2(\Gamma)$,
(3) there is an equivalent dual LUR norm on JL_2^*.

From Theorem 1 we can obtain a stronger result.

Theorem 3. *There is an equivalent dual norm on the space JL^* that is LUR and p-UR with $\mathbb{F}$, where $\mathbb{F}$ is the canonical imbedding of $c_0 \oplus \ell_2(\Gamma)$ into $\ell_\infty \oplus \ell_2(\Gamma) \cong JL_2^{**}$.*

It is shown in [5, pp. 299–305], that if K is a so-called "two arrow" compact space, then $C([0,1]) \subset C(K), C(K)/C([0,1]) = c_0([0,1])$ and $C(K)$ has no equivalent Gâteaux smooth norm. It means that there is no dual equivalent strictly convex norm on $C(K)^*$. It means that $C(K)^*$ does not admit an equivalent dual URED norm, although both $C([0,1])^*$ and $c_0([0,1])^*$ do admit an equivalent dual p-UR norms.

3. Scattered Eberlein compact spaces

Theorem 4. *Let K be a scattered compact space such that $K = \bigcup_{n=1}^\infty K_n$, and for all $n \in \mathbb{N}$ let $C(K_n)^*$ admit an equivalent dual p-UR norm with $\mathbb{F}_n = \{e_k; k \in K_n\}$. Then $C(K)^*$ admits an equivalent dual p-UR norm with $\mathbb{F} = \{e_k; k \in K\}$.*

Proof. This proof is similar to the proof of Theorem 2.7.16 in [5], which states, that the space $L_1(\Omega)$ admits a norm that is LUR and URED.

As in [9], we can define the operator $T : C(K) \to \sum_{\ell_2} C(K_n)$ by the formula $T(f) = \left(\frac{1}{n^2} f|_{K_n}\right)$. For $k \in K$ put $N(k) = \{n \in \mathbb{N}; k \in K_n\}$. For $k \in K, n \in N(k)$ let $\widetilde{k}_n$ denote a copy of k in K_n. For $A \subset K$ put $\widetilde{A} = \{\widetilde{k}_n; k \in A, n \in N(k)\}$. By Rudin's Theorem (see [7]) $C(K)^*$ is isometric to the space $l_1(K)$ and the canonical norm $\|.\|_1$ is a dual norm on $C(K)^*$, the same holds for K_n's. Without loss of generality, we can assume, that the p-UR norms are uniformly close to the original norms on $C(K_n)^*$. Hence $(\sum_{\ell_2} C(K_n))^* \cong \sum_{\ell_2} \ell_1(K_n)$ and $(\sum_{\ell_2} C(K_n))^*$ admits an equivalent dual norm $\|.\|_\Sigma$, which is p-UR with $\mathbb{G} = \{e_{\widetilde{k}_n}; k \in K, n \in N(k)\}$. The dual operator $T^* : (\sum_{\ell_2} C(K_n))^* \to C(K)^*$ is given by

$$T^*(y^*) = \Big(\sum_{n \in N(k)} \frac{1}{n^2} y^*(\widetilde{k}_n)\Big)_{k \in K}.$$

The range of T^* is a dense set in $C(K)^*$. Now, we shall use the standard LUR renorming method. For $n \in \mathbb{N}$ and $x \in \ell_1(K)$ we define

$$|x|_n^2 = \inf\Big\{\|x - T^* y\|_1^2 + \frac{1}{n}\|y\|_\Sigma^2; y \in \sum_{\ell_2} C(K_n)^*\Big\},$$

$$|||x|||^2 = \|x\|_1^2 + \sum_{n=1}^\infty 2^{-n} |x|_n^2.$$

This is a dual norm and we will prove that it is p-UR. Choose $x_i, y_i \in l_1(K)$ such that $\|x_i\|_1 \leq 1, \|y_i\|_1 \leq 1$ and

$$\lim_{i\to\infty} \left(2|||x_i|||^2 + 2|||y_i|||^2 - |||x_i + y_i|||^2\right) = 0.$$

