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REPETITIONS IN THE FIBONACCI INFINITE WORD (*)

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Abstract. – *Let* φ *be the golden number; we prove that the Fibonacci infinite word contains no fractional power with exponent greater than* $2+\varphi$ *and we prove that for any real number* $\varepsilon>0$ *the Fibonacci infinite word contains a fractional power with exponent greater than* $2+\varphi-\varepsilon$.

Résumé. – *Soit* φ *le nombre d'or; nous prouvons que le mot infini de Fibmacci ne contient pas la puissance fractionnaire d'exposant supérieur à* $2+\varphi$, *et nous prouvons qu'il contient des puissances d'exposant supérieur à* $2+\varphi-\varepsilon$, *quel que soit le nombre réel* $\varepsilon>0$.

INTRODUCTION

Many papers are concerned with the existence of integer powers in "long enough" words or in infinite words; a classical combinatorial property is wether a given infinite word is k power-free or not, with k natural number.

No word on a two letters alphabet can avoid a square but it is well known that the Thue infinite word **t** on a two letter alphabet does not contain cubes and that the Thue infinite word **m** on a three letter alphabet does not contain squares (*see* [9], [10]).

The notion of overlap-free word and more generally the notion of fractional power are considered in many papers (*see* for instance [4], [7], [9], [10]).

In this paper we prove that the Fibonacci infinite word contains no fractional power with exponent greater than $2+((\sqrt{5}+1)/2)$ and that for any real number $\varepsilon>0$ the Fibonacci infinite word contains a fractional power with exponent greater than $2+((\sqrt{5}+1)/2)-\varepsilon$.

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To our knowledge this is the first time that this property for a non rational value is looked for in a given infinite word.

DEFINITIONS AND PRELIMINARY RESULTS

We refer to [6] for the terminology.

Let A be an alphabet. We denote by A^* the *free monoid* on A. The elements of A^* are called *words* and the elements of A are called *letters*. We denote by 1 the empty word which is the identity of A^*; we also denote by $|v|$ the length of a word v.

A word v is a *factor* of a word w if there exist $u, u' \in A^*$ such that

$$w = uvu'$$

and we say that v is a *left factor* of w if u is the empty word.

If a word w is of the form

$$w = v \ldots v = v^k$$

with $u \neq 1$, we say that w is a *k-power* of v; k is called the *exponent* of the power and v is the *base* of the power.

If a word w is of the form

$$w = v \ldots vu = v^k u$$

with $u \neq 1$, $k \geqq 1$ and u left factor of v, we say that w is a *fractional power* of u of exponent $e = |w|/|v|$ and v is the base of the power.

An infinite word s on an alphabet A is a map from the set of positive integers into A; we denote by A^ω the set of all infinite words on the alphabet A.

A word $v \in A^*$ is a factor of the infinite word s if there exist $u \in A^*$, $s' \in A^\omega$ such that $s = uvs'$. If u is the empty word then v is a left factor of s.

The Fibonacci infinite word $\mathbf{f}$ on the alphabet $A = \{a, b\}$ is obtained by iterating the morphism $\psi : \{a, b\} \to \{a, b\}$ given by

$$\psi(a) = ab, \qquad \psi(b) = a$$

starting with the letter a (*see* [1]). Therefore

$$\mathbf{f} = abaababaabaabab\ldots$$

We define the sequence of the finite Fibonacci words by the rule:

$$\mathbf{f}_0 = b,$$

$$\mathbf{f}_{n+1} = \psi(\mathbf{f}_n).$$

It is easy to see that $\mathbf{f}_{n+2} = \mathbf{f}_{n+1}\mathbf{f}_n$ and, consequently, the sequence $|\mathbf{f}_n|$, $n \in \mathbb{N}$ is the sequence of Fibonacci numbers; moreover for any $n \geqq 1$, $\mathbf{f}_n$ is a left factor of $\mathbf{f}_{n+1}$ and of $\mathbf{f}$.

For $n \geqq 2$ we denote by $\mathbf{g}_n$ the word $\mathbf{f}_{n-2}\mathbf{f}_{n-1}$. *It is easy to see that for each* $n \geqq 2$ there exists a word $\mathbf{v}_n$ such that $\mathbf{f}_n = \mathbf{v}_n xy$ and $\mathbf{g}_n = \mathbf{v}_n yx$ with $x, y \in \{a, b\}$ and $x \neq y$ and also that $\mathbf{f}_{n+2} = \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1}$.

