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Repeated Angles in E_4

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Abstract. Let there be given n points in four-dimensional euclidean space E_4. We show that the number of occurrences of the angle α is $o(n^3)$ if α is not a right angle and $\Omega(n^3)$ otherwise.

For a configuration $\mathscr{C}$ of n points in d-dimensional euclidean space E_d, let $f_d(n, \alpha, \mathscr{C})$ denote the number of angles $A\hat{B}C$ that are equal to α, where $0 < \alpha < \pi$ radians and A, B, and C are points of $\mathscr{C}$. Let $f_d(n, \alpha)$ be the supremum of $f_d(n, \alpha, \mathscr{C})$ taken over all configurations $\mathscr{C}$ of n points. The function $f_d(n, \alpha)$ is integer valued and $0 \leq f_d(n, \alpha) \leq N = [n(n-1)(n-2)]/2$ so that the supremum is in fact a maximum.

Conway *et al.* [1] established the growth estimate $f_3(n, \alpha) = o(n^3)$ as $n \to \infty$ which they needed to establish the properties of certain angle-counting functions. P. Erdös asked me for which k and α $f_k(n, \alpha) = o(n^3)$. The modified "Lenz" construction:

$$X_i = (\lambda \mathcal{U}_i, \lambda \mathcal{U}_i, 0, 0, 0, 0), \qquad 1 \leq i \leq m,$$

$$Y_j = (0, 0, \mu \mathcal{U}_j, \mu \mathcal{U}_j, 0, 0), \qquad 1 \leq j \leq m,$$

$$Z_k = (0, 0, 0, 0, \nu \mathcal{U}_k, \nu \mathcal{U}_k), \qquad 1 \leq k \leq m,$$

where $\mathcal{U}_i^2 + \mathcal{V}_i^2 = 1$, $1 \leq i \leq m$, and λ, μ, $\nu > 0$ shows that $f_6(n, \alpha) \geq [n/3]^3$ for $0 < \alpha < \pi/2$.

In this note we show the following:

Theorem. *If* $\alpha \neq \pi/2$, *then* $f_4(n, \alpha) = o(n^3)$, *but* $f_4(n, \pi/2) \geq [n/3]^3$.

Proof if $\alpha = \pi/2$. Let $m = [n/3]$ and let $\mathcal{U}_i$, $\mathcal{V}_i$ be m solutions of $\mathcal{U}^2 + \mathcal{V}^2 = 1$. Let

$$X_i = (\mathcal{U}_i, \mathcal{V}_i, 0, 0), \qquad 1 \leq i \leq m,$$

$$Y_j = (1, 0, j, 0), \qquad 1 \leq j \leq m,$$

and

$$Z_k = (-1, 0, 0, k), \qquad 1 \le k \le m.$$

Then $(Y_j - X_i) \cdot (Z_k - X_i) = \mathcal{U}_i^2 - 1 + \mathcal{V}_i^2 = 0$ and the m^3 angles $Y_j\hat{X}_iZ_k$ are all right angles. □

Proof for $\alpha \ne \pi/2$. We shall assume $\alpha \ne \pi/2$ from now on, and we shall prove the stronger result $f_4(n, \alpha) = o(n^{3-\varepsilon})$, where $\varepsilon = \frac{1}{25}$. Suppose not. Then by the following combinatorial lemma of Erdös [2] there are 15 points X_i, Y_j, Z_k such that the 125 angles $Y_j\hat{X}_iZ_k$ all equal α, $1 \le i, j, k \le 5$. □

Combinatorial Lemma. *Let $H \subseteq A \times A \times A$, where $|A| = n$ and $|H| \ge n^{3-\varepsilon}$. Then there are subsets $A_i \subseteq A$, $1 \le i \le 3$, such that $|A_i| \ge k$ and $A_1 \times A_2 \times A_3 \subseteq H$, provided $k^2 \le 1/\varepsilon$. We use this lemma with $k = 5$ and $\varepsilon = \frac{1}{25}$.*

We need two additional lemmas.

Lemma 1. *The points $X_1, X_2, \ldots, X_5$ are not collinear.*

Lemma 2. *No three Y_j are collinear and no three Z_k are collinear.*

Proof of Lemma 1. The X_i are solutions to the vector equation

$$\{(X - Y_1) \cdot (X - Z_1)\}^2 = \cos^2 \alpha \{(X - Y_1) \cdot (X - Y_1)\}\{(X - Z_1) \cdot (X - Z_1)\}.$$

Suppose that the X_i lie on the line $X = C + t\mathcal{U}$. Substituting into the above equation, we obtain an equation in the scalar t of the fourth degree, since $\cos^2\alpha \ne 1$. Such an equation cannot have five solutions. □

Proof of Lemma 2. We shall show that no three Z_k are collinear, and the proof for the Y_j is similar. Suppose, without loss of generality, that Z_1, Z_2, and Z_3 lie on the line l. For a fixed X_i, the points X_i, Y_1 and the line l fit into an E_3, in which the locus of points Z such that $Y_1\hat{X}_iZ = \alpha$ is a cone with apex X_i. A line such as l that intersects the cone in three points must pass through the apex X_i. Hence the five points X_i all lie on l, contrary to the previous lemma. □

Proof of the Theorem. Let $X_1 = 0$ be the origin of coordinates and let $\hat{Y}_j$ denote a unit vector in the direction of Y_j and similarly for $\hat{Z}_k$. The points $\hat{Y}_j$ span an affine hull B which does not necessarily pass through $X_1 = 0$. The $\hat{Y}_j$ are not necessarily distinct, but no three can be the same, by Lemma 2, so there are at least three different ones. This forces B to have dimension two or more, since the $\hat{Y}_j$ lie on a unit sphere, and no three of them can be collinear.

Let $1 \le j, k, r \le 5$. Then $\hat{Y}_j \cdot \hat{Z}_k = \cos\alpha = \hat{Y}_r \cdot \hat{Z}_k$, so that $(\hat{Y}_j - \hat{Y}_r) \cdot \hat{Z}_k = 0$, and therefore $(\hat{Y}_j - \hat{Y}_r) \cdot Z_k = 0$. Thus B is orthogonal to the subspace H spanned by $X_1 = 0$ and the five points Z_k. If the dimension of H is three or more, then we have the absurdity of an orthogonal E_2 and E_3 in E_4. We also know by Lemma 2 that H is not a line. Hence H is a plane which together with Y_1 fits into three space. The cone, with apex $X_1 = 0$, which is the locus of points Z such that $Y_1\hat{X}_1 Z = \alpha$, cuts the plane H in a pair of (possibly coincident) lines, one of which must consequently contain three of the points Z_k, contrary to Lemma 2. □

References

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2. P. Erdös, On extremal problems of graphs and generalized graphs, Israel J. Math. **2** (1964), 183–190.

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