Then for all $n \in \mathbb{N}$,

$$\lim_{i\to\infty} \left(2|x_i|_n^2 + 2|y_i|_n^2 - |x_i + y_i|_n^2\right) = 0. \tag{1}$$

The infimum in the definition of $|.|_n$ is attained (see [5, p. 44]); e.g., for all $i, n \in \mathbb{N}$ there are $u_i^{(n)}, v_i^{(n)} \in \sum_{\ell_2} C(K_n)^*$ such that

$$\begin{aligned} |x_i|_n^2 &= \|x_i - T^* u_i^{(n)}\|_1^2 + \frac{1}{n}\|u_i^{(n)}\|_\Sigma^2, \\ |y_i|_n^2 &= \|y_i - T^* v_i^{(n)}\|_1^2 + \frac{1}{n}\|v_i^{(n)}\|_\Sigma^2. \end{aligned} \tag{2}$$

From (2) we get

$$\|u_i^{(n)}\|_\Sigma \leq n|x_i|_n \leq n\|x_i\|_1 \leq n, \tag{3}$$

and by the same manner we have $\|v_i^{(n)}\|_\Sigma \leq n$; therefore, for all $k \in K$ and $l \in N(k)$

$$(u_i^{(n)} - v_i^{(n)})(\widetilde{k}_l) \leq \|u_i^{(n)} - v_i^{(n)}\|_1 \leq c.\|u_i^{(n)} - v_i^{(n)}\|_\Sigma \leq 2cn, \tag{4}$$

where c is a constant of the equivalence of norms $\|.\|_1$ and $\|.\|_\Sigma$. From (1), (2), (3) we have

$$\lim_{i\to\infty} \left(2\|u_i^{(n)}\|_\Sigma^2 + 2\|v_i^{(n)}\|_\Sigma^2 - \|u_i^{(n)} + v_i^{(n)}\|_\Sigma^2\right) = 0.$$

The norm $\|.\|_\Sigma$ is p-UR and hence for all $k \in K$, $m \in \mathbb{N}$ and $n \in N(k)$

$$\lim_{i\to\infty} (u_i^{(m)} - v_i^{(m)})(\widetilde{k}_n) = 0. \tag{5}$$

We can assume (by passing to a subsequence), that $\lim_{i\to\infty} |x_i|_n = d_n$. For every $x \in l_1(K)$, $|x|_n$ is a nonincreasing sequence, hence there is $d = \lim_{n\to\infty} d_n$. By passing to a subsequence again, we can assume, moreover, that $\lim_{i\to\infty} |y_i|_n = d_n$. Choose $\varepsilon > 0$ and $k \in K$. Put $A = K \setminus \{k\}$. Let $m \in \mathbb{N}$ be such that $d_m < d + \varepsilon$. Then

$$|(x_i - y_i)(k)| \leq |(x_i - T^* u_i^{(m)})(k)| + |(T^* u_i^{(m)} - T^* v_i^{(m)})(k)| + |(T^* v_i^{(m)} - y_i)(k)|.$$

Considering the second term, we have

$$\begin{aligned} \left|T^*\big(u_i^{(m)} - v_i^{(m)}\big)(k)\right| &= \Big|\sum_{n\in N(k)} \frac{1}{n^2}\big(u_i^{(m)} - v_i^{(m)}\big)(\widetilde{k}_n)\Big| \\ &\leq \sum_{n\leq n_0, n\in N(k)} \Big|\frac{1}{n^2}\big(u_i^{(m)} - v_i^{(m)}\big)(\widetilde{k}_n)\Big| + \varepsilon, \end{aligned}$$

where n_0 depends only on ε (because of (4)) and the sum is finite and therefore tends to 0 for $i \to \infty$ because of (5). It remains to prove, that $|(x_i - T^* u_i^{(m)})(k)| < \varepsilon$. We can assume that $k \in K_{n_0}$. Put $y = s_i + (u_i^{(m)}|_{\widetilde{A}})$, where $s_i(l) = n_0^2 x_i(k)$ if $l = \tilde{k}_{n_0}$, and $s_i(l)=0$ otherwise. Considering this y in the definition of $|x_i|_n$ we get