The following fact is straigthforward

Fact. – If u is a left factor of $\mathbf{f}_n$ and also of $\mathbf{g}_{n-1}$ then u is a left factor of $\mathbf{v}_{n-1}$ and, consequently

$$|u| \leqq |\mathbf{v}_{n-1}| = |\mathbf{g}_{n-1}| - 2 = |\mathbf{f}_{n-1}| - 2.$$

In the sequel we will use the following results.

PROPOSITION 1 (Karhumäki [4]): *The Fibonacci infinite word* $\mathbf{f}$ *contains no 4-power.*

PROPOSITION 2 (Séébold [8]): *Let* $v \neq 1$; *if* v^2 *is a factor of the Fibonacci infinite word* $\mathbf{f}$ *then there exists n such that* $|v| = |\mathbf{f}_n|$; *more precisely* $v = wz$ *with* $zw = \mathbf{f}_n$ *for some words z and w*, $|w| > 0$, i. e. *v is a conjugate of* $\mathbf{f}_n$.

Now let $u \neq 1$, $u \in A^*$ and let $u = x_1 \ldots x_n$, $x_i \in A$; we denote by $\hat{u}$ the *mirror image* of u, that is $x_n \ldots x_1$.

We say that a factor u of $\mathbf{f}$ is *special* if ua and ub are both factors of $\mathbf{f}$.

PROPOSITION 3 (Berstel [1]): *If u is a special factor of the Fibonacci infinite word* $\mathbf{f}$ *then* $\hat{u}$ *is a left factor of* $\mathbf{f}$.

Since the sequence $|\mathbf{f}_n|$, $n \in \mathbb{N}$, is the sequence of Fibonacci numbers, we have the following proposition.

PROPOSITION 4 (Hardy and Wright [5]): *For any* $n > 1$

$$\frac{|\mathbf{f}_{n+1}| - 2}{|\mathbf{f}_n|} = \frac{|\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|} < \frac{\sqrt{5}+1}{2}$$

and

$$\lim_{n \to \infty} \frac{|\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|} = \frac{\sqrt{5}+1}{2}.$$

PROPOSITION 5 (de Luca [2]): *For each i the word* $\mathbf{f}_i$ *is primitive; therefore for each i the conjugates of* $\mathbf{f}_i$ *are distinct.*

RESULTS AND PROOFS

Let us prove the following lemma.

LEMMA: *No fractional power with exponent greater than* $1+(\sqrt{5}+1)/2$ *can be a left factor of the Fibonacci infinite word* $\mathbf{f}$. *More precisely, if* vvu *is a fractional power which is a left factor of* $\mathbf{f}$ *then* $v=\mathbf{f}_n$ *for some n and* $|vvu| \leqq |\mathbf{f}_n| + |\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2$.

Proof: Let vvu be a fractional power which is a left factor of $\mathbf{f}$.

By using Proposition 2 we have that $|v| = |\mathbf{f}_n|$ for some n, and, consequently vv is a left factor of $\mathbf{f}$ with length $2|\mathbf{f}_n|$. By inspection one can easily see that n is greater than or equal to 3.

As $\mathbf{f}_n$ is a left factor of $\mathbf{f}$ we have that $v=\mathbf{f}_n$ for some $n \geqq 3$. Thus vvu $= \mathbf{f}_n \mathbf{f}_n u$ and either u is a left factor of $\mathbf{f}_n$ or $\mathbf{f}_n$ is a left factor of u.

But for $n \geqq 3$ $\mathbf{f}_{n+2} = \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1}$ is a left factor of $\mathbf{f}$.

Hence, since $\mathbf{g}_{n-1}$ is not a left factor of $\mathbf{f}_n$, we have that u is necessarily a left factor of $\mathbf{g}_{n-1}$; by the fact

$$|u| \leqq |\mathbf{f}_{n-1}| - 2.$$

Thus $|vvu| \leqq |\mathbf{f}_n| + |\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2$ and, by Proposition 4,

$$\frac{|vvu|}{|v|} \leqq \frac{|\mathbf{f}_n| + |\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|} < 1 + \frac{\sqrt{5}+1}{2}, \quad \square$$

We are now ready to prove our main result.

PROPOSITION 6: *The Fibonacci infinite word* $\mathbf{f}$ *contains no fractional power with exponent greater than* $2+((\sqrt{5}+1)/2)$ *and, for any real number* $\varepsilon > 0$, *it contains a fractional power with exponent greater than* $2+((\sqrt{5}+1)/2)-\varepsilon$.

Proof: Let $vvvu$ be a fractional power factor of $\mathbf{f}$. As in $\mathbf{f}$ there are no 4 powers (Proposition 1) one can find in $\mathbf{f}$ a factor

$$u' xu'' u' xu'' u' xu'' u' y$$

where $u' xu'' = v$, u is a left factor of u', $u'' \in \{a, b\}^*$ and $x, y \in \{a, b\}$ with $x \neq y$.

It follows that $u' xu'' u' xu'' u'$ is a special factor of $\mathbf{f}$. By Proposition 3, $\hat{u}' \hat{u}'' x\hat{u}' \hat{u}'' x\hat{u}'$ is a left factor of $\mathbf{f}$. From the Lemma

$$\frac{|\hat{u}' \hat{u}'' x\hat{u}' \hat{u}'' x\hat{u}'|}{|\hat{u}' \hat{u}'' x|} = \frac{|vvu'|}{|v|} < 1 + \frac{\sqrt{5}+1}{2},$$

and, consequently,

$$\frac{|vvvu|}{|v|} \leqq \frac{|vvvu'|}{|v|} < 2 + \frac{\sqrt{5}+1}{2}.$$

At last, for $n \geqq 3$, $\mathbf{f}_{n+4} = \mathbf{f}_{n+1} \mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1} \mathbf{f}_{n-1} \mathbf{f}_n$.

Hence, for $n \geqq 3$, $\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{v}_{n-1}$ is always a factor of $\mathbf{f}$.

Since

$$\frac{|\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{v}_{n-1}|}{|\mathbf{f}_n|} = 2 + \frac{|\mathbf{f}_n| + |\mathbf{f}_{n-1}| - 2}{|\mathbf{f}_n|},$$

the second part of the proposition follows from Proposition 4. □

In the proof of the above proposition we used the fact that for $n \geqq 3$, $\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{v}_{n-1}$ is a factor of $\mathbf{f}$. As a consequence all words of the form $wzwzwz$ with $zw = \mathbf{f}_n$ and $|z| \leqq |\mathbf{v}_{n-1}|$ are factors of $\mathbf{f}$; by Proposition 5 all these words are distinct. Since $0 \leqq |z| \leqq \mathbf{v}_{n-1}|$, the number of these words is $|\mathbf{v}_{n-1}| + 1$.

Let us suppose that vvv is a factor of $\mathbf{f}$ and that $|v| = |\mathbf{f}_n|$ for some $n \geqq 3$. By proposition 2, $v = wz$, $|w| > 0$, and $zw = \mathbf{f}_n$.

Suppose that $|z| > |\mathbf{v}_{n-1}|$; since $\mathbf{f}_n = \mathbf{f}_{n-1} \mathbf{f}_{n-2} = \mathbf{v}_{n-1} yx \mathbf{f}_{n-2}$ with x, $y \in \{a, b\}$ and $x \neq y$, we can write $\mathbf{f}_n = \mathbf{v}_{n-1} yuw$ with $z = \mathbf{v}_{n-1} yu$ and, consequently, $vvv = w \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yu$.

We know that $\mathbf{f}_n \mathbf{f}_n \mathbf{f}_n \mathbf{g}_{n-1} = \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} xy$ is a factor of $\mathbf{f}$; thus $w \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} yuw \mathbf{v}_{n-1} = w \mathbf{v}_{n-1} (yuw \mathbf{v}_{n-1})^2$ is a special factor and by Proposition 3 its mirror image must be a prefix of $\mathbf{f}$. This is impossible by the Lemma because $|w| > 0$.

Hence we have proved the following proposition.

PROPOSITION 7: *For $n \geqq 3$ the number of distinct factors v of* $\mathbf{f}$ *with length* $|\mathbf{f}_n|$ *such that vvv is also a factor of* $\mathbf{f}$ *is exactly* $|\mathbf{v}_{n-1}|+1$. *More precisely they are all the words of the form wz with $zw=\mathbf{f}_n$ and $|z| \leqq |\mathbf{v}_{n-1}|$.*

OBSERVATION: As $2+((\sqrt{5}+1)/2)$ is an irrational number it cannot exist a fractional power with exponent equal to it.

In the Thue infinite word $\mathbf{t}$ on a two letters alphabet A there are clearly squares but there are no overlaps (that is factors like $xvxvy$, $x \in A$, $v \in A^*$). On the contrary it is easy to see that, for any $\varepsilon > 0$, in the Thue infinite word $\mathbf{m}$ on a three letters alphabet there exists a fractional power with exponent greater than $2-\varepsilon$ but it is a classical result that $\mathbf{m}$ is square free.

Remark: Proposition 6 and 7 were firstly proved by using techniques of Sturmian words. Following the suggestion of *P.* Séébold we tried to find a simpler proof; actually our proof is simpler than the previous one and use only elementary properties of the Fibonacci infinite word.

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