$$\begin{aligned}|x_i|_n^2 &\le \|(x_i-T^*u_i^{(m)})|_A\|_1^2+\frac{1}{n}\|s_i+(u_i^{(m)}|_{\widetilde{A}})\|_\Sigma^2\\ &\le \|(x_i-T^*u_i^{(m)})|_A\|_1^2+\frac{1}{n}(\|s_i\|_\Sigma+\|u_i^{(m)}|_{\widetilde{A}}\|_\Sigma)^2,\\ &\le \|(x_i-T^*u_i^{(m)})|_A\|_1^2+\frac{1}{n}(n_0^2c+mc^2)^2\end{aligned}$$

because

$$\|u_i^{(m)}|_{\widetilde{A}}\|_\Sigma \le c\|u_i^{(m)}|_{\widetilde{A}}\|_1 \le c\|u_i^{(m)}\|_1 \le mc^2,$$

where we used that the canonical norm $\|.\|_1$ on X^* is a lattice norm.

Hence for all $n\in\mathbb{N}$

$$\limsup_{i\to\infty}\Big\|\big(x_i-T^*u_i^{(m)}\big)|_A\Big\|_1^2 \ge \lim_{i\to\infty}|x_i|_n^2-\frac{1}{n}(cn_0^2+mc^2)^2.$$

Finally,

$$\limsup_{i\to\infty}\Big\|\big(x_i-T^*u_i^{(m)}\big)|_A\Big\|_1^2 \ge d^2.$$

For all $i\in\mathbb{N}$ we have

$$\begin{aligned}\Big|\big(x_i-T^*u_i^{(m)}\big)(k)\Big| &= \Big\|x_i-T^*u_i^{(m)}\Big\|_1-\Big\|\big(x_i-T^*u_i^{(m)}\big)|_A\Big\|_1\\ &\le |x_i|_m-\Big\|\big(x_i-T^*u_i^{(m)}\big)|_A\Big\|.\end{aligned}$$

Hence we get

$$\liminf_{i\to\infty}\Big|\big(x_i-T^*u_i^{(m)}\big)(k)\Big| \le d_m-d\le\varepsilon.$$

The same holds for the third term and this concludes the proof. □

Theorem 5. *Let K be a scattered Eberlein compact space. Then $C(K)^*$ admits an equivalent dual norm that is LUR and p-UR with $\mathbb{F}=\{e_k;k\in K\}$. In particular, $C(K)^*$ admits an equivalent dual norm that is LUR and URED.*

Proof. K. Alster proved in [1] that if K is a scattered Eberlein compact space, then K is a strong Eberlein compact space, e.g., $K\subset\{0,1\}^\Gamma$ for some Γ. Hence

$$K=\bigcup_{n=1}^{\infty}K_n,\ \text{ where }\ K_n=\Big\{x\in K;\operatorname{card}(\{\gamma\in\Gamma;x(\gamma)=1\})\le n\Big\}.$$

The K_n's are uniform Eberlein compact spaces, they are scattered and $K^{(n+1)}=\emptyset$. Hence by Corollary 2, $C(K_n)^*$ admits an equivalent dual norm that is both p-UR with $\mathbb{F}=\{e_k;k\in K_n\}$ and LUR. Thus we can use the preceding theorem to finish the proof. □

4. Open question

It is shown in [12, Th. 1] that if X has an unconditional Schauder basis and X^* admits an equivalent URED norm, then X^* admits an equivalent dual weak* uniformly rotund norm. Because there is a scattered Eberlein compact space K that is not uniform Eberlein compact (see, e.g., [2, Example 1.10]), the space $C(K)^*$ admits an equivalent dual p-UR (and hence URED norm) but does not admit any equivalent dual W*UR norm. But we do not know the answer to the following questions. Is there any reflexive Banach space X such that X admits an equivalent URED norm and does not admit any equivalent p-UR (and hence W*UR) norm?

Is there any Banach space that admits an equivalent URED norm and does not admit any p-UR norm?

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